

RECOGNITION OF LINEAR ACTIONS ON SPHERES

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ABSTRACT. Let G be a finite group acting smoothly on a homotopy sphere Σ^m . We wish to establish necessary and sufficient conditions for the given G -action on Σ to be topologically equivalent to a linear action. That is, we want to be able to decide whether or not there exists a G -homeomorphism $\gamma: \Sigma \rightarrow S^m(\rho)$, where $S^m(\rho) \subset \mathbf{R}^{m+1}(\rho)$ denotes the unit sphere in an orthogonal representation space $\mathbf{R}^{m+1}(\rho)$ for G . In order for a G -action on Σ to be topologically equivalent to a linear action it is clearly necessary that:

(i) For each subgroup H of G the fixed-point set Σ^H is homeomorphic to a sphere, or empty.

(ii) For any subgroups H and $H \subsetneq H_i$, $1 \leq i \leq k$, of G the pair $(\Sigma^H, \bigcup_{i=1}^k \Sigma^{H_i})$ is homeomorphic to a standard pair $(S^n, \bigcup_{i=1}^k S_i^{n_i})$, where each $S_i^{n_i}$, $1 \leq i \leq k$, is a standard n_i -subsphere of S^n .

In this paper we consider the case where the fixed-point set Σ^G is nonempty and all other fixed-point sets have dimension at least 5. In giving efficient sufficient conditions we do not need the full strength of condition (ii). We only need:

(ii)* For any subgroups H and $H \subsetneq H_i$, $1 \leq i \leq p$, of G such that $\dim \Sigma^{H_i} = \dim \Sigma^H - 2$, the pair $(\Sigma^H, \bigcup_{i=1}^p \Sigma^{H_i})$ is homeomorphic to a standard pair $(S^n, \bigcup_{i=1}^p S_i^{n-2})$, where each S_i^{n-2} , $1 \leq i \leq p$, is a standard $(n-2)$ -subsphere of S^n .

Our main results are then that, in the case when G is abelian, conditions (i) and (ii)* are necessary and sufficient for a given G -action on Σ to be topologically equivalent to a linear action, and in the case of an action of an arbitrary finite group the same holds under the additional assumption that any simultaneous codimension 1 and 2 fixed-point situation is simple. Our results generalize, for actions of finite groups, a well-known theorem of Connell, Montgomery and Yang, and are the first to also cover the case where codimension 2 fixed-point situations occur.

Let G be a finite group acting smoothly on a homotopy sphere Σ^m . (A homotopy sphere Σ^m is a smooth manifold which is homeomorphic to S^m .) We wish to establish necessary and sufficient conditions for the given G -action on Σ to be topologically equivalent to a linear action. That is, we want to be able to decide whether or not there exists a G -homeomorphism $\gamma: \Sigma \rightarrow S^m(\rho)$, where $S^m(\rho) \subset \mathbf{R}^{m+1}(\rho)$ denotes the unit sphere in an orthogonal representation space $\mathbf{R}^{m+1}(\rho)$ for G . In order for a given G -action on Σ to be topologically equivalent to a linear action it is clearly necessary that the following two conditions hold:

(i) For each subgroup H of G the fixed-point set Σ^H is homeomorphic to a sphere, or empty.

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(ii) For any subgroups H and $H \subset H_i$ ($1 \leq i \leq k$) of G the pair $(\Sigma^H, \bigcup_{i=1}^k \Sigma^{H_i})$ is homeomorphic to a standard pair $(S^n, \bigcup_{i=1}^k S_i^{n_i})$, where each $S_i^{n_i}$ ($1 \leq i \leq k$) is a standard n_i -subsphere of S^n .

By a standard q -subsphere of S^n ($-1 \leq q \leq n$), we mean an intersection of $S^n \subset \mathbf{R}^{n+1}$ with a $(q + 1)$ -dimensional linear subspace of \mathbf{R}^{n+1} . We also express condition (ii) by saying that the pair $(\Sigma^H, \bigcup_{i=1}^k \Sigma^{H_i})$ is *topologically standard* or that $\bigcup_{i=1}^k \Sigma^{H_i}$ is topologically standard in Σ^H .

In this paper we consider the case where the fixed-point set Σ^G is nonempty and all other fixed-point sets have dimension at least 5. In formulating convenient sufficient (and necessary) conditions for a given smooth G -action on Σ to be topologically equivalent to a linear action we do not need the full strength of condition (ii). It is enough to assume that (ii) holds in the cases where H_1, \dots, H_k are all the subgroups for which $\dim \Sigma^{H_i} = \dim \Sigma^H - 2$ and $H \subset H_i, i = 1, \dots, k$. Before giving our results we introduce some notation and terminology.

Let M be a smooth G -manifold and H a subgroup of G . Then M^H denotes the fixed-point set of H and we define

$$M^{(>H,j)} = \bigcup_{H' \in J(j)} M^{H'}, \quad j = 1, 2,$$

where $J(j)$ denotes the set of all subgroups H' of G such that $H \subset H'$ and $\dim M^{H'} = \dim M^H - j$. (Of course $M^{(>H,j)}$ can also be defined for other values of j , but we only need the cases $j = 1$ and 2 .) We say that M^H contains codimension j fixed-point subsets if $M^{(>H,j)} \neq \emptyset$ ($j = 1, 2$), and in this case we call the pair $(M^H, M^{(>H,j)})$ a *codimension j fixed-point situation*. A codimension j fixed-point situation $(M^H, M^{(>H,j)})$ is *simple*, if there exists a subgroup $H_j \supset H$ such that $M^{(>H,j)} = M^{H_j}$. If both $M^{(>H,1)} \neq \emptyset$ and $M^{(>H,2)} \neq \emptyset$ we call $(M^H; M^{(>H,1)}, M^{(>H,2)})$ a *simultaneous codimension 1 and 2 fixed-point situation*. (Observe that in general $M^{(>H,2)} \not\subset M^{(>H,1)}$.) A simultaneous codimension 1 and 2 fixed-point situation $(M^H; M^{(>H,1)}, M^{(>H,2)})$ is *simple* if there exist subgroups H_1 and H_2 such that $M^{(>H,1)} = M^{H_1}$ and $M^{(>H,2)} = M^{H_2}$. Finally, we say that a subgroup H *occurs* in M if there exists $x \in M$ such that $G_x = H$, where G_x denotes the isotropy subgroup at x .

In the case of an action of an arbitrary finite group our main result is:

THEOREM A. *Let G be a finite group acting smoothly on a homotopy sphere Σ such that $\Sigma^G \neq \emptyset$ and $\dim \Sigma^H \geq 5$ for every subgroup $H \neq G$ occurring in Σ , and such that any simultaneous codimension 1 and 2 fixed-point situation is simple. Then the action of G on Σ is topologically equivalent to a linear action if and only if the following two conditions hold:*

- (1) *Each fixed-point set Σ^H is homeomorphic to a sphere.*
- (2) *Each codimension 2 fixed-point situation $(\Sigma^H, \Sigma^{(>H,2)})$ is topologically standard.*

In the case when G is *abelian* we are able to get rid of the extra assumption that if simultaneous codimension 1 and 2 fixed-point situations occur they must be simple.

We prove:

THEOREM B. *Let G be a finite abelian group acting smoothly on Σ such that $\Sigma^G \neq \emptyset$ and $\dim \Sigma^H \geq 5$ for every subgroup $H \neq G$ occurring in Σ . Then the action of G is topologically equivalent to a linear action if and only if conditions (1) and (2) in Theorem A are satisfied.*

Addendum. In the case of a simple codimension 2 fixed-point situation $(\Sigma^H, \Sigma^{(>H,2)}) = (\Sigma^H, \Sigma^{H_2})$ the condition, in Theorems A and B, that (Σ^H, Σ^{H_2}) is topologically standard can be replaced by the condition:

$$(*) \quad \begin{cases} \pi_1(\Sigma^H - \Sigma^{H_2}) \cong \mathbb{Z}, \\ \pi_p(\Sigma^H - \Sigma^{H_2}) = 0, \text{ for all } p \geq 2. \end{cases}$$

PROOF OF ADDENDUM. Condition (*) gives a map $\alpha: S^1 \rightarrow \Sigma^H - \Sigma^{H_2}$ which induces an isomorphism in homotopy in all dimensions. Because $\Sigma^H - \Sigma^{H_2}$ is homeomorphic to an open subset of a sphere it can be triangulated, and hence it follows that α is a homotopy equivalence. Since $\dim \Sigma^H \geq 5$ it now follows by Corollary 9.3 in Stallings [17] that Σ^{H_2} is topologically standard in Σ^H . Conversely if Σ^{H_2} is topologically standard in Σ^H condition (*) clearly holds. \square

A well-known result of Connell, Montgomery and Yang from 1964 says that if a compact Lie group G acts smoothly on S^m such that the fixed-point set $(S^m)^G$ is diffeomorphic to S^k , where $0 \leq k \leq m - 3$, and all other orbits are of the same type and dimension r , where $m - r \geq 5$, then the action of G is topologically equivalent to a linear action of G on S^m . In fact, this result of Connell, Montgomery and Yang is a corollary to their corresponding result concerning actions on euclidean space, where the conclusion moreover is “differentiably equivalent to a linear action”. (See Connell-Montgomery-Yang [5 and 6], and cf. also Siebenmann [14]). It has been a problem to generalize the Connell-Montgomery-Yang result to the case where there are more than two orbit types and fixed-point sets of codimension 2 occur, cf. Problem C.2 in [2]. A generalization to the case with several orbit types but still assuming that there are no codimension 2 fixed-point situations is given, without proof, in Rothenberg [12, Theorem 3.14 and Corollary 3.15]. Our results, for actions of finite groups, are the first to also cover the case where codimension 2 fixed-point situations occur.

It is the codimension 2 fixed-point situations that are the critical ones and give the main problems. In fact, by Giffen [7], there exist smooth actions on spheres with simple codimension 2 fixed-point situations that are *not* topologically standard. Giffen shows, for example, that any finite cyclic group Z_n , $n \geq 2$, can act smoothly on S^5 such that $(S^5)^{Z_n} \cong S^3$ is knotted in S^5 . Such an action cannot be topologically equivalent to a linear action.

Let $(\Sigma^H, \Sigma^{(>H,2)})$ be a codimension 2 fixed-point situation, where $\Sigma^{(>H,2)} = \bigcup_{i=1}^r \Sigma^{H_i}$. Assume that each Σ^{H_i} , $1 \leq i \leq r$, is topologically standard in Σ^H . We do not know whether or not it follows from this assumption that $\Sigma^{(>H,2)}$ must be topologically standard in Σ^H (cf. the formulation of Problem C.2 in [2]).

We shall now give a very brief outline of the proof of our main results and of the contents of this paper. In §1 we explain our notation and terminology and also introduce some preliminary facts that are used (often without specific reference) later on in the paper. In §2 we prove an equivariant engulfing theorem, Theorem 2.4, which is an equivariant analogue of Stallings' engulfing theorem [16]. The main result in §3 is Proposition 3.2 which gives necessary and sufficient conditions for an equivariant p.l. manifold M (for the definition see §1 below) to be G -homeomorphic to some euclidean representation space $\mathbf{R}^m(\rho)$. The proof of Proposition 3.2 uses the equivariant engulfing theorem and also Proposition 3.4, which is an equivariant version of the theorem of Morton Brown [3] on the monotone union of open cells. Proposition 3.2 will be applied to $M = \Sigma - \{b\}$, where $b \in \Sigma^G \neq \emptyset$. It is in §4 that we establish the main technical results that allow us to handle codimension 2 fixed-point situations and the interference of codimension 1 and codimension 2 fixed-point situations. The results in §4 are to be applied to the smooth G -manifold $M = \Sigma - \{b\}$. Lemma 4.3 is related to the case of a simple simultaneous codimension 1 and 2 fixed-point situation and is needed for the proof of Theorem A. Lemma 4.4 deals with the interference of codimension 1 and codimension 2 fixed-point situations in the case when G is abelian and is a step in the proof of Theorem B. (In case Lemma 4.4 could be established for an arbitrary finite group, we would obtain the result in Theorem B for all finite groups.)

The next result, Lemma 4.5, is the key to the handling of codimension 2 fixed-point situations, in the case of both Theorems A and B. Having established Lemmas 4.1–4.4, this Lemma 4.5 now shows that from the assumption of a certain homotopical well-behavior of the codimension 2 fixed-point situations follows a certain homotopical well-behavior of the whole situation (for a precise statement see Lemma 4.5). Finally in 5.1 we give the concluding part of the proofs of Theorems A and B. Here there are two facts that must be verified. First it is shown that the assumption that each codimension 2 fixed-point situation is topologically standard implies the homotopical well-behavior of the codimension 2 fixed-point situations needed for an application of Lemma 4.5. Then Lemma 4.5 implies the homotopical well-behavior of the whole situation, and this fact is needed in order for us to be able to apply Proposition 3.2. The other fact that we need to know, in order to be able to apply Proposition 3.2, is that M is equivariantly unraveled at infinity (see Definition 3.1). The second thing shown in 5.1 is exactly this fact that M is equivariantly unraveled at infinity. This property of $M = \Sigma - \{b\}$ is a consequence of the fact that we are dealing with a smooth action on Σ . We have then shown that M satisfies the conditions in the assumption of Proposition 3.2 and hence by Proposition 3.2 we have that M is G -homeomorphic to some euclidean representation space $\mathbf{R}^m(\rho)$. Since Σ is the one-point compactification of M , it now follows that Σ is G -homeomorphic to $S^m(\rho \oplus \text{id})$.

The equivariant engulfing theorem, Theorem 2.4, and a less general and weaker form of Theorem A were announced in [11].

1. Notation, terminology and other preliminaries. In the rest of this paper G will always be a finite group. If X is a G -space and H is a subgroup of G , then X^H denotes the fixed-point set of H , and by $X^{>H}$ we denote the set of points with

isotropy subgroup strictly greater than H , i.e., $X^{>H} = \{x \in X \mid H \subsetneq G_x\}$. Furthermore we denote

$$X_H = X^H - X^{>H}.$$

Thus X_H is the set of points in X with isotropy subgroup equal to H . We say that a subgroup H of G occurs in X if there exists $x \in X$ with $G_x = H$, i.e., H occurs in X if and only if $X_H \neq \emptyset$.

In §§2 and 3 we will use some equivariant p.l. techniques. The definition of an *equivariant simplicial complex*, and *equivariant p.l. manifold*, and other related matters can already be found in [9 and 10], but for the convenience of the reader they will be repeated here. *In order to prevent misunderstandings the reader should pay attention to all these definitions.* By a simplicial complex we mean a geometric simplicial complex, that is, the topological realization of an abstract simplicial complex considered as a topological space together with the structure given by the simplexes. In this paper all simplicial complexes will be countable, locally finite and finite-dimensional. A simplicial G -complex consists of a simplicial complex K together with a G -action $\varphi: G \times X \rightarrow X$ such that the map $g: X \rightarrow X$ is a simplicial homeomorphism for every $g \in G$. We define an *equivariant simplicial complex* to be a simplicial G -complex K which satisfies the following two conditions.

1. For every subgroup H of G we have that if $s = \langle v_0, \dots, v_m \rangle$ is a simplex of K and $s' = \langle h_0 v_0, \dots, h_m v_m \rangle$, where $h_i \in H$, $i = 0, \dots, m$, also is a simplex of K then there exists $h \in H$ such that $h v_i = h_i v_i$, $i = 0, \dots, m$.

2. For any simplex s of K the vertices v_0, \dots, v_n of s can be ordered in such a way that we have $G_{v_n} \subset \dots \subset G_{v_0}$.

In condition 1 the vertices v_0, \dots, v_m need not be distinct. We call G_{v_n} the *principal isotropy subgroup* of the simplex s and G_{v_0} the *maximal isotropy subgroup* of s . The above conditions are purely technical since the second barycentric subdivision of any simplicial G -complex is an equivariant simplicial complex. (An abstract simplicial G -complex satisfying condition 1 above is called regular by Bredon [1, p. 116].) Moreover it is easy to see that in the presence of condition 2 condition 1 can be replaced by the following.

1'. If $s = \langle v_0, \dots, v_m \rangle$ is a simplex of K and $s' = \langle g_0 v_0, \dots, g_m v_m \rangle$, where $g_i \in G$, $i = 1, \dots, m$, also is a simplex of K then there exists $g \in G$ such that $g v_i = g_i v_i$, $i = 0, \dots, m$.

Condition 1 itself is a stronger condition than 1', but a simplicial G -complex satisfies conditions 1 and 2 if and only if it satisfies conditions 1' and 2 (see Lemma 1.1 and example (a) after Lemma 1.1 in [10]). Observe that in particular an equivariant simplicial complex K satisfies the following. For any simplex s of K and any $g \in G$ we have that if $g(s) = s$ then $g y = y$ for all $y \in s$. For any simplex s of K we have that if $x \in s$ and $g x \in s$ then $g x = x$, and moreover if $x \in \mathring{s}$ and $g x = x$ then $g y = y$ for every $y \in s$. If $s = \langle v_0, \dots, v_n \rangle$, where $G_{v_n} \subset \dots \subset G_{v_0}$, is an n -simplex of the equivariant simplicial complex K then every point y in the set $\langle v_0, \dots, v_i \rangle - \langle v_0, \dots, v_{i-1} \rangle$, $0 \leq i \leq n$, has isotropy subgroup equal to G_i . *Observe in particular that for the principal isotropy subgroup $H = G_{v_n}$ of s we have $H x = x$ for all $x \in s$.*

The orbit space K/G of an equivariant simplicial complex K has the structure of an ordinary simplicial complex such that the natural projection $p: K \rightarrow K/G$ is simplicial and p maps each simplex of K homeomorphically onto the corresponding image simplex of K/G .

