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LOCAL ANALYTICITY IN WEIGHTED L^1-SPACES AND APPLICATIONS TO STABILITY PROBLEMS FOR VOLTERRA EQUATIONS

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ABSTRACT. We study the qualitative properties of the solutions of linear convolution equations such as $x' + x * \mu = f$, $x + a * x = f$ and $x * \mu = f$. We are especially concerned with finding conditions which ensure that these equations have resolvents which belong to, or are determined up to a term belonging to, certain weighted L^1-spaces. Our results are obtained as consequences of more general Banach algebra results on functions that are locally analytic with respect to the elements of a weighted L^1-space. In particular, we derive a proposition of Wiener-Lévy type for weighted L^1-spaces which underlies all subsequent results. Our method applies equally well to equations more general than those mentioned above. We unify and sharpen the results of several recent papers on the asymptotic behavior of Volterra convolution equations of the types mentioned above, and indicate how many of them can be extended to the Fredholm case. In addition, we give necessary and sufficient conditions on the perturbation term f for the existence of bounded or integrable solutions x in some critical cases when the corresponding limit equations have nontrivial solutions.

1. Introduction. This paper is motivated by recent studies on the asymptotic behavior for large t of the solutions x of the linear Volterra integral and integrodifferential equations

$$x(t) + \int_0^t x(t-s)a(s)\,ds = f(t), \qquad t \in R^+ \equiv [0,\infty), \tag{1.1}$$

$$x'(t) + \int_0^t x(t-s)\,d\mu(s) = f(t), \qquad t \in R^+,\ x(0) = x_0. \tag{1.2}$$

As is well known, the solutions of these equations are given by

$$x(t) = f(t) - \int_0^t f(t-s)r_1(s)\,ds, \qquad t \in R^+, \tag{1.3}$$

$$x(t) = x_0 r_2(t) + \int_0^t f(t-s)r_2(s)\,ds, \qquad t \in R^+, \tag{1.4}$$

respectively, where the resolvent r_1 and differential resolvent r_2 are the solutions of

$$r_1(t) + \int_0^t r_1(t-s)a(s)\,ds = a(t), \qquad t \in R^+, \tag{1.5}$$

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$$(1.6) \qquad r_2'(t) + \int_0^t r_2(t-s)\, d\mu(s) = 0, \qquad t \in R^+,\, r_2(0) = 1.$$

If $r_1(t)$ or $r_2(t)$ decays sufficiently rapidly as $t \to \infty$, then many asymptotic properties of f are inherited by the solution x. Therefore, we are in particular concerned with finding conditions on a or μ which ensure that r_1 or r_2 belongs to, or is determined up to a term belonging to, certain weighted L^1-spaces.

The problem of the rate of decay of the resolvents r_1 and r_2 has been examined previously in a variety of ways. The general theory we present is based on a new variation of a known Banach algebra result [**3**, p. 82]; it shows that many of the apparently diverse transform techniques used previously do actually have a common element. Once recognized, this fact can be used not only to unify earlier results, but also to strengthen and extend these results and to obtain new results.

Although our motivation comes from the Volterra equations (1.1) and (1.2), the technique that we use does not rely on the Volterra nature of (1.1) and (1.2). As a consequence of this fact, almost all of the theory applies equally well to Fredholm equations of convolution type. The only difference is that one replaces a weighted L^1-space on R^+ with a weighted L^1-space on the whole line $R \equiv (-\infty, \infty)$. For the most part, we phrase our results so that they include both cases.

We begin §2 with a discussion of those aspects of the Gelfand-Shilov theory (as developed in [**3**]) that we will use. The weighted spaces with which we work are described, and the concept of local analyticity with respect to the elements of our Banach algebras is defined. Next, as an easy consequence of a general Banach algebra result [**3**, p. 82], we state and prove our basic Proposition 2.3 which underlies all the subsequent results. As an immediate application of Proposition 2.3, we deduce a variant of one form of the classical Wiener-Lévy theorem [**15**, p. 63]; this variant was originally established and applied to Volterra equations by D. F. Shea and S. Wainger [**16**, Theorem 2, Condition (10)]. In addition, [**16**, Theorem 3] is also deduced from Proposition 2.3. (The main result of [**16**, Theorem 2 with Condition (6)] is not a consequence of the methods of this paper; see the discussion preceding Proposition 7.6 for more details.)