If $\mathbf{R}^m(\rho)$ is an orthogonal representation space for G we denote

$$D^m(\rho) = \text{convex hull of } \{\pm ge_i \mid g \in G, i = 1, \dots, m\},$$

and $\partial D^m(\rho)$ denotes the boundary of $D^m(\rho)$. Here e_1, \dots, e_m denote the standard unit vectors in \mathbf{R}^m . (In [9 and 10] we have denoted $S^{m-1}(\rho) = \partial D^m(\rho)$, but in this paper we will reserve the notation $S^{m-1}(\rho)$ for the round G -sphere $S^{m-1}(\rho) = \{x \in \mathbf{R}^m(\rho) \mid \|x\| = 1\}$.) Both $\partial D^{m-1}(\rho)$ and $D^m(\rho)$ can be given the structure of rectilinear equivariant simplicial complexes in $\mathbf{R}^m(\rho)$ (see [9, p. 205]), and any two rectilinear equivariant simplicial complexes K_1 and K_2 in $\mathbf{R}^m(\rho)$ with $|K_1| = |K_2| = \partial D^m(\rho)$ (or $= D^m(\rho)$) have a common equivariant simplicial subdivision (see [9, pp. 202–203]).

An equivariant simplicial complex K is an m -dimensional *equivariant combinatorial manifold* (without boundary) if for every vertex v of K there exists an orthogonal representation $\rho_v: G_v \rightarrow O(m)$ and a p.l. G_v -homeomorphism

$$\alpha: \text{Lk}(v, K) \rightarrow \partial D^m(\rho_v).$$

Let S be a G -space. An *equivariant triangulation* of S consists of an equivariant simplicial complex K and a G -homeomorphism $t: K \rightarrow S$. Two equivariant triangulations $t_1: K_1 \rightarrow S$ and $t_2: K_2 \rightarrow S$ are called equivalent if $t_2^{-1}t_1: K_1 \rightarrow K_2$ is a p.l. G -homeomorphism. Given an equivariant triangulation t we have the class of all equivariant triangulations of S equivalent to t , called the preferred class of equivariant triangulations of S determined by t . Two equivariant triangulations t_1 and t_2 of S determine the same class of preferred triangulations of S if and only if t_1 and t_2 are equivalent. An *equivariant polyhedron* P is a G -space P together with a (non-empty!) preferred class of equivariant triangulations of P . An equivariant subpolyhedron Q of P is a G -subset of P such that for some preferred equivariant triangulation $h: K \rightarrow P$ there exists an equivariant subcomplex L of K such that $h(L) = Q$. In particular it follows that Q is a closed subset of P . An *m -dimensional equivariant p.l. manifold* M is an equivariant polyhedron such that in the class of preferred equivariant triangulations of M there exists an equivariant triangulation $t: K \rightarrow M$ with K an m -dimensional equivariant combinatorial manifold. In fact, in this case, for any preferred equivariant triangulation $t': L \rightarrow M$ we have that L is an m -dimensional equivariant combinatorial manifold.

Observe that if M is an equivariant p.l. manifold then M^H is a (locally flat) p.l. submanifold of M for every subgroup H of G . If M is an equivariant p.l. manifold we use M_H^α to denote a component of M_H and M_β^H to denote a component of M^H . Observe that if M_β^H is the component of M^H containing M_H^α then $\dim M_\beta^H = \dim M_H^\alpha$, since M is an equivariant p.l. manifold. If K is a G -subset of M we often denote $[K_H]^\alpha = K_H \cap M_H^\alpha$ and $[K_H]_\beta = K_H \cap M_\beta^H$.

We will often use the fact that equivariant p.l. manifolds, as well as smooth G -manifolds, have the property given by the following lemma.

LEMMA. Let M be an equivariant p.l. manifold (or a smooth G -manifold), and let H be a subgroup of G such that M^H is connected. Then there exists a subgroup \bar{H} of G , with $H \subset \bar{H}$, such that $M^{\bar{H}} = M^H$ and \bar{H} occurs in M . Moreover \bar{H} is uniquely determined, and if W is any nonempty open subset of M^H there exists $x \in W$ with $G_x = \bar{H}$.

PROOF. That smooth G -manifolds have this property is a well-known fact and the argument for the case of equivariant p.l. manifolds is the same, i.e., it runs as follows: Let \mathfrak{S} denote the family of all subgroups H' of G such that $H \subset H'$ and $\dim M^{H'} = \dim M^H = n$. Then \mathfrak{S} is nonempty and if $H' \in \mathfrak{S}$ we have $M^{H'} \subset M^H$. Let $(M^{H'})_0$ denote a component of $M^{H'}$ such that $\dim(M^{H'})_0 = n$. Since $(M^{H'})_0$ is a connected submanifold of M^H of the same dimension as M^H we have that $(M^{H'})_0$ is open in M^H . Since $(M^{H'})_0$ is closed in M' and M' is closed in M^H we have that $(M^{H'})_0$ is closed in M^H . Thus $(M^{H'})_0$ is both open and closed in M^H and hence $(M^{H'})_0 = M^H$, since M^H is connected. Because $M^{H'} \subset M^H$ it now follows that $M^{H'} = M^H$. We have shown that $H' \in \mathfrak{S}$ implies that $M^{H'} = M^H$. Therefore it follows that if $H', H'' \in \mathfrak{S}$ then also $\langle H', H'' \rangle \in \mathfrak{S}$, where $\langle H', H'' \rangle$ denotes the subgroup of G generated by H' and H'' . Hence there exists a largest subgroup \bar{H} in \mathfrak{S} .

We now claim that if W is any nonempty open subset of M^H there exists $x \in W$ such that $G_x = \bar{H}$. This is seen as follows. Let \tilde{H} be any subgroup such that $\bar{H} \subsetneq \tilde{H}$. By the maximality of \bar{H} we have $\tilde{H} \notin \mathfrak{S}$, and since $H \subset \tilde{H}$ we must then have $\dim M^{\tilde{H}} < n$. If $n = 0$ we have $M^H = \{a\}$, and then also $M^{\tilde{H}} = \{a\}$ and $W = \{a\}$. Since $M^{\tilde{H}} = \emptyset$, we have $G_a = \bar{H}$. Now assume $n > 0$. Give M an equivariant triangulation, which we again denote by M , such that some vertex v of $M^{\tilde{H}}$ lies in W . Let $\alpha: \text{St}(v, M) \rightarrow D^m(\rho_v)$ be a p.l. G_v -homeomorphism, where $\rho_v: G_v \rightarrow O(m)$. We have $\bar{H} \subset G_v$ and $\dim[\mathbf{R}^m(\rho_v)]^{\bar{H}} = n$. Since for any subgroup \tilde{H} such that $\bar{H} \subsetneq \tilde{H}$ we have $\dim M^{\tilde{H}} < n$ it follows that $[\mathbf{R}^m(\rho_v)]^{\tilde{H}}$ equals a finite union of linear subspaces of dimension $\leq n - 1$. Hence $W' \cap [\mathbf{R}^m(\rho_v)]_{\bar{H}} \neq \emptyset$, for any neighbourhood W' of the origin in $[\mathbf{R}^m(\rho_v)]^{\tilde{H}}$, and therefore $W \cap \text{St}(v, M)_{\bar{H}} \neq \emptyset$. Any $x \in W \cap \text{St}(v, M)_{\bar{H}}$ then satisfies $G_x = \bar{H}$ and $x \in W$.

To see that \bar{H} is uniquely determined let \bar{H} and $\bar{\bar{H}}$ be two subgroups of G such that $M^{\bar{H}} = M^H = M^{\bar{\bar{H}}}$, and both \bar{H} and $\bar{\bar{H}}$ occur in M . Let $x \in M$ be such that $G_x = \bar{H}$. Then $x \in M^{\bar{H}} \cap M^{\bar{\bar{H}}} = M^{\langle \bar{H}, \bar{\bar{H}} \rangle}$, where $\langle \bar{H}, \bar{\bar{H}} \rangle$ denotes the subgroup of G generated by \bar{H} and $\bar{\bar{H}}$. Thus $G_x \supset \langle \bar{H}, \bar{\bar{H}} \rangle \supset \bar{H}$, and hence $\langle \bar{H}, \bar{\bar{H}} \rangle = \bar{H}$. Likewise $\langle \bar{H}, \bar{\bar{H}} \rangle = \bar{\bar{H}}$, and hence $\bar{H} = \bar{\bar{H}}$. \square

Let us also here give a remark, that will be used in 5.1, on the notion of $\Sigma^{(>H,2)}$ being topologically standard in Σ^H . Recall from the introduction that $\Sigma^{(>H,2)}$ is topologically standard in Σ^H means that there exists a homeomorphism $\alpha: \Sigma^H \rightarrow S^n$ such that $\alpha(\Sigma^{(>H,2)})$ equals a union of standard $(n - 2)$ -dimensional subspheres of S^n . Let H_1, \dots, H_r be such that $\Sigma^{(>H,2)} = \bigcup_{i=1}^r \Sigma^{H_i}$. Now let $\bar{L}_i, 1 \leq i \leq r$, be $(n - 1)$ -dimensional linear subspaces of \mathbf{R}^{n+1} such that $\alpha(\Sigma^{H_i}) = S^n \cap \bar{L}_i$. By our assumption Σ^G is homeomorphic to a nonempty sphere and hence Σ^G contains at least two points. Let $a, b \in \Sigma^G$, where $a \neq b$. We have $\bar{b} = \alpha(b) \in \bigcap_{i=1}^r \alpha(\Sigma^{H_i})$,

and hence in particular $0 \neq \bar{b} \in \bar{L}_0 = \bigcap_{i=1}^r \bar{L}_i$. Thus \bar{L}_0 is a linear subspace of \mathbf{R}^{n+1} with $\dim \bar{L}_0 \geq 1$. Choose coordinates in \mathbf{R}^{n+1} such that $\bar{b} = (0, \dots, 0, 1)$ and write $\mathbf{R}^{n+1} = \mathbf{R}^n \oplus \mathbf{R}\bar{b}$. Since $\mathbf{R}\bar{b} \subset \bar{L}_0 \subset \bar{L}_i$, for $1 \leq i \leq r$, it follows that each \bar{L}_i is of the form $\bar{L}_i = L_i \oplus \mathbf{R}\bar{b}$, where L_i is an $(n - 2)$ -dimensional linear subspace of \mathbf{R}^n . Now define a homeomorphism $\gamma: S^n - \{\bar{b}\} \rightarrow \mathbf{R}^n$ by

$$\gamma(x, x_{n+1}) = \begin{cases} x \in \mathbf{R}^n, & \text{if } x_{n+1} \leq 0, \\ \frac{x}{\|x\|^2}, & \text{if } 0 \leq x_{n+1} < 1. \end{cases}$$

Then $((S^n - \{b'\}) \cap \bar{L}_i) = L_i$, for $1 \leq i \leq r$. Since $a' = \gamma\alpha(a) \in L_i$, for all i , it follows that the affine map $t: \mathbf{R}^n \rightarrow \mathbf{R}^n$, defined by $t(x) = x - a'$, satisfies $t(L_i) = L_i$, $1 \leq i \leq r$. Therefore

$$\alpha' = t \circ \gamma \circ \alpha: \Sigma^H - \{b\} \xrightarrow{\cong} \mathbf{R}^n$$

is a homeomorphism with $\alpha'(a) = 0$ and such that $\alpha'(\Sigma^{H_i} - \{b\}) = L_i$, an $(n - 2)$ -dimensional linear subspace of \mathbf{R}^n , for $1 \leq i \leq r$, and thus $\alpha'(\Sigma^{(>H,2)} - \{b\}) = \bigcup_{i=1}^r L_i$. It is in this form that the assumption that $\Sigma^{(>H,2)}$ is topologically standard in Σ^H is put to use in the proof of 5.1.

We will occasionally deal with nonconnected manifolds. All notions we use for nonconnected manifolds have their obvious interpretation, i.e., they hold for all components of the manifolds.

Let us point out one final notation, used in the proof of 2.4. Let s be a simplex with vertices v_0, \dots, v_q , where $q \geq 1$, and let $0 < \epsilon < 1/(q + 1)$. Then s_ϵ denotes the smaller simplex inside s given by

$$s_\epsilon = \left\{ x \in s \mid x = \sum_{i=0}^q t_i v_i, \epsilon \leq t_i < 1, \sum_{i=0}^q t_i = 1 \right\}.$$

2. The equivariant engulfing theorem. In this section we establish an equivariant analogue of Stallings' engulfing theorem [16]. First we consider equivariant collapsing and prove the easy but crucial Proposition 2.3 below.

Let L be an equivariant subcomplex of the equivariant simplicial complex K . We say that there is an *elementary G-simplicial collapse* from K to L if $K - L$ consists of two equivariant simplexes Gs and Gs_1 with s_1 a face of s and such that s_1 and s have the same principal isotropy subgroup and $s = vs_1$, where v is a vertex of s . We say that K collapses *G-simplicially* to K_0 if there is a sequence $K = K_r \supset K_{r-1} \supset \dots \supset K_0$ of equivariant subcomplexes of K such that there is an elementary *G-simplicial collapse* from K_i to K_{i-1} , for each $i = 1, \dots, r$.

Let Q be an equivariant subpolyhedron of the equivariant polyhedron P , and let $p: P \rightarrow P/G$ denote the projection onto the orbit space. We say that there is an *elementary equivariant collapse* (more precisely, an elementary equivariant m -dimensional collapse of type (H)) from P to Q if the following holds. There is a p.l. m -ball B^m in P such that $p|: B^m \rightarrow p(B^m)$ is a homeomorphism and a p.l. $(m - 1)$ -ball $B^{m-1} \subset \partial B^m$ such that $Q \cap B^m = B^{m-1}$ and $P = Q \cup GB^m$ and all points in

$B^m - B^{m-1}$ have the same isotropy subgroup H . We say that P collapses equivariantly to P_0 if there exists a sequence $P = P_t \supset P_{t-1} \supset \dots \supset P_0$ of equivariant subpolyhedra of P such that there is an elementary equivariant collapse from P_j to P_{j-1} , for $j = 1, \dots, t$.

Let s be an ordinary simplex of the equivariant simplicial complex K and let H be the principal isotropy subgroup of s . Then $\text{st}(s; K)$ and $\text{lk}(s; K)$ are H -equivariant simplicial complexes and if $g \notin H$ we have $\text{st}(s; K) \cap \text{st}(gs; K) = \emptyset$. Thus, if $b \in \mathring{s}$ and we star s at b and all the other simplexes $g_1s, \dots, g_k s$ at $g_1b, \dots, g_k b$, respectively, we get a well-defined G -equivariant subdivision K' of K . In this case we say that K' is obtained from K by an elementary G -starring at $Gb \in G\mathring{s}$. We say that K^* is an equivariant stellar subdivision of K if there exists a sequence $K, K', K'', \dots, K^{(r)} = K^*$, where each $K^{(i)}$ is obtained from $K^{(i-1)}$ by an elementary G -starring, $i = 1, \dots, r$. If K^* is an equivariant stellar subdivision of K the induced subdivision K^*/G of K/G is stellar, and in fact conversely if $(K/G)^*$ is a stellar subdivision of K/G then the induced equivariant subdivision K^* of K is an equivariant stellar subdivision of K .

LEMMA 2.1. Assume that K collapses G -simplicially to K_0 and let K^* be an equivariant stellar subdivision of K . Then K^* collapses G -simplicially to K_0^* .

PROOF. It is enough to consider the case when there is an elementary G -simplicial collapse from K to K_0 and where the equivariant stellar subdivision is an elementary G -starring. The proof is then completely analogous to the standard proof in the ordinary case, see for example Hudson [8, Lemma 2.2]. \square

LEMMA 2.2. Let K_0 be an equivariant subcomplex of the equivariant simplicial complex K , such that the equivariant polyhedron $|K|$ collapses equivariantly to $|K_0|$. Then there exists an equivariant subdivision K^* of K such that K^* collapses G -simplicially to K_0^* .

PROOF. Let us first show that if L_0 is an equivariant subcomplex of the equivariant simplicial complex L such that there is an elementary equivariant collapse from $|L|$ to $|L_0|$, then there exists an equivariant subdivision L^* of L such that L_0^* is an equivariant stellar subdivision of L_0 and such that L^* collapses G -simplicially to L_0^* . By the assumption there is a p.l.-ball B^m in $|L|$, such that B^m injects into the orbit space and $B^m \cap |L_0| = B^{m-1}$ is an $(m - 1)$ -ball in ∂B^m and $|L| = |L_0| \cup GB$, and moreover all points in $B^m - B^{m-1}$ have the same isotropy subgroup. Let $p: |L| \rightarrow |L|/G$ denote the projection onto the orbit space. Then $p(B^m)$ is a p.l. m -ball in $|L|/G$ and $|L_0|/G \cap p(B^m) = p(|L_0| \cap B^m) = p(B^{m-1})$ is a p.l. $(m - 1)$ -ball in $\partial(p(B^m))$ and $|L|/G = |L_0|/G \cup p(B^m)$, i.e., there is an elementary collapse from $|L|/G = |L/G|$ to $|L_0|/G = |L_0/G|$. Hence (see e.g. Hudson [8, Lemma 2.3]) there is a subdivision $(L/G)^*$ of L/G such that $(L_0/G)^*$ is a stellar subdivision of L_0/G and such that $(L/G)^*$ collapses simplicially to $(L_0/G)^*$. Let L^* denote the induced equivariant subdivision of L . Then L_0^* is an equivariant stellar subdivision of L_0 . Let $(L/G)^* = R_q \searrow R_{q-1} \searrow \dots \searrow R_0 = (L_0/G)^*$ be a sequence of elementary simplicial collapses. Since all points in $|L/G| - |L_0/G|$ have the same orbit

type it follows immediately that $L^* = L_q^* \searrow L_{q-1}^* \searrow \cdots \searrow L_0^*$, where $L_i^* = p^{-1}(R_i)$, $i = 0, \dots, q$, is a sequence of elementary G -simplicial collapses.

The lemma now follows, by induction, from the above and Lemma 2.1. \square

PROPOSITION 2.3. *Let M be an equivariant p.l. manifold without boundary and let $Q \subset P$ be equivariant subpolyhedra of M , such that P collapses equivariantly to Q . Let U be an open G -subset of M such that $Q \subset U$. Then there exist a compact G -subset E and an ambient G -isotopy $h, \text{rel}((M - E) \cup Q)$ such that $P \subset h_1(U)$.*

PROOF. Let $L \subset K \subset J$ be an equivariant triangulation of $Q \subset P \subset M$, such that K collapses G -simplicially to L . Then J is an equivariant combinatorial manifold. Next observe that for any r -simplex σ of J , such that $\sigma \subset J^H$, we have that $\text{lk}(\sigma; J)$ is p.l. H -homeomorphic to a p.l. H -sphere $S^{m-r-1}(\rho)$, where $\mathbf{R}^{m-r}(\rho)$ is an orthogonal representation space for H and $m = \dim M = \dim J$. This follows, by induction, from the fact that $\text{lk}(\sigma; J) = \text{lk}(w; \text{lk}(\sigma_1; J))$, where w is a vertex of σ and σ_1 is the face of σ opposite w .

It is enough to consider the case where there is an elementary G -simplicial collapse from K to L . Let G_{s_1} and G_s be the equivariant simplexes of $K - L$, where s_1 is a face of s , and s_1 and s have the same principal isotropy subgroup, say H , and $s = s_1 v$, where v is a vertex of s . Thus v is fixed under H . We have $\dot{s}_1 v \subset L \subset U$. Let $b \in \dot{s}_1$ be the barycenter of s_1 , and choose $c \in bv$, where $c \neq v$, so close to v that $\dot{s}_1 v c \subset U$.

Since $q = \dim s_1 \leq m - 1$, we have $\text{lk}(s_1; J) \neq \emptyset$. Let $\alpha: \text{lk}(s_1; J) \rightarrow S^{m-q-1}(\rho)$ be a p.l. H -homeomorphism where $S^{m-q-1}(\rho)$ is the p.l. H -sphere in the orthogonal representation space $\mathbf{R}^{m-q}(\rho)$. Since $\alpha(v)$ is fixed under H we have $\dim(\mathbf{R}^{m-q}(\rho))^H \geq 1$. Let V be the 1-dimensional linear subspace of $\mathbf{R}^{m-q}(\rho)$ generated by $\alpha(v)$. Now let $\mathbf{R}^{m-q}(\rho) = V \oplus W$, where W is an H -invariant $(m - q - 1)$ -dimensional linear subspace of $\mathbf{R}^{m-q}(\rho)$. Denote $\bar{v} = \alpha(v) / \|\alpha(v)\| \in V$. There exists a p.l. H -homeomorphism $\gamma: S^{m-q-1}(\rho) \rightarrow S(V \oplus W) = \{\bar{v}, -\bar{v}\}S(W)$, such that $\gamma(\alpha(v)) = \bar{v}$. We denote $\beta = \gamma\alpha: \text{lk}(s_1; J) \rightarrow S(V \oplus W)$. Let $\bar{s}_1 \subset \mathbf{R}^q$ denote a copy of s_1 linearly imbedded in \mathbf{R}^q (with trivial H -action). Define

$$\varphi = \text{id} * \beta: \text{st}(s_1; J) = s_1 \text{lk}(s_1; J) \rightarrow \bar{s}_1 S(V \oplus W) \quad (\subset \mathbf{R}^q \oplus V \oplus W).$$

Then φ is a p.l. H -homeomorphism and $\varphi(s) = \varphi(s_1 v) = \bar{s}_1 \bar{v}$. Let us further denote $U' = U \cap \text{st}(s_1; J)$ and $\varphi(c) = \bar{c} \in \mathbf{R}^q \oplus V$ and $\varphi(b) = \bar{b} \in \mathbf{R}^q$. Then $\dot{s}_1 \bar{v} \bar{c} \subset \varphi(U')$. Now let

$$\eta: \bar{s}_1 \{\bar{v}, -\bar{v}\} \rightarrow \bar{s}_1 \{\bar{v}, -\bar{v}\}$$

be a p.l. homeomorphism, keeping the boundary fixed, such that $\bar{s}_1 \bar{v} \subset \eta(\dot{s}_1 \bar{v} \bar{c})$. [For example by defining $\eta(\bar{v}) = \bar{v}$, $\eta(\bar{c}) = \bar{b}$, $\eta(\bar{b}) = -\bar{c}$, $\eta(-\bar{v}) = -\bar{v}$ and $\eta|_{\dot{s}_1} = \text{id}$ and extending linearly to the corresponding simplexes.] We define

$$\omega = \eta * \text{id}: \bar{s}_1 \{\bar{v}, -\bar{v}\} S(W) \rightarrow \bar{s}_1 \{\bar{v}, -v\} S(W).$$

Then $\xi = \varphi^{-1} \omega \varphi: \text{st}(s_1; J) \rightarrow \text{st}(s_1; J)$ is a p.l. H -homeomorphism keeping the boundary fixed, such that $\xi(U') \supset \varphi^{-1} \omega(\dot{s}_1 \bar{v} \bar{c}) \supset \varphi^{-1}(\bar{s}_1 \bar{v}) = s$. By Lemma 2.10 in [10] there is an ambient p.l. H -isotopy h'_i of $\text{st}(s_1; J) \text{ rel } \partial(\text{st}(s_1; J)) = \dot{s}_1 \text{lk}(s_1; J)$

from id to ξ . Extending h_t equivariantly we get an ambient p.l. G -isotopy of $G \text{st}(s_1; J) \text{ rel } G(\partial \text{st}(s_1; J))$ and by extending this by the identity outside $G \text{st}(s_1; J)$ we get a well-defined ambient p.l. G -isotopy h_t of $M = |J| \text{ rel } ((M - E) \cup L)$, where $E = G \text{st}(s_1; J)$, such that $h_1(U) \supset |K| = P$. \square

We are now ready to prove the equivariant engulfing theorem.

THEOREM 2.4. *Let M be an equivariant p.l. manifold, without boundary. Let U be an open G -subset of M and let $L \subset K$ be equivariant subpolyhedra of M such that $L \subset U$ and $\overline{K - L}$ is compact, and $\dim [K_H]^\alpha \leq \dim M_H^\alpha - 3$, for every subgroup H and component M_H^α of M_H for which $[(K - L)_H]^\alpha \neq \emptyset$. Assume that $(M_H, [U_H]^\alpha)$ is $r(H, \alpha)$ -connected and $\dim [(K - L)_H]^\alpha \leq r(H, \alpha)$ for every subgroup H occurring in M and any component M_H^α of M_H . Then there exist a compact G -subset $E \subset M$ and an ambient p.l. G -isotopy h_t of $M \text{ rel } ((M - E) \cup L)$ such that $K \subset h_1(U)$.*

PROOF. Let, for every subgroup H of G occurring in M and any component M_H^α of M_H , $n(H, \alpha)$ be an integer such that $-3 \leq n(H, \alpha) \leq \dim M_H^\alpha - 3$ and $n(gHg^{-1}, g\alpha) = n(H, \alpha)$ for every $g \in G$, and let $n_0(\ , \)$ be another similar integer-valued function, satisfying the same properties as $n(\ , \)$, and such that $n_0(H, \alpha) \leq n(H, \alpha) \leq n_0(H, \alpha) + 1$ for all H and α . We assume inductively that the theorem holds if $r(H, \alpha) \leq n_0(H, \alpha)$ for all H and α , and we shall prove that it then holds if $r(H, \alpha) \leq n(H, \alpha)$ for all H and α . The theorem holds if $n_0(H, \alpha) \leq -1$ for all H and α .

If $h: M \rightarrow M$ is a G -homeomorphism which is G -isotopic to the identity we have $h(M_H^\alpha) = M_H^\alpha$ and $h([U_H]^\alpha) = [h(U)_H]^\alpha$ for all H and α . Hence it follows, by induction, that it is enough to consider the case when

$$K - L = \mathring{A} = G\mathring{s},$$

where A is an equivariant simplex, in some equivariant triangulation of $L \subset K$. Then $G\mathring{s} = \mathring{A} \subset L \subset U$. Let H be the principal isotropy subgroup of s and let M_H^α denote the component of M_H containing \mathring{s} , and let M_B^β be the component of M^H containing M_H^α . Then $\mathring{s} \subset [(K - L)_H]^\alpha$ and we are assuming that

$$\dim s = \dim [(K - L)_H]^\alpha \leq n(H, \alpha) \leq \dim M_H^\alpha - 3.$$

If $\dim [(K - L)_H]^\alpha \leq n_0(H, \alpha)$, the inductive assumption applies and there is nothing to prove. Hence we can assume that $n(H, \alpha) = n_0(H, \alpha) + 1$, and that

$$\dim s = n(H, \alpha) = p.$$

Then the pair $(M_H^\alpha, [U_H]^\alpha)$ is p -connected. We have $\mathring{s} \subset U$, and since U is open there exists $\epsilon > 0$ such that

$$s - \mathring{s}_\epsilon \subset U.$$

Let us denote $\sigma = s_\epsilon$ and $B = G\sigma$. We now define

$$L_1 = L \cup (A - \mathring{B}).$$

Then L_1 is a subpolyhedron of K and $K - L_1 = \mathring{B}$. We have $\sigma \subset \mathring{s} \subset M_H^\alpha$ and since $\mathring{\sigma} \subset U$ it follows that $\mathring{\sigma} \subset [U_H]^\alpha$. Observe that the only isotropy subgroup in σ is H .

Also $B \subset \dot{A}$ and $\dot{B} \subset U$, and the only orbit type occurring in B is (H) . Now consider the inclusion

$$i: (\sigma, \dot{\sigma}) \rightarrow (M_H^\alpha, [U_H]^\alpha).$$

Since $\dim \sigma = p$ and $(M_H^\alpha, [U_H]^\alpha)$ is p -connected there exists a homotopy $\xi: \sigma \times I \rightarrow M_H$ rel $\dot{\sigma}$ from the inclusion i to a map $\xi_1: \sigma \rightarrow [U_H]^\alpha \subset M_H^\alpha$. By p.l. approximation there exists a p.l. map

$$\eta: \sigma \times I \rightarrow M_\beta^H$$

such that

1. $\eta(\sigma \times I) \subset M_H^\alpha$,
2. $\eta|_{\sigma \times \{0\}} = \xi|_{\sigma \times \{0\}} = i$,
3. $\eta(\dot{\sigma} \times I \cup \sigma \times \{1\}) \subset [U_H]^\alpha$.

Conditions 1 and 3 are obtained by letting η approximate ξ well enough. Since $\dim \sigma = \dim s \leq \dim M_H^\alpha - 3$ we have in particular $\dim(\sigma \times I) \leq \dim M_H^\alpha = \dim M_\beta^H$, and hence we can approximate η arbitrarily well by a nondegenerate p.l. map satisfying conditions 1–3 above. Thus we may as well henceforth assume that η moreover is nondegenerate.

A triangulation $(\sigma \times I)^*$ of $\sigma \times I$ induces in an obvious way an equivariant triangulation $(B \times I)^*$ of $B \times I$. Now let $(\sigma \times I)^*$ be a triangulation of $\sigma \times I$, which induces a triangulation on $\sigma \times \{0\}$, and M^* an equivariant triangulation of M such that η embeds the simplexes of $(\sigma \times I)^*$ and the G -map $\bar{\eta}: (B \times I)^* \rightarrow M^*$, defined by $\bar{\eta}(gx, t) = g\eta(x, t)$ for all $g \in G$ and $(x, t) \in \sigma \times I$, is simplicial. Since $\bar{\eta}$ is isovariant and nondegenerate in the ordinary sense it now follows by Lemma 1.7 in [10] that $\bar{\eta}$ is equivariantly nondegenerate, i.e., $\bar{\eta}$ embeds every equivariant simplex of $(B \times I)^*$. By Theorem 6 in Whitehead [19] there exists a subdivision $(\sigma \times I)^{**}$ of $(\sigma \times I)^*$ such that $(\sigma \times I)^{**}$ collapses simplicially to $(\dot{\sigma} \times I \cup \sigma \times \{1\})^{**}$. Clearly $\bar{\eta}$ also embeds every equivariant simplex of $(B \times I)^{**}$.

Let s_1, \dots, s_k be closed simplexes of $(\sigma \times I)^{**} - (\dot{\sigma} \times I \cup \sigma \times \{1\})^{**}$ such that, denoting

$$t_j = (\dot{\sigma} \times I \cup \sigma \times \{1\})^{**} \cup s_1 \cup \dots \cup s_j, \quad 1 \leq j \leq k, \quad \text{and}$$

$$t_0 = (\dot{\sigma} \times I \cup \sigma \times \{1\})^{**},$$

we have $t_k = (\sigma \times I)^{**}$ and $t_j \searrow t_{j-1}$ by one elementary simplicial collapse, $j = 1, \dots, k$. Thus we have

$$t_{j-1} \cap s_j = v_j \dot{\tau}_j, \quad \text{where } s_j = v_j \tau_j, \quad 1 \leq j \leq k.$$

Let t_j^p denote the p -skeleton of t_j . Observe that

$$t_0 = t_0^p, \quad \text{and} \quad (\sigma \times \{0\})^{**} \subset t_k^p.$$

Moreover we also have

$$t_{j-1}^p \cap s_j = v_j \dot{\tau}_j$$

i.e.

$$t_{j-1}^p \cup s_j \searrow t_{j-1}^p$$

by one elementary simplicial collapse, $j = 1, \dots, k$. [We have $t_{j-1}^p \cup s_j \neq t_j^p$ in the cases when $\dim s_j = p + 1$.]

Let us define

$$\begin{aligned} S_i &= Gs_i, & i &= 1, \dots, k, \\ T_j &= Gt_j, & j &= 0, \dots, k, \end{aligned}$$

and let $T_j^p = Gt_j^p$ denote the p -skeleton of T_j . Then S_1, \dots, S_k are closed equivariant simplexes of $(B \times I)^{**} - (\dot{B} \times I \cup B \times \{1\})^{**}$. We have

$$T_0 = T_0^p = (\dot{B} \times I \cup B \times \{1\})^{**}, \quad \text{and} \quad (B \times \{0\})^{**} \subset T_k^p$$

and

$$T_{j-1}^p \cup S_j \searrow T_{j-1}^p, \quad j = 1, \dots, k,$$

by one elementary G -simplicial collapse.

Let us consider the equivariant polyhedron

$$P = L_1 \cup_{\dot{B} \times \{0\}} (B \times I) = K \cup_{B \times \{0\}} (B \times I).$$

There is an equivariant triangulation P^{**} of P which is an extension of $(B \times I)^{**}$ and which induces K^{**} , L_1^{**} and L^{**} (in fact $L^{**} = L$). We have the elementary G -simplicial collapses

$$L_1^{**} \cup T_{j-1}^p \cup S_j \searrow L_1^{**} \cup T_{j-1}^p, \quad j = 1, \dots, k.$$

Now consider the G -map

$$\hat{\eta}: P^{**} \rightarrow M$$

where $\hat{\eta}|K^{**}$ equals the inclusion of K^{**} into M and $\hat{\eta}|(B \times I)^{**} = \bar{\eta}$. Then $\hat{\eta}$ is an equivariantly nondegenerate proper p.l. G -map. By Proposition 3.9 in [10] there is an isovariant proper p.l. ε -homotopy rel K from $\hat{\eta}$ to an equivariantly nondegenerate proper p.l. G -map $\bar{\varphi}: P^{**} \rightarrow M$ such that $\bar{\varphi}|P^{**} - K^{**} \rightarrow M$ is in equivariant strong general position with respect to M and $K_0 \subset M$, where $K_0 = G[K_H]^\alpha$. Here ε denotes an arbitrary given G -invariant function $\varepsilon: P \rightarrow \mathbf{R}_+$. We have $\bar{\varphi}(\sigma \times I) \subset M_H^\alpha$, and by choosing $\bar{\varphi}$ to approximate $\hat{\eta}$ well enough we get $\bar{\varphi}(\dot{\sigma} \times I \cup \sigma \times \{1\}) \subset [U_H]^\alpha$. Moreover we have that if τ_1 and τ_2 are any two distinct simplexes of $P^{**} - K^{**}$ such that $\bar{\varphi}(\tau_i) \subset M_\beta^H, i = 1, 2$, then

- (i) $\dim(\bar{\varphi}(\dot{\tau}_1) \cap \bar{\varphi}(\dot{\tau}_2)) \leq \dim \tau_1 + \dim \tau_2 - \dim M_\beta^H,$
- (ii) $\dim(\bar{\varphi}(\dot{\tau}_1) \cap K_0) \leq \dim \tau_1 + \dim[(K_0)_H]_\beta - \dim M_\beta^H.$

Here $[(K_0)_H]_\beta = (K_0)_H \cap M_\beta^H$.