In §3 we define the concept of a zero of finite integral order of a locally analytic function, the reciprocal notion of a pole, and the even more fundamental concept of a locally analytic function being smooth of finite integral order at a given point. Smoothness of a locally analytic function at points on the boundary of its domain is the crucial property in many of our results. Here we also state and prove our L^1-quotient and L^1-remainder results dealing, respectively, with dividing out or subtracting off singular rational terms at a finite number of poles of a locally analytic function.

The key property of smoothness of a locally analytic function is examined in §4. In Lemma 4.3 we give a necessary and sufficient condition for the Laplace transform $\hat{\mu}$ of a measure μ, finite with respect to a weight, to be smooth of prescribed order at a point in its domain. In Lemma 4.2 we show that smoothness of a locally analytic function at a point can arise either from the smoothness of the transforms $\hat{\mu}_j$

occurring in its representation at the point or from the order of dependence of the locally analytic function on these transforms at the point.

§5 contains applications of the theory established in the preceding sections. We first discuss and extend the quotient and remainder results of [**8**, **10**, **13**]. In particular, we observe that the global conditions imposed on the kernels in these papers (such as moment or moment and monotonicity conditions) are not necessary, but instead are sufficient to yield, by Lemma 4.3, the required smoothness of their Laplace transforms. Next, we use the L^1-quotient theorem to give the new Propositions 5.1 and 5.2 which settle the question of precisely which type of bounded (or integrable) perturbations in (1.1), (1.2) give rise to bounded (or integrable) solutions x. Finally, in Proposition 5.3 and the subsequent discussion we give the connection between Propositions 5.1, 5.2 and the L^1-remainder theorems previously discussed.

In §6 we explore the possibility of showing that a function known to belong to one weighted space actually belongs to another space with a larger weight. A key concept here is that of the order of dependence of an analytic function on each of its arguments at a fixed point. The basic result, Theorem 6.1, is used to obtain some recent results of Jordan and Wheeler on the rates of decay of the resolvents r_1 and r_2 [**12**, Theorems 1 and 2].

In §7 the concept of extended local analyticity is introduced and the basic result of this section, Theorem 7.2, is given. We then use this theorem to obtain necessary and sufficient conditions for the resolvent r_1 or r_2 to belong to a weighted L^1-space when the kernel a in (1.1) does not belong to that space, or the measure μ in (1.2) does not belong to the corresponding weighted space of Borel measures. A recent result of G. Gripenberg [**5**] on the integrability of the resolvent of a system of Volterra integral equations is discussed in relation to Theorem 7.2. We then return to the paper of Shea and Wainger and discuss their principal result, namely their extension of the Wiener-Lévy Theorem [**16**, Theorem 2, Condition (6)]; in particular it is shown that the general case of their theorem follows from the particular case yielding $r_1 \in L^1(R^+)$ combined with our Theorem 7.2. We also complete the discussion of [**8**] begun in §5. After briefly discussing some results of several authors on the asymptotic rates of decay of the resolvents r_1 or r_2, we conclude §7 with a result about locally analytic functions with fractional order zeros or singularities on the boundaries of their domains, and an application to a perturbation problem for linear Volterra equations.

Finally, in the last section we define the notion of pseudo local analyticity with respect to a weighted space of measures on R or R^+, and obtain an analogue of Proposition 2.3 in this setting. We conclude with a brief discussion of how this result can be used to analyze the equation

$$\int_0^t x(t-s)\,d\mu(s) = f(t), \qquad t \in R^+. \tag{1.7}$$

2. Weighted L^1-spaces and locally analytic functions. We begin with a presentation of the weighted L^1-spaces and measure spaces that we use. For more details, see [**2**, pp. 56–65; **3**, pp. 113–116 and **9**, pp. 141–150].