Claim. We now claim that for each $j = 0, \dots, k$ there exist a compact G -subset E_j and an ambient p.l. G -isotopy $h_1^{(j)}$ of M rel $((M - E_j) \cup L_1)$ such that

$$\bar{\varphi}(T_j^p) \subset h_1^{(j)}(U).$$

We prove this claim by induction in j . For $j = 0$ we have $\bar{\varphi}(T_0^p) = \bar{\varphi}(T_0) \subset U$, so we can take $h_1^{(0)} = \text{id}$. Assume the claim holds for the value $i - 1$, so that $\bar{\varphi}(T_{i-1}^p) \subset h_1^{(i-1)}(U)$. We shall show that it holds for the value i . Consider a set of the form

$$S(\bar{\varphi}|s_i \cup \tau),$$

where τ is any ordinary simplex of $L_1^{**} \cup T_{i-1}^p$. (Here $S(\varphi|_{s_i} \cup \tau)$ denotes the singular set of the map $\varphi|_{s_i} \cup \tau$, see e.g. Rushing [13, p. 28].) Since $\bar{\varphi}$ is nondegenerate it is dimension preserving. Hence it follows from (i) that if τ is a simplex of $T_{i-1}^p - L_1^{**}$ we have

$$\dim S(\bar{\varphi}|_{s_i} \cup \tau) \leq \dim s_i + \dim \tau - \dim M_\beta^H \leq p + 1 + p - \dim M_\beta^H \leq p - 2,$$

since $p \leq \dim M_H^\alpha - 3 = \dim M_\beta^H - 3$. If τ is a simplex of L_1^{**} we have $S(\bar{\varphi}|_{s_i} \cup \tau) = \emptyset$ unless $\tau = \bar{\varphi}(\tau) \subset M_H^\alpha$, i.e., $\tau \subset [K_H]^\alpha \subset K_0$. In this case we have by (ii)

$$\begin{aligned} \dim S(\bar{\varphi}|_{s_i} \cup \tau) &\leq \dim(\bar{\varphi}(s_i) \cap K_0) \\ &\leq \dim s_i + \dim[(K_0)_H]_\beta - \dim M_\beta^H \leq p - 2, \end{aligned}$$

since $\dim K_0 = \dim[K_H]^\alpha \leq \dim M_H^\alpha - 3 = \dim M_\beta^H - 3$.

Let us denote

$$\Sigma'_i = \left[\bigcup_i S(\bar{\varphi}|_{s_i} \cup \tau) \right] \cap s_i$$

where the union is over all ordinary simplexes τ of $L_1^{**} \cup T_{i-1}^p$. Then Σ'_i is a subpolyhedron of s_i and $\dim \Sigma'_i \leq p - 2$. Since $t_{i-1}^p \cup s_i$ collapses (by an elementary simplicial collapse) to t_{i-1}^p there exists (see Lemma 1.6.3 in Rushing [13]) a subpolyhedron $\Lambda'_i \subset s_i$ with $\Sigma'_i \subset \Lambda'_i$ such that $t_{i-1}^p \cup s_i$ collapses to $t_{i-1}^p \cup \Lambda'_i$ and $\dim \Lambda'_i \leq p - 1$. Hence in some subdivision $(t_{i-1}^p \cup s_i)^\oplus$ of $t_{i-1}^p \cup s_i$, such that $(t_{i-1}^p \cup \Lambda'_i)^\oplus$ is a subcomplex, we have that $(t_{i-1}^p \cup s_i)^\oplus$ collapses simplicially to $(t_{i-1}^p \cup \Lambda'_i)^\oplus$. Let $\Lambda_i = G\Lambda'_i \subset S_i$ and $\Sigma_i = G\Sigma'_i \subset \Lambda_i$. Then $(T_{i-1}^p \cup S_i)^\oplus$ collapses G -simplicially to $(T_{i-1}^p \cup \Lambda_i)^\oplus$. The equivariant subdivision $(T_{i-1}^p \cup S_i)^\oplus$ can be extended to an equivariant subdivision $(L_1 \cup T_{i-1}^p \cup S_i)^\oplus$ of $L_1^{**} \cup T_{i-1}^p \cup S_i$. Then $(L_1 \cup T_{i-1}^p \cup S_i)^\oplus$ collapses G -simplicially to $(L_1 \cup T_{i-1}^p \cup \Lambda_i)^\oplus$. Let

$$(L_1 \cup T_{i-1}^p \cup S_i)^\oplus = R_d \searrow R_{d-1} \searrow \cdots \searrow R_0 = (L_1 \cup T_{i-1}^p \cup \Lambda_i)^\oplus$$

be a sequence of elementary G -simplicial collapses. Let us now show that there is an elementary equivariant collapse from $\bar{\varphi}(R_e)$ to $\bar{\varphi}(R_{e-1})$ for $e = 1, \dots, d$. Let $R_e - R_{e-1}$ consist of the two equivariant simplexes Gw and Gw_1 with w_1 a face of w and $w = aw_1$, where a is a vertex of w , and where we have chosen the ordinary simplex w such that $w \subset s_i$. (Thus both w_1 and w have principal isotropy subgroup equal to H .) We have $w \cap R_{e-1} = aw_1$, and $\bar{\varphi}| : Gw \rightarrow M$ is a p.l. G -embedding. We claim that $p| : \bar{\varphi}(w) \rightarrow M/G$ is injective. For if $x, y \in w$ and $p(\bar{\varphi}(x)) = p(\bar{\varphi}(y))$ we have $\bar{\varphi}(x) = g\bar{\varphi}(y)$ for some $g \in G$ and hence $\bar{\varphi}(x) = \bar{\varphi}(gy)$ implies $x = gy$. Since $y \in w$ and $gy = x \in w$ it follows that $g \in H$ and $x = y$. Next we claim that $x \in w - R_{e-1}$ implies that $\bar{\varphi}(x) \notin \bar{\varphi}(R_{e-1})$. For if $z \in R_{e-1}$ and $\bar{\varphi}(x) = \bar{\varphi}(z)$ we cannot have $z \in L_1 \cup T_{i-1}^p$ since $x \in s_i - \Lambda'_i \subset s_i - \Sigma'_i$, and hence $z \in S_i$ which since $\bar{\varphi}|_{S_i}$ is a G -embedding implies that $x = z$, a contradiction. It now follows that $\bar{\varphi}(w) \cap \bar{\varphi}(R_{e-1}) = \bar{\varphi}(w \cap R_{e-1}) = \bar{\varphi}(aw_1)$. Thus $\bar{\varphi}(w)$ is a q -ball ($q = \dim w$) in $\bar{\varphi}(R_e)$ which injects into the orbit space such that $\bar{\varphi}(w) \cap \bar{\varphi}(R_{e-1}) = \bar{\varphi}(aw_1)$ is a $(q - 1)$ -ball in $\partial(\bar{\varphi}(w))$ and $\bar{\varphi}(R_e) = \bar{\varphi}(R_{e-1}) \cup G\bar{\varphi}(w)$. This shows that there is an elementary equivariant collapse from $\bar{\varphi}(R_e)$ to $\bar{\varphi}(R_{e-1})$.

Thus $\bar{\varphi}(L_1 \cup T_{i-1}^p \cup S_i)$ collapses equivariantly to $\bar{\varphi}(L_1 \cup T_{i-1}^p \cup \Lambda_i)$. Since $\bar{\varphi}(\sigma \times I) \subset M_H$ it follows that $[(\bar{\varphi}(L_1 \cup T_{i-1}^p \cup \Lambda_i) - \bar{\varphi}(L_1 \cup T_{i-1}^p))]_H \neq \emptyset$ only

if $H' = gHg^{-1}$ and $\gamma = g\alpha$, for some $g \in G$. Since also $\dim(\bar{\varphi}(L_1 \cup T_{i-1}^p \cup \Lambda_i) - \bar{\varphi}(L_1 \cup T_{i-1}^p)) \leq p - 1$, we have by our main inductive assumption that the theorem holds in this case, i.e., there exist a compact G -subset \hat{E}_i and an ambient p.l. G -isotopy $\hat{h}^{(i)} \text{ rel}((M - \hat{E}_i) \cup \bar{\varphi}(L_1 \cup T_{i-1}^p))$ such that

$$\hat{h}_1^{(i)}(h_1^{(i-1)}(U)) \supset \bar{\varphi}(L_1 \cup T_{i-1}^p \cup \Lambda_i).$$

Since $\bar{\varphi}(L_1 \cup T_{i-1}^p \cup S_i)$ collapses equivariantly to $\bar{\varphi}(L_1 \cup T_{i-1}^p \cup \Lambda_i)$ it now follows by Proposition 2.3 that there exist a compact G -subset \hat{E}_i and an ambient p.l. G -isotopy $\hat{h}^{(i)} \text{ rel}((M - \hat{E}_i) \cup \bar{\varphi}(L_1 \cup T_{i-1}^p \cup \Lambda_i))$ such that

$$\hat{h}_1^{(i)}(\hat{h}_1^{(i)} h_1^{(i-1)}(U)) \supset \bar{\varphi}(L_1 \cup T_{i-1}^p \cup S_i).$$

Now let $E_i = \hat{E}_i \cup \tilde{E}_i \cup E_{i-1}$ and $h^{(i)} = \hat{h}^{(i)} \circ \tilde{h}^{(i)} \circ h^{(i-1)}$, then $h^{(i)}$ is an ambient p.l. G -isotopy such that $h_1^{(i)}(U) \supset \bar{\varphi}(L_1 \cup T_{i-1}^p \cup S_i) \supset \bar{\varphi}(L_1 \cup T_i^p)$. This completes the inductive proof of our "claim". Thus $h = h^{(k)}$ is an ambient p.l. G -isotopy $\text{rel}((M - E_k) \cup L)$ such that

$$h_1(U) \supset \bar{\varphi}(L_1 \cup T_k^p) \supset \bar{\varphi}(L_1 \cup (B \times \{0\})) = \bar{\varphi}(K) = K. \quad \square$$

We will also use the following version of the equivariant engulfing theorem. (This version of the equivariant engulfing theorem corresponds to the version of the ordinary engulfing theorem given in Theorem 6.1 in Stallings [17].)

THEOREM 2.5. *Let M be an equivariant p.l. manifold, without boundary. Let U be an open G -subset of M and C a closed G -subset of M such that $C \subset U$. Let K be an equivariant subpolyhedron of M such that $K - U$ is compact. Assume that*

$$([(M - C)_H]^\alpha, [(U - C)_H]^\alpha)$$

is $r(H, \alpha)$ -connected and $\dim[(K - C)_H]^\alpha \leq r(H, \alpha) \leq \dim M_H^\alpha - 3$ for every subgroup H and component M_H^α of M_H for which $[(K - C)_H]^\alpha \neq \emptyset$. Then there exist a compact G -subset $E \subset M - C$ and an ambient p.l. G -isotopy h_1 of $M \text{ rel}(M - E)$ such that $K \subset h_1(U)$.

PROOF. Denote $M_1 = M - C$, $U_1 = U - C$ and $K_1 = K - C$. Then K_1 is an equivariant subpolyhedron of the equivariant p.l. manifold M_1 . Let K_1 be given some equivariant triangulation and let L_1 denote the equivariant subcomplex of K_1 consisting of all closed simplexes of K_1 that are contained in U_1 . Then $L_1 \subset U_1$ and $\overline{K_1 - L_1} = N(K_1 - U_1; K_1)$ is compact since $K_1 - U_1 = K - U$ is compact. The proof is now completed by applying Theorem 2.4 to the situation (M_1, U_1, L_1, K_1) . (Observe the following: Let H be a subgroup of G and $(M_1)_H^\gamma$ a component of $(M_1)_H$ such that $[(K_1 - L_1)_H]^\gamma \neq \emptyset$. Let M_H^α be the component of M_H containing $(M_1)_H^\gamma$. Then $\emptyset \neq [(K_1)_H]^\gamma \subset [(K_1)_H]^\alpha$ and hence, by the assumption in 2.5 and the fact that $\dim M_H^\alpha = \dim (M_1)_H^\gamma$, we have $\dim[(K_1)_H]^\gamma \leq \dim[(K_1)_H]^\alpha \leq \dim M_H^\alpha - 3 = \dim (M_1)_H^\gamma - 3$.) \square

3. Recognition of linear actions on euclidean space. In this section we prove Proposition 3.2 which gives necessary and sufficient conditions for an equivariant p.l. manifold M to be G -homeomorphic to some euclidean representation space. Proposition 3.2 below is in fact given in a more general form than we would need. In

the situation that we use Proposition 3.2 we are already given that the underlying manifold is euclidean space and that all fixed-point sets are homeomorphic to euclidean space. There is also a version of the result in Proposition 3.2 that gives necessary and sufficient conditions for M to be p.l. G -homeomorphic to some euclidean representation space, but we have no use for this p.l. version in this paper, and hence we omit it.

DEFINITION 3.1. Let M be an equivariant p.l. manifold and \mathfrak{F} a family of subgroups of G , occurring in M , which is closed under conjugation. We say that M is \mathfrak{F} -equivariantly unraveled at infinity if given any compact subset A of M there exists a compact G -subset B of M , with $A \subset B$, such that $(M_H, (M - B)_H)$ is 2-connected for every $H \in \mathfrak{F}$ for which M^H is noncompact. In case \mathfrak{F} is the family of all subgroups occurring in M we simply use the expression “equivariantly unraveled at infinity” instead of “ \mathfrak{F} -equivariantly unraveled at infinity”.

PROPOSITION 3.2. Let M be an equivariant p.l. manifold, without boundary, such that $\dim M^H \geq 5$ for every subgroup $H \neq G$ occurring in M . Then M is G -homeomorphic to some orthogonal representation space $\mathbf{R}^m(\rho)$ if and only if the following three conditions are satisfied.

- I. The case $0 \leq \dim M^G \leq 4$.
 - 1. M^G is homeomorphic to \mathbf{R}^k , where $k = \dim M^G$.
 - 2. M is \mathfrak{F}' -equivariantly unraveled at infinity, where $\mathfrak{F}' = \mathfrak{F} - \{G\}$ and \mathfrak{F} denotes the family of all subgroups occurring in M .
 - 3. There is a closed G -invariant neighbourhood D of M^G in M , with D G -homeomorphic to some $\mathbf{R}^k \times B^{m-k}(\rho')$, such that (M_H, D_H) is q -connected for all q and every $H \in \mathfrak{F}'$.
- II. The case $\dim M^G \geq 5$.
 - 1. M^G is contractible.
 - 2. M is equivariantly unraveled at infinity.
 - 3. There exists an open G -subset U of M G -homeomorphic to some $\mathbf{R}^m(\rho)$ such that (M_H, U_H) is q -connected for all q and every $H \in \mathfrak{F}'$. (Observe that in this case $(M_G, U_G) = (M^G, U^G)$ is also automatically q -connected for all q .)

PROOF. The “only if” part is clear.

In order to prove the “if” part we will first prove that given any compact subset A of M there exists an equivariant ball E in M such that $A \subset E$.

I. The case $0 \leq \dim M^G \leq 4$. Let $k = \dim M^G$ and let $\alpha: \mathbf{R}^k \times B^{m-k}(\rho') \rightarrow D$ be a G -homeomorphism. Then we have $M^G = D^G = \alpha(\mathbf{R}^k \times \{0\})$. Let us define

$$C = \alpha(\mathbf{R}^k \times B_{1/2}^{m-k}(\rho')), \quad \text{and}$$

$$W = M - \alpha(\mathbf{R}^k \times B_{1/3}^{m-k}(\rho')),$$

where $B_r^{m-k}(\rho') = \{x \in \mathbf{R}^{m-k}(\rho') \mid \|x\| \leq r\}$. Then W is open in M and $\text{Int } C = \alpha(\mathbf{R}^k \times \mathring{B}_{1/2}^{m-k}(\rho'))$. Observe that $M = \text{Int } C \cup W$. Let us denote $U = \text{Int } D = \alpha(\mathbf{R}^k \times \mathring{B}^{m-k}(\rho'))$. Then $M^G \subset C \subset U$.

Now let A be an arbitrary compact subset of M . Since M is \mathfrak{F}' -equivariantly unraveled at infinity there exists a compact G -subset B , where $A \subset B$, such that

$(M_H, (M - B)_H)$ is 2-connected for every subgroup $H \neq G$ occurring in M for which M^H is noncompact. If $1 \leq \dim M^G \leq 4$, then M^G is noncompact, and hence in this case M^H is noncompact for every subgroup H . If $\dim M^G = 0$ then $M^G = \{a\}$, a point, and we can assume that B is chosen such that $A \cup \{a\} \subset B$. Then we have $(M - B)^G = \emptyset$. We claim that M^H is noncompact for every subgroup $H \neq G$ occurring in M . Let H_1 be a subgroup of G which occurs in M and which is maximal in the set of all subgroups different from G occurring in M . Then $M^{H_1} = M_{H_1} \cup M^G = M_{H_1} \cup \{a\}$ and $D^{H_1} = D_{H_1} \cup D^G = D_{H_1} \cup \{a\}$. Let us denote $k_1 = \dim M^{H_1} \geq 5$. If M^{H_1} were compact we would have $H_{k_1}(M^{H_1}; Z_2) \cong Z_2$. Since (M_{H_1}, D_{H_1}) is q -connected for all, it follows by excision that $H_{k_1}(M^{H_1}, D^{H_1}; Z_2) \cong H_{k_1}(M_{H_1}, D_{H_1}; Z_2) = 0$. Therefore by the exact homology sequence of the pair (M^{H_1}, D^{H_1}) , we get $H_{k_1}(M^{H_1}; Z_2) \cong H_{k_1}(M^{H_1}, D^{H_1}; Z_2) = 0$. Thus M^{H_1} is noncompact. It now follows that M^H is noncompact for every subgroup $H \neq G$ occurring in M .

Let T be an equivariant triangulation of M which is subordinate to the open cover $\{\text{Int } C, W\}$. Let L be the subcomplex of T which consists of all closed simplexes of T that are contained in W . Since $W^G = \emptyset$ we have $L^G = \emptyset$. Let us denote

$$V = M - B.$$

Then $M - V = B$ is compact. Let L^2 be the 2-skeleton of L . Thus $L^2 - V$ is compact and (M_H, V_H) is 2-connected for every H which occurs in L . Hence there exist, by Theorem 2.4, a compact G -subset E'_1 of M and a G -homeomorphism v of M such that $L^2 \subset v(V)$ and $v|_{M - E'_1} = \text{id}$. Let $E_1 = E'_1 \cup B$. Then we have $v|_{M - E_1} = \text{id}$ and $M - E_1 \subset V$. We now define P to be the union of all closed simplexes of T that are contained in $M - E_1$, and

$$K = L^2 \cup P.$$

Then, since $L^2 \subset v(V)$ and $P \subset M - E_1 = v(M - E_1) \subset v(V)$, it follows that

$$K \subset v(V).$$

Now define

$$J = T \div K,$$

i.e., J is the maximal subcomplex of the first barycentric subdivision T' of T which does not intersect K . We have

$$J \subset T - K \subset T - P \subset N(E_1; T),$$

and hence J is compact. Next observe that

$$(*) \quad J - C \subset (T \div L^2) - C.$$

We claim that for every subgroup H of G and any component M_H^α of M_H we have

$$(**) \quad \dim[(J - C)_H]^\alpha \leq \dim M_H^\alpha - 3.$$

Because of $(*)$ it is enough to prove $(**)$ with $(T \div L^2) - C$ in place of $J - C$. Let s' be a simplex of $T \div L^2$ with principal isotropy subgroup equal to H and such that $s' \subset M_H^\alpha$ and $s' \not\subset C$. Then s' is a simplex of T' such that $s' \cap L^2 = \emptyset$. Let s be the carrier of s' in T , i.e. s is the smallest simplex of T which contains s' . Then we also

have $s \not\subset C$ and hence in particular $s \not\subset \text{Int } C$. Thus $s \subset W$, since T is subordinate to the open cover $\{\text{Int } C, W\}$. Since the principal isotropy subgroup of s also equals H we have $s \subset M_H$. Since $s \subset W$ we have $s \subset L$, and hence the 2-skeleton s^2 of s lies in L^2 . Then $s' \subset s$ and $s' \cap L^2 = \emptyset$ imply that

$$\dim s' \leq \dim s - 3 \leq \dim M_H^G - 3.$$

The pair $(M - C, U - C)$ is G -homeomorphic to $(M - M^G, U - M^G)$. Thus $((M - C)_H, (U - C)_H)$ is homeomorphic to (M_H, U_H) for every $H \neq G$. Hence $((M - C)_H, (U - C)_H)$ is q -connected for all q and every subgroup H occurring in $J - C$. It now follows, by Theorem 2.5, that there is a G -homeomorphism u of M such that $J \subset u(U)$.

Since $K \subset v(V)$ and $J \subset u(U)$ and $J = T \div K$ there exists a G -homeomorphism $w: |T'| \rightarrow |T'|$ such that

$$w(u(U)) \cup v(V) = |T'| = M.$$

Then

$$h = v^{-1} \circ w \circ u: M \rightarrow M$$

is a G -homeomorphism such that $h(U) \cup V = M$, and hence

$$A \subset B \subset h(U).$$

Since A is compact and $h(U) \subset h(D) = (h \circ \alpha)(\mathbf{R}^k \times B^{m-k}(\rho'))$ there exists $r > 0$ such that

$$A \subset (h \circ \alpha)(B_r^k \times B^{m-k}(\rho')) = E,$$

where $B_r^k = \{x \in \mathbf{R}^k \mid \|x\| \leq r\}$. Then E is G -homeomorphic with $B^m(\rho)$, where

$$\rho = \underbrace{\text{id} \oplus \cdots \oplus \text{id}}_k \oplus \rho',$$

and we have thus proved that A is contained in an equivariant ball E in M .

In the case when $\dim M^G \geq 5$ the proof of the fact that any compact subset A of M is contained in an equivariant ball E is completely similar to the above proof and in fact simpler. The sets C and W are not needed at all and L is chosen to be T .

Thus it follows that in both cases I and II we have that $M = \bigcup_{i=1}^\infty E_i$, where $E_1 \subset \mathring{E}_2 \subset E_2 \subset \mathring{E}_3 \subset \dots$ is an increasing sequence of equivariant m -balls and each E_i is G -homeomorphic to $B^m(\rho)$, $i = 1, 2, 3, \dots$. Hence Proposition 3.4 below implies that M is G -homeomorphic to $\mathbf{R}^m(\rho)$. \square

The proof of Proposition 3.4 below is completely analogous to the proof of the “monotone union of open n -cells theorem” by Morton Brown [3]. If S and S' are two embedded nonintersecting $(m - 1)$ -spheres in an m -ball we denote by $[S, S']$ the closure of the region between S and S' . The crucial lemma is the following.

LEMMA 3.3. *Let $j: B^n(\rho) \rightarrow B^n(\rho)$ be a G -embedding such that $j(B^n(\rho)) \subset \mathring{B}^n(\rho)$, and let $A \subset \mathring{B}^n(\rho)$ be compact. Then there exists a G -homeomorphism $h: B^n(\rho) \rightarrow B^n(\rho)$ such that $j(B^n(\rho)) \cup A \subset hj(B_{1/2}^n(\rho))$ and $[j(S_{1/2}^{n-1}(\rho)), hj(S_{1/2}^{n-1}(\rho))]$ is G -homeomorphic to $S^{n-1}(\rho) \times [0, 1]$.*

PROOF. We first prove the lemma in the case that $j(0) = 0$. Let $\epsilon > 0$ be such that $B_\epsilon^n(\rho) \subset j(B_{1/2}^n(\rho))$ and $j(B^n(\rho)) \cup A \subset B_{1-\epsilon}^n(\rho)$. Let $h: B^n(\rho) \rightarrow B^n(\rho)$ be a G -homeomorphism such that $h|_{B_{\epsilon/2}^n(\rho)} = \text{id}$, $h(B_\epsilon^n(\rho)) = B_{1-\epsilon}^n(\rho)$ and $h|_{S^{n-1}(\rho)} = \text{id}$. Then $j(B^n(\rho)) \cup A \subset B_{1-\epsilon}^n(\rho) = h(B_\epsilon^n(\rho)) \subset hj(B_{1/2}^n(\rho))$. It remains to prove that $[j(S_{1/2}^{n-1}(\rho)), hj(S_{1/2}^{n-1}(\rho))]$ is G -homeomorphic to $S^{n-1}(\rho) \times [0, 1]$.

Choose a G -homeomorphism $f': j(B^n(\rho)) \rightarrow j(B^n(\rho))$ such that $f'|_{j(S^{n-1}(\rho))} = \text{id}$, $f'(j(S_{3/4}^{n-1}(\rho))) = j(S_{1/2}^{n-1}(\rho))$ and $f'(j(B_{1/2}^n(\rho))) \subset \dot{B}_{\epsilon/2}^n(\rho)$, and extend f' by the identity map to a G -homeomorphism $f: B^n(\rho) \rightarrow B^n(\rho)$. Then we have the following sequence of G -homeomorphisms

$$\begin{aligned} S^{n-1}(\rho) \times [0, 1] &\xrightarrow{\cong} [S_{1/2}^{n-1}(\rho), S_{3/4}^{n-1}(\rho)] \xrightarrow{j} [j(S_{1/2}^{n-1}(\rho)), j(S_{3/4}^{n-1}(\rho))] \\ &\xrightarrow{f} [fj(S_{1/2}^{n-1}(\rho)), j(S_{1/2}^{n-1}(\rho))] \xrightarrow{h} [fj(S_{1/2}^{n-1}(\rho)), hj(S_{1/2}^{n-1}(\rho))] \\ &\xrightarrow{f^{-1}} [j(S_{1/2}^{n-1}(\rho)), hj(S_{1/2}^{n-1}(\rho))]. \end{aligned}$$

Now consider the case when $j(0) = b \neq 0$. Then $G_b = G$ and the line $\mathbf{R}b$ is pointwise fixed under the action of G , i.e. $\mathbf{R}b \subset (\mathbf{R}^n(\rho))^G$. Then we can write $\mathbf{R}^n(\rho) = \mathbf{R}^{n-1}(\rho) \oplus \mathbf{R}b$ and it follows immediately that there exists a G -homeomorphism $k: B^n(\rho) \rightarrow B^n(\rho)$ such that $k(b) = 0$. Then applying the above considered case to the G -imbedding $k \circ j$ and the compact set $k(A)$ we find a G -homeomorphism $h': B^n(\rho) \rightarrow B^n(\rho)$ such that $kj(B^n(\rho)) \cup k(A) \subset h'kj(B_{1/2}^n(\rho))$ and $[kj(S_{1/2}^{n-1}(\rho)), h'kj(S_{1/2}^{n-1}(\rho))]$ is G -homeomorphic to $S^{n-1}(\rho) \times [0, 1]$. Thus the G -homeomorphism $h = k^{-1} \circ h' \circ k$ is of the required form. \square

PROPOSITION 3.4. *Let M be a topological m -manifold with an action of G . Assume that $M = \bigcup_{i=1}^\infty E_i$, where $E_1 \subset \dot{E}_2 \subset E_2 \subset \dot{E}_3 \subset \dots$ and there exist G -homeomorphisms $f_i: B^m(\rho) \rightarrow E_i$ and $\dot{E}_i = f_i(\dot{B}^m(\rho))$, for all i . Then M is G -homeomorphic to $\mathbf{R}^m(\rho)$.*

PROOF. Let us denote $E_1 = E$, $f_1(B_{1/2}^m(\rho)) = E(1/2) \subset E$ and $f_1(S_{1/2}^{m-1}(\rho)) = \partial E(1/2)$, and let \cong stand for "is G -homeomorphic to". Consider the G -embedding $E = E_1 \hookrightarrow E_2$. Then Lemma 3.3 implies that there exists a G -homeomorphism $k_2: E_2 \rightarrow E_2$ such that $E \subset k_2(E(1/2))$ and $[\partial E(1/2), k_2\partial E(1/2)] \cong S^{m-1}(\rho) \times [0, 1]$. Then consider the G -embedding $k_2|: E \rightarrow E_3$, $x \mapsto k_2(x)$. By Lemma 3.3 there exists a G -homeomorphism $k_3: E_3 \rightarrow E_3$ such that $E_2 \subset k_3k_2(E(1/2))$ and

$$[k_2\partial E(1/2), k_3k_2\partial E(1/2)] \cong S^{m-1}(\rho) \times [0, 1].$$

Next consider the G -embedding $k_3k_2|: E \rightarrow E_4$, and so on. In this way we see that there exist G -homeomorphisms k_i of E_i , where $k_1 = \text{id}$, such that denoting

$$s_i = k_i \circ \dots \circ k_2 \circ k_1: E \rightarrow M$$

we have that

$$\begin{cases} E_{i-1} \subset s_i(E(1/2)), \\ [s_{i-1}\partial E(1/2), s_i\partial E(1/2)] \cong S^{m-1}(\rho) \times [0, 1], \end{cases} \quad i = 2, 3, \dots$$

Thus $M = \bigcup_{i=1}^{\infty} s_i(E(1/2))$. It now follows that there is a continuous bijective G -map $h: \mathbf{R}^m(\rho) \rightarrow M$, and hence, by invariance of domain, h is a G -homeomorphism. \square

4. The main lemmas. In this section we prove the main lemmas that allow us to handle codimension 2 fixed-point situations and the interference of codimension 1 and codimension 2 fixed-point situations. We will often encounter nonconnected manifolds. Any notion we use for a nonconnected manifold W is defined in the obvious way, i.e., by the requirement that it holds for every component V of W . For example, we say that a nonconnected manifold W is orientable if every component V of W is orientable, and $p: \tilde{W} \rightarrow W$ is a universal covering of W if $p|_V: p^{-1}(V) \rightarrow V$ is a universal covering of V for every component V of W .

LEMMA 4.1. *Let M be an orientable manifold and W an open subset of M . Let \mathcal{U} be a finite family of orientable closed submanifolds of M which is closed under intersection. Assume that $i_*: H_q(W) \rightarrow H_q(M)$ and $i'_*: H_q(W \cap N) \rightarrow H_q(N)$ are isomorphisms for all $q \geq 0$, and every $N \in \mathcal{U}$, where i and i' denote inclusions. Then*

$$j_*: H_q\left(W - \bigcup_{i=1}^r (W \cap N_i)\right) \rightarrow H_q\left(M - \bigcup_{i=1}^r N_i\right),$$

where j denotes the inclusion, is an isomorphism for all $q \geq 0$ and any $N_1, \dots, N_r \in \mathcal{U}$.

PROOF. Using homology Mayer-Vietoris sequences and the 5-lemma we see that it is enough to consider the case $r = 1$. Let $N \in \mathcal{U}$. Once again using homology Mayer-Vietoris sequences and the 5-lemma we see that it is enough to consider one component of N . Let N_0 be a component of N and denote $W \cap N_0 = W_0$, $C = M - W$ and $C_0 = C \cap N_0 = N_0 - W_0$. Using exact homology sequences and the 5-lemma it follows that it is enough to prove that $j_*: H_q(W, W - W_0) \rightarrow H_q(M, M - N_0)$ are isomorphisms for all q . We have the commutative diagram

$$\begin{array}{ccccc} H_q(W, W - W_0) & \xrightarrow[\cong]{c_*} & H_q(M - C_0, M - N_0) & \xrightarrow[\cong]{\gamma} & \overline{H}_c^{m-q}(N_0, C_0) \\ j_* \downarrow & & \downarrow & & \downarrow j_*^* \\ H_q(M, M - N_0) & \xrightarrow[\cong]{\text{id}} & H_q(M, M - N_0) & \xrightarrow[\cong]{\gamma} & \overline{H}_c^{m-q}(N_0) \end{array}$$

where $m = \dim M$. Here $\overline{H}_c^*()$ denotes Alexander-Spanier cohomology with compact supports and integer coefficients. The maps γ are duality isomorphisms, see e.g. Spanier [15, Theorem 6.9.10], and c_* is an isomorphism by excision. Since we also have the commutative diagram

$$\begin{array}{ccc} H_p(W_0) = H_p(N_0 - C_0) & \xrightarrow[\cong]{\gamma'} & \overline{H}_c^{n-p}(N_0, C_0) \\ (i_0)_* \downarrow \cong & & \downarrow j_*^* \\ H_p(N_0) & \xrightarrow[\cong]{\gamma'} & \overline{H}_c^{n-p}(N_0) \end{array}$$

where $n = \dim N_0$, the proof is complete. \square

LEMMA 4.2. Let M denote euclidean space \mathbf{R}^m with a smooth action of G such that each fixed-point set M^H is homeomorphic to euclidean space $\mathbf{R}^{n(H)}$ and $M^G \neq \emptyset$, and $\dim M^H \geq 4$ for each subgroup $H \neq G$ occurring in M . Let H, H_1, \dots, H_p be subgroups of G such that $H \subset H_i$ and $\dim M^{H_i} = \dim M^H - 1$, for $1 \leq i \leq p$. Then every component of $M^H - \bigcup_{i=1}^p M^{H_i}$ is homeomorphic to euclidean space $\mathbf{R}^{n(H)}$. In particular it follows that if U is a smooth slice at $a \in M^G$ the map $i_\# : \pi_q(U^H - \bigcup_{i=1}^p U^{H_i}) \rightarrow \pi_q(M^H - \bigcup_{i=1}^p M^{H_i})$, induced by the inclusion, is an isomorphism for every $q \geq 0$.

PROOF. We may assume that the subgroups H, H_1, \dots, H_p are such that they occur in their respective fixed-point sets and that $M^{H_i} \neq M^{H_j}$ for $i \neq j$. Since $M^{H_i} \cong \mathbf{R}^{n(H)-1}$ is closed in $M^H \cong \mathbf{R}^{n(H)}$ it follows by the Jordan-Brouwer theorem that $M^H - M^{H_i}$ consists of two components M_+^i and M_-^i , $1 \leq i \leq p$. Let U_+^i and U_-^i be the two components of $U^H - U^{H_i}$, where $U_+^i \subset M_+^i$ and $U_-^i \subset M_-^i$, $1 \leq i \leq p$. We now claim the following. Each component of $M^H - \bigcup_{i=1}^p M^{H_i}$ is of the form $M_{\sigma(1)}^1 \cap \dots \cap M_{\sigma(p)}^p$, where $\sigma : \{1, \dots, p\} \rightarrow \{+, -\}$, and each of these components is homeomorphic to $\mathbf{R}^{n(H)}$, and moreover the inclusion $i : U^H - \bigcup_{i=1}^p U^{H_i} \rightarrow M^H - \bigcup_{i=1}^p M^{H_i}$ induces a one-to-one correspondence between components. We shall prove this claim by induction in p . Since $M^{H_1} \cong \mathbf{R}^{n(H)-1}$ and M^{H_1} is locally flat and closed in $M^H \cong \mathbf{R}^{n(H)}$, where $n(H) \geq 4$, it follows by Cantrell [4] (the corollary on p. 718 in [4], see also §3.3 in Rushing [13]) that the two components M_+^1 and M_-^1 of $M^H - M^{H_1}$ are both homeomorphic to $\mathbf{R}^{n(H)}$. Moreover $i : U^H - U^{H_1} \rightarrow M^H - M^{H_1}$ induces a one-to-one correspondence of the components.

Now let $p \geq 2$ and assume that our claim holds for the values 1 to $p - 1$. Observe that in this case $\dim M^{H_1} \geq 4$, for $\dim M^{H_1} \leq 3$ would imply $M^{H_1} = M^G$ and similarly $M^{H_2} = M^G$, which contradicts the fact that $M^{H_1} \neq M^{H_2}$. Clearly every component of $M^H - \bigcup_{i=1}^p M^{H_i}$ is contained in some $M_{\sigma(1)}^1 \cap \dots \cap M_{\sigma(p)}^p$, so we must prove that every nonempty $M_{\sigma(1)}^1 \cap \dots \cap M_{\sigma(p)}^p$ is connected and homeomorphic to $\mathbf{R}^{n(H)}$. Let $N = M_{\sigma(1)}^1 \cap \dots \cap M_{\sigma(p)}^p \neq \emptyset$. Then $N' = M_{\sigma(1)}^1 \cap \dots \cap M_{\sigma(p-1)}^{p-1} \neq \emptyset$ and hence by our inductive assumption N' is a component of $M^H - \bigcup_{j=1}^{p-1} M^{H_j}$ and $N' \cong \mathbf{R}^{n(H)}$. Let us denote $L = M^{H_p}$ and let the two components of $M^{H_p} - (M^{H_p} \cap M^{H_j}) = M^{H_p} - M^{\langle H_p, H_j \rangle}$ be denoted by L_+^j and L_-^j , where $L_+^j \subset M_+^j$ and $L_-^j \subset M_-^j$, $1 \leq j \leq p - 1$. Then, by our inductive assumption, each component of $M^{H_p} - \bigcup_{j=1}^{p-1} M^{\langle H_p, H_j \rangle}$ is of the form $L_{\tau(1)}^1 \cap \dots \cap L_{\tau(p-1)}^{p-1}$, where $\tau : \{1, \dots, p - 1\} \rightarrow \{+, -\}$, and each nonempty $L_{\tau(1)}^1 \cap \dots \cap L_{\tau(p-1)}^{p-1}$ is homeomorphic to $\mathbf{R}^{n(H_p)}$. We have

$$\begin{aligned}
 (*) \quad (N' \cap M_+^p) \dot{\cup} (N' \cap M_-^p) &= N' - M^{H_p} = N' - \left(M^{H_p} - \bigcup_{j=1}^{p-1} M^{\langle H_p, H_j \rangle} \right) \\
 &= N' - \bigcup_{\tau} \left(L_{\tau(1)}^1 \cap \dots \cap L_{\tau(p-1)}^{p-1} \right) \\
 &= N' - \left(L_{\sigma(1)}^1 \cap \dots \cap L_{\sigma(p-1)}^{p-1} \right).
 \end{aligned}$$

If $L' = L_{\sigma(1)}^1 \cap \dots \cap L_{\sigma(p-1)}^{p-1} = \emptyset$ we have $N' \cap M^{H_p} = \emptyset$ and hence, since N' is connected, either $N' \subset M_+^p$ or $N' \subset M_-^p$. Since $N = N' \cap M_{\sigma(p)}^p \neq \emptyset$ we have

$N' \subset M_{\sigma(p)}^p$, and hence in this case $N = N' \cap M_{\sigma(p)}^p = N'$, which is connected and homeomorphic to $\mathbf{R}^{n(H)}$. If $L' \neq \emptyset$ we have $N' \cap M^{H_p} \neq \emptyset$, and hence $N' \cap M_+^p \neq \emptyset$ and $N' \cap M_-^p \neq \emptyset$. Since $L' \subset N'$ and $L' \cong \mathbf{R}^{n(H)-1}$ is locally flat and closed in $N' \cong \mathbf{R}^{n(H)}$ it again follows by Cantrell [4] that the two components of $N' - L'$ are both homeomorphic to $\mathbf{R}^{n(H)}$. Since (*) implies that the two components of $N' - L'$ are $N' \cap M_+^p$ and $N' \cap M_-^p$, it now follows that $N = N' \cap M_{\sigma(p)}^p$ is homeomorphic to $\mathbf{R}^{n(H)}$.

The map between the sets of components of $U^H - \cup_{i=1}^p U^{H_i}$ and $M^H - \cup_{i=1}^p M^{H_i}$, induced by the inclusion, assigns to a (nonempty) component $U_{\sigma(1)}^1 \cap \dots \cap U_{\sigma(p)}^p$ the component $M_{\sigma(1)}^1 \cap \dots \cap M_{\sigma(p)}^p$ and hence this map is clearly injective. It remains to show that it is surjective, namely that $N = M_{\sigma(1)}^1 \cap \dots \cap M_{\sigma(p)}^p \neq \emptyset$ implies $V = U_{\sigma(1)}^1 \cap \dots \cap U_{\sigma(p)}^p \neq \emptyset$. Since $N' = M_{\sigma(1)}^1 \cap \dots \cap M_{\sigma(p-1)}^p \neq \emptyset$ we have by the inductive assumption that $V' = U_{\sigma(1)}^1 \cap \dots \cap U_{\sigma(p-1)}^p \neq \emptyset$. Let W_+^j and W_-^j be the two components of $U^{H_p} - U^{\langle H_p, H_j \rangle}$, where $W_+^j \subset L_+^j$ and $W_-^j \subset L_-^j$, $1 \leq j \leq p-1$. Then we get, in analogy with (*),

$$(**) \quad (V' \cap U_+^p) \dot{\cup} (V' \cap U_-^p) = V' - U^{H_p} = V' - (W_{\sigma(1)}^1 \cap \dots \cap W_{\sigma(p-1)}^{p-1}).$$

If $L' = \emptyset$ we have, as we saw above, that $N' \subset M_{\sigma(p)}^p$. Therefore $V' \subset N' \subset M_{\sigma(p)}^p$ and $V' \subset M_{\sigma(p)}^p \cap U = U_{\sigma(p)}^p$, and hence $V = V' \cap U_{\sigma(p)}^p = V' \neq \emptyset$. If $L' \neq \emptyset$ it follows by the inductive assumption that $W' = W_{\sigma(1)}^1 \cap \dots \cap W_{\sigma(p-1)}^{p-1} \neq \emptyset$, and hence by (**) we have that $V' \cap U^{H_p} \neq \emptyset$, which in turn implies that both $V' \cap U_+^p \neq \emptyset$ and $V' \cap U_-^p \neq \emptyset$. Thus $V = V' \cap U_{\sigma(p)}^p \neq \emptyset$. This completes the induction and the proof of Lemma 4.2. \square

LEMMA 4.3. *Let M be as in 4.2, and let H, H_1 and H_2 be subgroups of G , where $H \subset H_1$ and $H \subset H_2$, such that $\dim M^{H_1} = \dim M^H - 1$ and $\dim M^{H_2} = \dim M^H - 2$. Let U be a smooth slice at a point $a \in M^G$ and assume that, for any subgroups $K \subset K_2$ with $\dim M^{K_2} = \dim M^K - 2$, the map*

$$i'_\# : \pi_q(U^K - U^{K_2}) \rightarrow \pi_q(M^K - M^{K_2}),$$

induced by the inclusion i' , is an isomorphism for every $q \geq 1$. Then the map $j'_\# : \pi_q(U^H - (U^{H_1} \cup U^{H_2})) \rightarrow \pi_q(M^H - (M^{H_1} \cup M^{H_2}))$, induced by the inclusion j' , is an isomorphism for every $q \geq 0$.

PROOF. Since M^{H_1} is closed in M^H it follows by the Jordan-Brouwer separation theorem that $M^H - M^{H_1}$ consists of two components M_+ and M_- such that $\overline{M}_+ \cup \overline{M}_- = M^H$ and $\overline{M}_+ \cap \overline{M}_- = M^{H_1}$. Likewise $U^H - U^{H_1}$ consists of two components U_+ and U_- , where $U_+ \subset M_+$, such that $\overline{U}_+ \cup \overline{U}_- = U^H$ and $\overline{U}_+ \cap \overline{U}_- = U^{H_1}$. The spaces $U_+ - U^{H_2}$ and $M_+ - M^{H_2}$ are connected, and we shall prove that

$$j'_\# : \pi_q(U_+ - U^{H_2}) \rightarrow \pi_q(M_+ - M^{H_2})$$

is an isomorphism for every $q \geq 1$. The proof in the case of the minus-components is of course completely analogous.

If $U^{H_2} \subset U^{H_1}$ we also have $M^{H_2} \subset M^{H_1}$ and hence in this case $U_+ - U^{H_2} = U_+$ and $M_+ - M^{H_2} = M_+$. Since $M^{H_1} \cong \mathbf{R}^{n(H)-1}$ is locally flat and closed in $M^H \cong \mathbf{R}^{n(H)}$ it follows by Cantrell [4] that M_+ is homeomorphic to $\mathbf{R}^{n(H)}$. Since also $U_+ \cong \mathbf{R}^{n(H)}$ our claim is clear in this case.

Let us now assume that $U^{H_2} \not\subset U^{H_1}$. Let $\alpha': \mathbf{R}^m(\rho) \rightarrow U$ be a G -diffeomorphism, and denote $n = \dim(\mathbf{R}^m(\rho))^H = \dim U^H$. Then $\dim(\mathbf{R}^m(\rho))^{H_1} = n - 1$, $\dim(\mathbf{R}^m(\rho))^{H_2} = n - 2$ and $(\mathbf{R}^m(\rho))^{H_2} \not\subset (\mathbf{R}^m(\rho))^{H_1}$. By a linear change of coordinates in $\mathbf{R}^n = (\mathbf{R}^m(\rho))^H$ we get the situation that

$$\alpha = \alpha' | : \mathbf{R}^n \xrightarrow{\cong} U^H$$

is a homeomorphism such that

$$\alpha(\mathbf{R}^{n-1}) = U^{H_1},$$

$$\alpha(\hat{\mathbf{R}}^{n-2}) = U^{H_2},$$

$$\alpha(\mathbf{R}_+^n) = \bar{U}_+ \quad (\text{here the closure is with respect to } U),$$

where $\mathbf{R}^{n-1} = \mathbf{R}^{n-1} \times \{0\} \subset \mathbf{R}^n$, $\hat{\mathbf{R}}^{n-2} = \mathbf{R}^{n-3} \times \{0\} \times \{0\} \times \mathbf{R} \subset \mathbf{R}^n$ and $\mathbf{R}_+^n = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_n \geq 0\}$. For $X \subset \mathbf{R}^n$ we set $B(X) = \{x \in X \mid \|x\| \leq 1\}$, and denote $B(\mathbf{R}^n) = B^n$, $B(\mathbf{R}^{n-1}) = B^{n-1}$, $B(\hat{\mathbf{R}}^{n-2}) = \hat{B}^{n-2}$ and $B(\mathbf{R}_+^n) = B_+^n$. Now define

$$\alpha(B^n) = A^n \subset U^H,$$

$$\alpha(B^{n-1}) = A^{n-1} \subset U^{H_1},$$

$$\alpha(\hat{B}^{n-2}) = \hat{A}^{n-2} \subset U^{H_2},$$

$$\alpha(B_+^n) = A_+^n \subset \bar{U}_+.$$

Since the inclusions $B^n - \hat{B}^{n-2} \hookrightarrow \mathbf{R}^n - \hat{\mathbf{R}}^{n-2}$ and $B_+^n - \hat{B}^{n-2} \hookrightarrow \mathbf{R}_+^n - \hat{\mathbf{R}}^{n-2}$ are homotopy equivalences it follows that the inclusions

- (1) $A^n - \hat{A}^{n-2} \hookrightarrow U^H - U^{H_2},$
- (2) $A_+^n - \hat{A}^{n-2} \hookrightarrow \bar{U}_+ - U^{H_2}$

are homotopy equivalences. By (1) and our assumption, it follows that

$$(3) \quad i_{\#}: \pi_q(A^n - \hat{A}^{n-2}) \xrightarrow{\cong} \pi_q(M^H - M^{H_2})$$

is an isomorphism for every $q \geq 1$. Now $A^n - \hat{A}^{n-2}$ is a smooth submanifold, with boundary, of $M^H - M^{H_2}$. The boundary $\partial(A^n - \hat{A}^{n-2})$ is closed in $M^H - M^{H_2}$ and $(M^H - M^{H_2}) - \partial(A^n - \hat{A}^{n-2})$ consists of two components. Hence it follows that $M^H - M^{H_2}$ can be triangulated in such a way that $A^n - \hat{A}^{n-2}$ is a subcomplex of $M^H - M^{H_2}$. Since (3) holds it now follows that $A^n - \hat{A}^{n-2}$ is a strong deformation retract of $M^H - M^{H_2}$. Let

$$F: (M^H - M^{H_2}) \times I \rightarrow M^H - M^{H_2}$$

be a homotopy $\text{rel}(A^n - \hat{A}^{n-2})$ from the identity to the map $i \circ r$, where

$$r: M^H - M^{H_2} \rightarrow A^n - \hat{A}^{n-2}$$

is a retraction and $i: A^n - \hat{A}^{n-2} \hookrightarrow M^H - M^{H_2}$ is the inclusion.

We have $M^{H_1} \cap M^{H_2} = M^{H_3}$, where $H_3 = [H_1, H_2]$ denotes the subgroup generated by H_1 and H_2 , and since $\dim M^{H_3} = n - 3$ we have by assumption that $i_{\#}: \pi_q(U^{H_1} - U^{H_3}) \rightarrow \pi_q(M^{H_1} - M^{H_3})$ is an isomorphism for all $q \geq 1$. Therefore it follows, in the same way as above, that $A^{n-1} - A^{n-3}$ is a strong deformation retract of $M^{H_1} - M^{H_3}$, where $A^{n-3} = A^{n-1} \cap \hat{A}^{n-2}$. Hence $A^n - \hat{A}^{n-2}$ is a strong deformation retract of $(A^n - \hat{A}^{n-2}) \cup (M^{H_1} - M^{H_3})$, and since the corresponding inclusion as well as the inclusion i are homotopy equivalences it follows that the inclusion

$(A^n - \hat{A}^{n-2}) \cup (M^{H_1} - M^{H_3}) \hookrightarrow M^H - M^{H_2}$ is a homotopy equivalence. Thus $(\hat{A} - \hat{A}^{n-2}) \cup (M^{H_1} - M^{H_3})$ is a strong deformation retract of $M^H - M^{H_2}$, and in particular there exists a retraction

$$\bar{r}: M^H - M^{H_2} \rightarrow (A^n - \hat{A}^{n-2}) \cup (M^{H_1} - M^{H_3}).$$

Let

$$r'_0: B^n - \hat{B}^{n-2} \rightarrow B_+^n - \hat{B}^{n-2}$$

be the retraction defined by

$$\begin{cases} r'_0|_{B_+^n - B^{n-2}} = \text{id}, \\ r'_0(x_1, \dots, x_n) = (x_1, \dots, -x_n) \quad \text{for } (x_1, \dots, x_n) \in B_-^n - \hat{B}^{n-2}. \end{cases}$$

Then

$$r_0 = \alpha \circ r'_0 \circ \alpha^{-1}: A^n - \hat{A}^{n-2} \rightarrow A_+^n - \hat{A}^{n-2}$$

is a retraction, and extending by the identity map we get a retraction

$$\bar{r}_0: (A^n - \hat{A}^{n-2}) \cup (M^{H_1} - M^{H_3}) \rightarrow (A_+^n - \hat{A}^{n-2}) \cup (M^{H_1} - M^{H_3}).$$

We now define a retraction

$$\tilde{r}: M^H - M^{H_2} \rightarrow \bar{M}_+ - M^{H_2}$$

by

$$\begin{cases} \tilde{r}|_{\bar{M}_+ - M^{H_2}} = \text{id}, \\ \tilde{r}|_{\bar{M}_- - M^{H_2}} = (\bar{r}_0 \circ \tilde{r})|_{(\bar{M}_- - M^{H_2})}. \end{cases}$$

Then the composite

$$(\bar{M}_+ - M^{H_2}) \times I \xrightarrow{\tilde{i} \times \text{id}} (M^H - M^{H_2}) \times I \xrightarrow{F} M^H - M^{H_2} \xrightarrow{\tilde{r}} \bar{M}_+ - M^{H_2}$$

is a homotopy $\text{rel}(A_+^n - \hat{A}^{n-2})$ from the identity to $i_+ \circ r_0 \circ r \circ \tilde{i}$, where $i_+: A_+^n - \hat{A}^{n-2} \hookrightarrow \bar{M}_+ - M^{H_2}$ and $\tilde{i}: \bar{M}_+ - M^{H_2} \hookrightarrow M^H - M^{H_2}$ denote inclusions, and $r_0 \circ r \circ \tilde{i}: \bar{M}_+ - M^{H_2} \rightarrow A_+^n - \hat{A}^{n-2}$ is a retraction. Thus

$$(i_+)_\# : \pi_q(A_+^n - \hat{A}^{n-2}) \xrightarrow{\cong} \pi_q(\bar{M}_+ - M^{H_2})$$

is an isomorphism for every q . Together with (2) this now implies that

$$\bar{j}_\# : \pi_q(\bar{U}_+ - U^{H_2}) \xrightarrow{\cong} \pi_q(\bar{M}_+ - M^{H_2})$$

is an isomorphism for all q . Since $\bar{M}_+ - M^{H_2}$ is a manifold with boundary whose interior is $M_+ - M^{H_2}$ it follows, using a collar of the boundary, that the inclusion $M_+ - M^{H_2} \hookrightarrow \bar{M}_+ - M^{H_2}$ is a homotopy equivalence. Also $U_+ - U^{H_2} \hookrightarrow \bar{U}_+ - U^{H_2}$ is a homotopy equivalence and hence

$$j_\# : \pi_q(U_+ - U^{H_2}) \xrightarrow{\cong} \pi_q(M_+ - M^{H_2})$$

is an isomorphism for all q . \square

LEMMA 4.4. Assume that G is a finite abelian group, and let M be as in 4.2. Let H be a subgroup of G , and let H_1, \dots, H_p and K_1, \dots, K_r be subgroups of G containing H such that $\dim M^{H_i} = \dim M^H - 1$, $1 \leq i \leq p$, and $\dim M^{K_i} = \dim M^H - 2$, $1 \leq i \leq r$. Let U be a smooth slice at a point $a \in M^G$ and assume that the map $i_{\#}: \pi_q(U^H - \cup_{i=1}^r U^{K_i}) \rightarrow \pi_q(M^H - \cup_{i=1}^r M^{K_i})$, induced by the inclusion i , is an isomorphism for every $q \geq 1$. Then the map

$$j_{\#}: \pi_q \left(U^H - \left(\bigcup_{i=1}^p U^{H_i} \cup \bigcup_{i=1}^r U^{K_i} \right) \right) \rightarrow \pi_q \left(M^H - \left(\bigcup_{i=1}^p M^{H_i} \cup \bigcup_{i=1}^r M^{K_i} \right) \right),$$

induced by the inclusion j , is an isomorphism for every $q \geq 0$.

PROOF. First of all we may assume that $M^{H_i} \neq M^{H_j}$, if $i \neq j$. Furthermore we may assume that all the groups H, H_1, \dots, H_p and K_1, \dots, K_r are chosen such that they occur in their respective fixed-point sets. Since M^{H_i} is closed in M^H it follows by the Jordan-Brouwer separation theorem that $M^H - M^{H_i}$ consists of two components M_+^i and M_-^i such that $\bar{M}_+^i \cup \bar{M}_-^i = M^H$ and $\bar{M}_+^i \cap \bar{M}_-^i = M^{H_i}$, $1 \leq i \leq p$. Likewise $U^H - U^{H_i}$ consists of two components U_+^i and U_-^i , where $U_+^i \subset M_+^i$, such that $\bar{U}_+^i \cup \bar{U}_-^i = U^H$ and $\bar{U}_+^i \cap \bar{U}_-^i = U^{H_i}$, $1 \leq i \leq p$.

Since G is abelian any $g \in G$ induces a homeomorphism $g: M^H \rightarrow M^H$, and if $g \in H_i$ we have $g|_{M^{H_i}} = \text{id}$. We now claim that for any $g_i \in H_i \setminus H$ we have

$$(1) \quad \begin{cases} g_i(U_+^i) = U_-^i, \\ g_i(U_-^i) = U_+^i, \end{cases} \quad 1 \leq i \leq p,$$

and

$$(2) \quad \begin{cases} g_i(M_+^i) = M_-^i, \\ g_i(M_-^i) = M_+^i, \end{cases} \quad 1 \leq i \leq p.$$

To see this let $\alpha: \mathbf{R}^m(\rho) \rightarrow U$ be a G -homeomorphism. Then $(\mathbf{R}^m(\rho))^H = \mathbf{R}^n(\tau)$ is a representation space for H_i and $\alpha(\mathbf{R}^n(\tau)) = U^H$. Furthermore $\mathbf{R}^n(\tau) = \mathbf{R}^{n-1} \oplus \mathbf{R}(\tau')$, where H_i acts trivially on \mathbf{R}^{n-1} . If the action of g_i on $\mathbf{R}(\tau')$ is trivial we would have $(\mathbf{R}^n(\tau))^{\langle H, \langle g_i \rangle \rangle} = \mathbf{R}^n(\tau)$ and hence $U^{\langle H, \langle g_i \rangle \rangle} = U^H$. But since H occurs in M and hence also in U and $H \subsetneq \langle H, \langle g_i \rangle \rangle$, this is a contradiction. Hence g_i acts on $\mathbf{R}(\tau')$ by $x \mapsto -x$ and it follows that (1) and (2) hold.

Next we claim that

$$(3) \quad \begin{cases} g_i(M_+^j) = M_+^j, \\ g_i(M_-^j) = M_-^j, \end{cases} \quad 1 \leq j \leq p \text{ and } j \neq i,$$

and

$$(4) \quad \begin{cases} g_i(U_+^j) = U_+^j, \\ g_i(U_-^j) = U_-^j, \end{cases} \quad 1 \leq j \leq p \text{ and } j \neq i.$$

This is seen as follows. Since $M^{H_i} \neq M^{H_j}$, and M^{H_i} and M^{H_j} are connected and closed in M^H , it follows that $M^{H_i} \cap M^{H_j} = M^{\langle H_i, H_j \rangle}$ is $(n - 2)$ -dimensional. Choose $y \in M^{H_i} - M^{\langle H_i, H_j \rangle} \neq \emptyset$. Then $y \in M^H - M^{H_j} = M^J_+ \dot{\cup} M^J_-$. Because G is abelian we have the homeomorphism $g_i: M^H - M^{H_j} \rightarrow M^H - M^{H_j}$. Since $g_i(y) = y$ it now follows that $g_i(M^J_+) = M^J_+$ and $g_i(M^J_-) = M^J_-$. This proves (3) and (4).

As we showed in the proof of Lemma 4.2 any component of $M^H - \cup_{i=1}^p M^{H_i}$ is of the form $M^1_{\sigma(1)} \cap \dots \cap M^p_{\sigma(p)}$, where $\sigma: \{1, \dots, p\} \rightarrow \{+, -\}$, and moreover the inclusion $U^H - \cup_{i=1}^p U^{H_i} \hookrightarrow M^H - \cup_{i=1}^p M^{H_i}$ induces a one-to-one correspondence between the components. In fact, since at least one of the intersections $M^1_{\sigma(1)} \cap \dots \cap M^p_{\sigma(p)}$ must be nonempty, it follows by (2) and (3) that all 2^p of them are nonempty. Likewise every component of $U^H - \cup_{i=1}^p U^{H_i}$ is of the form $U^1_{\sigma(1)} \cap \dots \cap U^p_{\sigma(p)}$, and there are 2^p of them. Since $\dim M^{K_i} = \dim M^H - 2$, $1 \leq i \leq r$, it follows that each component of $M^H - (\cup_{i=1}^p M^{H_i} \cup \cup_{i=1}^r M^{K_i})$ is of the form $N - \cup_{i=1}^r M^{K_i}$, where N is a component of $M^H - \cup_{i=1}^p M^{H_i}$, and analogously for U . Hence it follows that the inclusion

$$j: U^H - \left(\bigcup_{i=1}^p U^{H_i} \cup \bigcup_{i=1}^r U^{K_i} \right) \rightarrow M^H - \left(\bigcup_{i=1}^p M^{H_i} \cup \bigcup_{i=1}^r M^{K_i} \right)$$

induces a one-to-one correspondence between the components. Let us now denote $V = U^1_+ \cap \dots \cap U^p_+$ and $N = M^1_+ \cap \dots \cap M^p_+$. Our claim is that

$$(5) \quad j_{\#}: \pi_q \left(V - \bigcup_{i=1}^r U^{K_i} \right) \rightarrow \pi_q \left(N - \bigcup_{i=1}^r M^{K_i} \right)$$

is an isomorphism for every $q \geq 1$. The proof of this fact for any other pair of components is of course completely similar, and in fact using (1)–(4) it follows that the validity of the case in (5) implies the validity of all the other cases.

First of all we claim that

$$(6) \quad \bar{N} = \bar{M}^1_+ \cap \dots \cap \bar{M}^p_+,$$

and that \bar{N} is a topological manifold with boundary whose interior equals N . This is seen as follows. The inclusion $\bar{N} \subset \cup_{i=1}^p \bar{M}^i_+$ is clear. Now let $y \in \cap_{i=1}^p \bar{M}^i_+$. We may assume that $y \in \cap_{i=1}^t M^{H_i} \cap \cap_{i=t+1}^p M^i_+$, where $1 \leq t \leq p$. Let W_y be a smooth slice at y in the smooth G -manifold M^H . Then the pair $(W_y, \cup_{i=1}^t W_y^{H_i})$ is homeomorphic to a pair $(\mathbf{R}^n(\tau), \cup_{i=1}^t \mathbf{R}^n(\tau)^{H_i})$, where $\tau: G_y \rightarrow O(n)$. Thus we see that y belongs to the closure of each of the 2^t components of $W_y - \cup_{i=1}^t W_y^{H_i}$, and hence it follows that $y \in \cap_{i=1}^t \bar{M}^i_+$ and therefore $y \in \cap_{i=1}^t \bar{M}^i_+ \cap \cap_{i=t+1}^p M^i_+ \subset \cap_{i=1}^p \bar{M}^i_+$. Thus (6) is valid. Moreover the neighbourhood $W_y \cap \bar{N}$ of y in \bar{N} is homeomorphic to a set of the form $\{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_{n-t+1} \geq 0, \dots, x_n \geq 0\}$. Hence \bar{N} is a topological manifold with boundary and $\partial \bar{N} = \bar{N} - N$.

Similarly we have $\bar{V} = \bar{U}^1_+ \cap \dots \cap \bar{U}^p_+$ and \bar{V} is a topological manifold with boundary whose interior equals V . Since $\bar{N} - \cup_{i=1}^r M^{K_i}$ is open in \bar{N} it follows that $\bar{N} - \cup_{i=1}^r M^{K_i}$ is a topological manifold with boundary whose interior equals $N - \cup_{i=1}^r M^{K_i}$. Similarly $\bar{V} - \cup_{i=1}^r U^{K_i}$ is a topological manifold with boundary

whose interior equals $V - \bigcup_{i=1}^r U^{K_i}$. Hence it follows (by using a collar of the boundary) that the inclusions

$$(7) \quad \begin{cases} V - \bigcup_{i=1}^r U^{K_i} \hookrightarrow \bar{V} - \bigcup_{i=1}^r U^{K_i}, \\ N - \bigcup_{i=1}^r M^{K_i} \hookrightarrow \bar{N} - \bigcup_{i=1}^r M^{K_i} \end{cases}$$

are homotopy equivalences. Therefore in order to prove that (5) is an isomorphism for every $q \geq 1$ it is enough to prove that

$$(8) \quad \bar{j}_\# : \pi_q \left(\bar{V} - \bigcup_{i=1}^r U^{K_i} \right) \rightarrow \pi_q \left(\bar{N} - \bigcup_{i=1}^r M^{K_i} \right)$$

is an isomorphism for every $q \geq 1$.

We shall now conclude the proof of the lemma by showing that (8) is an isomorphism for every $q \geq 1$. Choose $g_i \in H_i \setminus H, i = 1, \dots, p$. Define a retraction

$$r_1 : M^H - \bigcup_{i=1}^r M^{K_i} \rightarrow \bar{M}_+^1 - \bigcup_{i=1}^r M^{K_i}$$

by

$$\begin{cases} r_1(x) = x, & x \in \bar{M}_+^1 - \bigcup_{i=1}^r M^{K_i}, \\ r_1(x) = g_1(x), & x \in \bar{M}_-^1 - \bigcup_{i=1}^r M^{K_i}. \end{cases}$$

Since G is abelian we have $g_1(M^{K_i}) = M^{K_i}, 1 \leq i \leq r$, and it follows that r_1 is a well-defined retraction. Moreover r_1 restricts to give a retraction

$$r_1| : U^H - \bigcup_{i=1}^r U^{K_i} \rightarrow \bar{U}_+^1 - \bigcup_{i=1}^r U^{K_i}.$$

Now we define

$$r_2 : \bar{M}_+^1 - \bigcup_{i=1}^r M^{K_i} \rightarrow (\bar{M}_+^1 \cap \bar{M}_+^2) - \bigcup_{i=1}^r M^{K_i}$$

by

$$\begin{cases} r_2(x) = x, & x \in (\bar{M}_+^1 \cap \bar{M}_+^2) - \bigcup_{i=1}^r M^{K_i}, \\ r_2(x) = g_2(x), & x \in (\bar{M}_+^1 \cap \bar{M}_-^2) - \bigcup_{i=1}^r M^{K_i}. \end{cases}$$

Then r_2 restricts to a retraction $r_2| : \bar{U}_+^1 - \bigcup_{i=1}^r U^{K_i} \rightarrow (\bar{U}_+^1 \cap \bar{U}_+^2) - \bigcup_{i=1}^r U^{K_i}$. Continuing in this way we get a retraction

$$R = r_p \circ \dots \circ r_1 : M^H - \bigcup_{i=1}^r M^{K_i} \rightarrow \bar{N} - \bigcup_{i=1}^r M^{K_i},$$

such that also

$$R| : U^H - \bigcup_{i=1}^r U^{K_i} \rightarrow \bar{V} - \bigcup_{i=1}^r U^{K_i}$$

is a retraction. Now consider the following diagram:

$$\begin{array}{ccc} \pi_q \left(U^H - \bigcup_{i=1}^r U^{K_i} \right) & \xrightarrow[\cong]{i_{\#}} & \pi_q \left(M^H - \bigcup_{i=1}^r M^{K_i} \right) \\ i_{\#}^{\uparrow} \downarrow (R|)_{\#} & & i_{\#}^{\prime\prime} \uparrow \downarrow R_{\#} \\ \pi_q \left(\bar{V} - \bigcup_{i=1}^r U^{K_i} \right) & \xrightarrow{\bar{j}_{\#}} & \pi_q \left(\bar{N} - \bigcup_{i=1}^r M^{K_i} \right) \end{array}$$

Here $i_{\#}$ is an isomorphism by assumption. The maps $i_{\#}^{\prime\prime}$ and $i_{\#}^{\uparrow}$ are injective and $(R|)_{\#}$ and $R_{\#}$ are surjective. Since $i_{\#}^{\prime\prime} \circ \bar{j}_{\#} = i_{\#} \circ i_{\#}^{\uparrow}$ is injective it follows that $\bar{j}_{\#}$ is injective. Since $\bar{j}_{\#} \circ (R|)_{\#} = R_{\#} \circ i_{\#}$ is surjective it follows that $\bar{j}_{\#}$ is surjective. Hence $\bar{j}_{\#}$ is an isomorphism for all $q \geq 1$. \square

LEMMA 4.5. Let M be as in 4.2, and assume that either:

- (A) Any simultaneous codimension 1 and 2 fixed-point situation is simple, or
- (B) G is abelian.

Let U be a smooth slice at a point $a \in M^G$ and assume that for any subgroup H of G and any subgroups $K_i \supseteq H$ such that $\dim M^{K_i} = M^H - 2$, $1 \leq i \leq r$, the map

$$i_{\#} : \pi_q \left(U^H - \bigcup_{i=1}^r U^{K_i} \right) \rightarrow \pi_q \left(M^H - \bigcup_{i=1}^r M^{K_i} \right),$$

induced by the inclusion i , is an isomorphism for every $q \geq 1$. Then the map $j_{\#} : \pi_q(U_H) \rightarrow \pi_q(M_H)$, induced by the inclusion j , is an isomorphism for every subgroup H and every $q \geq 0$.

PROOF. For any subgroup H of G we define ‘‘Assertion H ’’ as follows.

Assertion H : For any subgroups H_1, \dots, H_s of G such that $H_i \supseteq H$, $1 \leq i \leq s$, the map

$$j_{\#} : \pi_q \left(U^H - \bigcup_{i=1}^s U^{H_i} \right) \rightarrow \pi_q \left(M^H - \bigcup_{i=1}^s M^{H_i} \right),$$

induced by the inclusion j , is an isomorphism for every $q \geq 0$.

Now let H be an arbitrary subgroup of G and assume inductively that ‘‘Assertion K ’’ is valid for all subgroups K of G that are strictly greater than H . Observe that ‘‘Assertion G ’’ holds. We shall prove that ‘‘Assertion H ’’ is valid. Let H_1, \dots, H_s be any subgroups of G such that $H_i \supseteq H$ and $\dim M^{H_i} \leq \dim M^H - 1$, $1 \leq i \leq s$. (If $\dim M^{H_i} = \dim M^H$, for some i , then $M^{H_i} = M^H$ and $U^{H_i} = U^H$ and there is nothing to prove in this case.)

Let us denote $\dim M^H = n$, and let H_1, \dots, H_s be ordered in such a way that

- (1) $\dim M^{H_i} = n - 1$, for $1 \leq i \leq p$,
- (2) $\dim M^{H_i} = n - 2$, for $p + 1 \leq i \leq r$,
- (3) $\dim M^{H_i} \leq n - 3$, for $r + 1 \leq i \leq s$.

Here $0 \leq p \leq r \leq s$. Let us denote

$$U_2 = U^H - \bigcup_{i=1}^r U^{H_i}, \quad M_2 = M^H - \bigcup_{i=1}^r M^{H_i},$$

and

$$U_3 = U^H - \bigcup_{i=1}^s U^{H_i}, \quad M_3 = M^H - \bigcup_{i=1}^s M^{H_i}.$$

To prove that ‘‘Assertion H ’’ is valid is thus to prove that

$$(*) \quad j_{\#}: \pi_q(U_3) \rightarrow \pi_q(M_3)$$

is an isomorphism for every $q \geq 0$.

First of all we claim that

$$(4) \quad i_{\#}: \pi_q(U_2) \rightarrow \pi_q(M_2)$$

is an isomorphism for all $q \geq 0$. If G is nonabelian we have by our assumption (A) that at least one of the following three cases holds.

- (i) $p = 0$,
- (ii) $p = r$,
- (iii) $p = 1$ and $r = p + 1 = 2$.

In case (i) we have by the assumption in Lemma 4.5, and the fact that in this case U_2 and M_2 are connected, that (4) is an isomorphism for every $q \geq 0$. In case (ii) it follows by Lemma 4.2 and in case (iii) by Lemma 4.3 that (4) is an isomorphism for all $q \geq 0$. (In case $p = r = 0$ we have $U_2 = U^H$ and $M_2 = M^H$, and hence (4) is an isomorphism for all q .) If G is abelian it follows by Lemma 4.4 that (4) is an isomorphism for all $q \geq 0$.

Since (3) holds it follows, by general position, that the maps

$$(5) \quad \begin{cases} i'_t: \pi_t(U_3) \xrightarrow{\cong} \pi_t(U_2), \\ i''_t: \pi_t(M_3) \xrightarrow{\cong} \pi_t(M_2), \end{cases} \quad t = 0, 1,$$

are isomorphisms. Hence using (4) we see that

$$(6) \quad j_t: \pi_t(U_3) \rightarrow \pi_t(M_3), \quad t = 0, 1,$$

are isomorphisms.

Let $p: \tilde{M}_2 \rightarrow M_2$ be a universal covering of M_2 , i.e., for any component N of M_2 $p|: p^{-1}(N) \rightarrow N$ is a universal covering of the connected manifold N . Since $i_{\#}: \pi_t(U_2) \rightarrow \pi_t(M_2)$ is an isomorphism for $t = 0$ and 1 it follows that $p|: p^{-1}(U_2) \rightarrow U_2$ is a universal covering of U_2 . Let us denote $\tilde{U}_2 = p^{-1}(U_2)$. It follows from (4) that the inclusion $\tilde{i}: \tilde{U}_2 \rightarrow \tilde{M}_2$ induces isomorphisms $\tilde{i}_{\#}: \pi_q(\tilde{U}_2) \rightarrow \pi_q(\tilde{M}_2)$ for all $q \geq 0$, and hence

$$(7) \quad \tilde{i}_{\#}: H_q(\tilde{U}_2) \rightarrow H_q(\tilde{M}_2)$$

is an isomorphism for every $q \geq 0$.

We have

$$(8) \quad \begin{aligned} U_3 &= U_2 - \bigcup_{k=r+1}^s (U_2 \cap U^{H_k}), \\ M_3 &= M_2 - \bigcup_{k=r+1}^s (M_2 \cap M^{H_k}), \end{aligned}$$

where we in particular have $H_k \supseteq H, k = r + 1, \dots, s$.

Let us now observe that for any subgroup K of G such that $K \supseteq H$ all the following hold. First of all we have that

$$\begin{aligned} U_2 \cap U^K &= U^K - \bigcup_{i=1}^r (U^{H_i} \cap U^K) = U^K - \bigcup_{i=1}^r U^{\langle H_i, K \rangle}, \\ M_2 \cap M^K &= M^K - \bigcup_{i=1}^r (M^{H_i} \cap M^K) = M^K - \bigcup_{i=1}^r M^{\langle H_i, K \rangle}. \end{aligned}$$

(Here $\langle H_i, K \rangle$ denotes the subgroup of G generated by H_i and K .) Since, by our inductive assumption, "Assertion K " holds it follows that

$$(9) \quad j_*: \pi_q(U_2 \cap U^K) \xrightarrow{\cong} \pi_q(M_2 \cap M^K)$$

is an isomorphism for every $q \geq 0$. We consider the coverings $p|: p^{-1}(U_2 \cap U^K) \rightarrow U_2 \cap U^K$ and $p|: p^{-1}(M_2 \cap M^K) \rightarrow M_2 \cap M^K$ and claim that the inclusion \hat{i} induces isomorphisms

$$(10) \quad \hat{i}_*: \pi_q(p^{-1}(U_2 \cap U^K)) \xrightarrow{\cong} \pi_q(p^{-1}(M_2 \cap M^K))$$

for every $q \geq 0$. For any component N' of $M_2 \cap M^K$ the homotopy sequences of the coverings $p|: p^{-1}(U \cap N') \rightarrow U \cap N'$ and $p|: p^{-1}(N') \rightarrow N'$ and the 5-lemma (plus a separate argument in degree 0) together with (9) imply that $\hat{i}_*: \pi_q(p^{-1}(U \cap N')) \rightarrow \pi_q(p^{-1}(N'))$ is an isomorphism for every $q \geq 0$. Hence (10) is an isomorphism for every $q \geq 0$ and thus

$$(11) \quad \hat{i}_*: H_q(p^{-1}(U_2 \cap U^K)) \xrightarrow{\cong} H_q(p^{-1}(M_2 \cap M^K))$$

is an isomorphism for every $q \geq 0$.

Furthermore observe that the submanifold $p^{-1}(M_2 \cap M^K)$ of \tilde{M}_2 is closed in \tilde{M}_2 , and

$$(12) \quad \tilde{U}_2 \cap p^{-1}(M_2 \cap M^K) = p^{-1}(U_2 \cap U^K).$$

The manifold \tilde{M}_2 is orientable since every component of it is simply connected. Moreover $M_2 \cap M^K$ is orientable since it is an open subset of the contractible manifold M^K , and hence $p^{-1}(M_2 \cap M^K)$ is orientable since it is a covering of $M_2 \cap M^K$ (for any component N of $M_2 \cap M^K$, each component of $p^{-1}(N)$ is a connected orientable manifold). Hence

$$\mathfrak{U} = \{ p^{-1}(M_2 \cap M^K) \mid K \supseteq H \}$$

is a family of orientable closed submanifolds of \tilde{M}_2 which is closed under intersection. Since (7), (11) and (12) hold we have by Lemma 4.1 and (8) that

$$(13) \quad j_*: H_q(p^{-1}(U_3)) \rightarrow H_q(p^{-1}(M_3))$$

is an isomorphism for every $q \geq 0$. It follows from (5) that $p|: p^{-1}(U_3) \rightarrow U_3$ and $p|: p^{-1}(M_3) \rightarrow M_3$ are universal coverings, i.e., for any components V_3 of U_3 and N_3 of M_3 we have that $p^{-1}(V_3)$ and $p^{-1}(N_3)$ are simply connected. Hence (13) implies that $j_*: \pi_q(p^{-1}(U_3)) \rightarrow \pi_q(p^{-1}(M_3))$ is an isomorphism for every $q \geq 0$, and therefore

$$j_*: \pi_q(U_3) \xrightarrow{\cong} \pi_q(M_3)$$

is an isomorphism for every $q \geq 0$. Thus we have proved our claim in (*), and this proves that "Assertion H" is valid. This completes the induction and the proof of Lemma 4.5. \square

5. Proof of the main results. We are now ready to give the concluding part of the proofs of our main results, and in 5.1 below we prove both Theorems A and B.

THEOREM 5.1. *Let G be a finite group acting smoothly on a homotopy sphere Σ^m such that $\Sigma^G \neq \emptyset$ and $\dim \Sigma^H \geq 5$ for each subgroup $H \neq G$ occurring in Σ . Assume that either:*

- (A) *Any simultaneous codimension 1 and 2 fixed-point situation is simple, or*
- (B) *G is abelian.*

Then the action is topologically equivalent to a linear action if and only if each fixed-point set Σ^H is homeomorphic to a sphere and each codimension 2 fixed-point situation $(\Sigma^H, \Sigma^{(>H,2)})$ is topologically standard.

PROOF. The "only if" part is clear as we already noted in the introduction. We shall prove that the conditions are sufficient. First we give the proof in the case when $\dim \Sigma^G \geq 5$. Let $b \in \Sigma^G$ and $M = \Sigma - \{b\}$. Then M is a smooth G -manifold, whose underlying smooth manifold is \mathbf{R}^m , and each fixed-point set Σ^H is homeomorphic to euclidean space $\mathbf{R}^{n(H)}$, where $n(H) = \dim \Sigma^H$. (In fact, in this case, each Σ^H is diffeomorphic to $\mathbf{R}^{n(H)}$, due to the uniqueness up to diffeomorphism of differentiable structures on \mathbf{R}^n , $n \geq 5$.)

Let $a \in M^G$ and let U be a smooth slice at $a \in M$. We wish to show that the condition in the assumption of Lemma 4.5 is satisfied. Let H be a subgroup of G and H_1, \dots, H_r subgroups of G containing H such that $\dim M^{H_k} = \dim M^H - 2$, $1 \leq k \leq r$. Our claim is that

$$(1) \quad i_*: \pi_q\left(U^H - \bigcup_{k=1}^r U^{H_k}\right) \rightarrow \pi_q\left(M^H - \bigcup_{k=1}^r M^{H_k}\right),$$

where i denotes the inclusion, is an isomorphism for every $q \geq 1$. This is seen as follows. Since $(\Sigma^H, \Sigma^{(>H,2)})$ is topologically standard it follows that there exists a homeomorphism

$$\beta: (M^H, M^{(>H,2)}) \xrightarrow{\cong} \left(\mathbf{R}^n, \bigcup_{j=1}^t L_j^{n-2}\right)$$

with $\beta(a) = 0$, such that if $K \supset H$ and $\dim M^K = \dim M^H - 2$ we have $\beta(M^K) = L_{j_0}^{n-2}$, for some j_0 . Here $n = \dim M^H$ and each L_j^{n-2} , $1 \leq j \leq t$, is an $(n - 2)$ -dimensional linear subspace of \mathbf{R}^n . (See §1.) Hence β gives a homeomorphism

$$\beta: \left(M^H, \bigcup_{k=1}^r M^{H_k} \right) \xrightarrow{\cong} \left(\mathbf{R}^n, \bigcup_{k=1}^r L_k^{n-2} \right)$$

where $\beta(M^{H_k}) = L_k^{n-2}$, $1 \leq k \leq r$, and $\beta(a) = 0$. It now follows that $a \in M^H$ has arbitrarily small neighbourhoods W in M^H such that the inclusion

$$(2) \quad i_0: W - \bigcup_{k=1}^r (W \cap M^{H_k}) \hookrightarrow M^H - \bigcup_{k=1}^r M^{H_k}$$

is a homotopy equivalence. Choose such a W with $W \subset U^H$. Then we get the inclusion

$$(3) \quad i': W - \bigcup_{k=1}^r (W \cap M^{H_k}) \hookrightarrow U^H - \bigcup_{k=1}^r U^{H_k}.$$

Let $\alpha: \mathbf{R}^n(\rho) \rightarrow U$ be a G -homeomorphism. Choose $\varepsilon > 0$ such that $\alpha(\mathring{B}_\varepsilon^n(\rho)^H) \subset W$, where $\mathring{B}_\varepsilon^n(\rho) = \{x \in \mathbf{R}^n(\rho) \mid \|x\| < \varepsilon\}$, and denote $U_1 = \alpha(\mathring{B}_\varepsilon^n(\rho))$. Then we have the inclusion

$$(4) \quad i'': U_1^H - \bigcup_{k=1}^r U^{H_k} \hookrightarrow W - \bigcup_{k=1}^r (W \cap M^{H_k}).$$

The inclusion

$$(5) \quad i_1 = i' \circ i'': U_1^H - \bigcup_{k=1}^r U^{H_k} \hookrightarrow U^H - \bigcup_{k=1}^r U^{H_k}$$

is a homotopy equivalence. Since $(i_1)_\# = i'_\# \circ i''_\#$ it follows that the inclusion i' in (3) induces an epimorphism in homotopy in every degree. Since

$$\begin{aligned} (i_0)_\# &= i_\# \circ i'_\#: \pi_q \left(W - \bigcup_{k=1}^r (W \cap M^{H_k}) \right) \rightarrow \pi_q \left(U - \bigcup_{k=1}^r U^{H_k} \right) \\ &\rightarrow \pi_q \left(M^H - \bigcup_{k=1}^r M^{H_k} \right), \end{aligned}$$

and i_0 is a homotopy equivalence and $i'_\#$ is an epimorphism in every degree it follows that $i_\#$ is an isomorphism in every degree, i.e., our claim that (1) is an isomorphism for every $q \geq 1$ is valid. Hence Lemma 4.5 implies that

$$j_\#: \pi_q(U_H) \rightarrow \pi(M_H)$$

is an isomorphism for every $q \geq 0$ and every subgroup H of G , i.e., the pair (M_H, U_H) is q -connected for all $q \geq 0$ and every H .

We want to apply part II of Proposition 3.2. Since M is a smooth G -manifold we have by Theorems 3.6 and 3.8 in [9] that there exists an equivariant triangulation (in fact, smooth) of M which is an equivariant combinatorial manifold. In this way M is given the structure of an equivariant p.l. manifold. Since M^G is contractible and we showed above that U satisfies the condition in the assumption of part II of

Proposition 3.2, it only remains to prove that M is equivariantly unraveled at infinity. So let A be any compact subset of M . Then GA is a compact subset of $M = \Sigma - \{b\}$. Hence there exists a smooth slice V at b in Σ such that $V \cap (GA \cup \{a\}) = \emptyset$. In particular V is G -homeomorphic to some representation space $\mathbf{R}^m(\tau)$ for G . We claim that $(M_H, (V - \{b\})_H)$ is 2-connected for every H . Let $N = \Sigma - \{a\}$. Then $V \subset N$ and the pair (N, V) is exactly of the same form as the pair (M, U) considered above, and hence the proof given above shows that (N_H, V_H) is q -connected for all $q \geq 0$ and every subgroup H . For $H \neq G$ we have

$$(M_H, (V - \{b\})_H) = (N_H, V_H),$$

so in this case $(M_H, (V - \{b\})_H)$ is q -connected for all q and hence in particular 2-connected. The pair $(M_G, (V - \{b\})_G) = (M^G, (V - \{b\})^G)$ is homeomorphic to a pair of the form $(S^k - \{\infty\}, V' - \{\infty\})$, where V' is an open neighbourhood of $\infty \in S^k$ which is homeomorphic to \mathbf{R}^k and $k \geq 5$. Hence $(M_G, (V - \{b\})_G)$ is at least 4-connected and hence in particular 2-connected. Thus M is equivariantly unraveled at infinity. It now follows by Proposition 3.2, part II, that M is G -homeomorphic to a representation space $\mathbf{R}^m(\rho)$. Since $M = \Sigma - \{b\}$ it follows that the G -space Σ is G -homeomorphic to $S^m(\rho \oplus \text{id})$. This completes the proof of Theorem 5.1 in the case when $\dim \Sigma^G \geq 5$.

In the case when $0 \leq \dim \Sigma^G \leq 4$ the proof is very similar to the above proof. As before we consider the smooth G -manifold $M = \Sigma - \{b\}$. By the equivariant tubular neighbourhood theorem (see Bredon [1, Theorem VI.2.2]) there exists a G -diffeomorphism $\varphi: E(\eta) \rightarrow U'$, where $E(\eta)$ denotes the total space of the normal bundle η of M^G in M and U' is an open G -invariant neighbourhood of M^G in M . Let $k = \dim M^G = \dim \Sigma^G$. Since M^G is homeomorphic to \mathbf{R}^k , it is in particular contractible and hence, by Corollary 2.5 in Wasserman [18], there exists an isomorphism of smooth G -vector bundles $\gamma: M^G \times \mathbf{R}^{m-k}(\rho') \rightarrow E(\eta)$. Let $\omega: \mathbf{R}^k \rightarrow M^G$ be a homeomorphism. Then

$$\alpha = \varphi \circ \gamma \circ (\omega \times \text{id}): \mathbf{R}^k \times \mathbf{R}^{m-k}(\rho') \rightarrow U'$$

is a G -homeomorphism and $D = \alpha(\mathbf{R}^k \times B^{m-k}(\rho'))$ is a closed G -neighbourhood of M^G in M . Let

$$(6) \quad U = \text{Int } D = \alpha(\mathbf{R}^k \times \mathring{B}^{m-k}(\rho')).$$

Now with this choice of the open G -subset U the proof proceeds as in the earlier case. Our first claim is that with U defined by (6) the maps in (1) are isomorphisms for all $q \geq 1$. The proof of this is almost identical with the one in the earlier case. The set $W \subset U^H$ is chosen as before and $\epsilon > 0$ is chosen such that

$$U_1 = \alpha\left(\left(\mathring{B}_\epsilon^k \times \mathring{B}_\epsilon^{m-k}(\rho')\right)^H\right) \subset W.$$

Then (2)–(5) is valid and the conclusion is that Lemma 4.5 implies that (M_H, U_H) is q -connected for all $q \geq 0$ and every subgroup H . Moreover the inclusion $U_H \hookrightarrow D_H$ is a homotopy equivalence for every H . Hence (M_H, D_H) is q -connected for all $q \geq 0$ and every subgroup H .

It follows that Proposition 3.2, part I, applies if we can show that M is \mathcal{F}' -equivariantly unraveled at infinity, where $\mathcal{F}' = \mathcal{F} - \{G\}$ and \mathcal{F} denotes the family of all subgroups occurring in M . But this we already proved when we showed that $(M_H, (V - \{b\})_H)$ is 2-connected (in fact q -connected for all $q \geq 0$) for every subgroup $H \neq G$. Thus M is G -homeomorphic to $\mathbf{R}^m(\rho)$, where $\rho = \underbrace{\text{id} \oplus \cdots \oplus \text{id}}_k \oplus \rho'$, and hence Σ is G -homeomorphic to $S^m(\rho \oplus \text{id})$. This completes the proof of Theorem 5.1. \square

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