We call the function ρ a weight function on R if ρ is Borel measurable, strictly positive, ρ and ρ^{-1} are locally bounded, and ρ is submultiplicative, i.e.

$$\rho(s+t) \leqslant \rho(s)\rho(t) \tag{2.1}$$

for $s, t \in R$. Observe that without loss of generality one may always take $\rho(0) = 1$ (redefine $\rho(0)$ if necessary). Some interesting weights that satisfy (2.1) are

$$\begin{aligned} \rho_1(t) &= (1+|t|)^{\delta}, \qquad t \in R, \delta \geqslant 0, \\ \rho_2(t) &= \big(1+\log(1+|t|)\big)^{\gamma}\rho_1(t), \qquad t \in R, \gamma \geqslant 0, \\ \rho_3(t) &= \exp\big(|t|^{\alpha}\big)\rho_2(t), \qquad t \in R, 0 \leqslant \alpha < 1, \\ \rho_4(t) &= \exp(\beta t)\rho_3(t), \qquad t \in R, \beta \in R. \end{aligned}$$

The space $L^1(R; \rho)$ consists of all complex measurable functions x on R for which

$$\|x\| \equiv \int |x(t)|\rho(t)\,dt < \infty, \tag{2.2}$$

where the integration is taken over R. For $x, y \in L^1(R; \rho)$, define the convolution of x and y by

$$x * y(t) = \int x(t-s)y(s)\,ds, \tag{2.3}$$

where as before we integrate over R. Then $L^1(R; \rho)$ becomes a normed ring (i.e., a Banach algebra) with convolution multiplication. We let $V(R; \rho)$ denote the ring one obtains from $L^1(R; \rho)$ by adjoining a unit.

Define

$$\begin{aligned} \rho_* &= -\inf_{t>0} \frac{\log \rho(t)}{t} = -\lim_{t\to\infty} \frac{\log \rho(t)}{t}, \\ \rho^* &= -\sup_{t<0} \frac{\log \rho(t)}{t} = -\lim_{t\to-\infty} \frac{\log \rho(t)}{t}. \end{aligned} \tag{2.4}$$

Then $-\infty < \rho_* \leqslant \rho^* < \infty$, and the maximal ideal space of $V(R; \rho)$ can be identified with $\overline{\Pi} = \Pi \cup \{\infty\}$, where

$$\Pi \equiv \big\{z \in \mathbf{C} \,|\, \rho_* \leqslant \operatorname{Re} z \leqslant \rho^*\big\}, \tag{2.5}$$

with the usual topology of the compactified plane [**3**, pp. 100, 113–115]. The Gelfand transform of $a \in L^1(R; \rho)$ equals its bilateral Laplace transform $\hat{a}(z) \equiv \int_{-\infty}^{\infty} e^{-zt}a(t)\,dt$ ($z \in \Pi$). Observe that the Laplace transform converges absolutely for $z \in \Pi$, because it follows from (2.4) that

$$L^1(R; \rho) \subset L^1(R, e^{-rt}) \quad \textit{for every } r \in R \textit{ satisfying } \rho_* \leqslant r \leqslant \rho^*. \tag{2.6}$$

If ρ is only defined on R^+ and satisfies (2.1) for $s, t \in R^+$, then we call ρ a weight on R^+. Trivially, the preceding examples of weights on R become weights on R^+, if they are restricted to R^+. The space $L(R^+; \rho)$ consists of those complex measurable functions x on R^+ which satisfy (2.2), where this time one integrates over R^+. Convolution is defined as in (2.3), except that now the integration is over $[0, t]$. Then

$L^1(R^+; \rho)$ is a normed ring [3, p. 115], and the ring obtained by adjoining a unit to $L^1(R^+; \rho)$ will be called $V(R^+; \rho)$.

Define ρ_* as in (2.4). Then $-\infty < \rho_* \leqslant \infty$. To avoid trivialities we assume throughout that $\rho_* < \infty$. The maximal ideal space of $V(R^+; \rho)$ can be identified with $\overline{\Pi} = \Pi \cup \{\infty\}$, where

$$\Pi \equiv \{z \in \mathbf{C} | \operatorname{Re} z \geqslant \rho_*\} \tag{2.7}$$

[3, p. 115; 9, p. 148]. Again, the Gelfand transform of $a \in L^1(R^+; \rho)$ equals its Laplace transform $\hat{a}(z) \equiv \int_0^\infty e^{-zt} a(t)\,dt$ $(z \in \Pi)$, and (2.6) holds for $r \geqslant \rho_*$ and with R replaced by R^+.

In the sequel, if we do not specifically state that we work on either R or R^+, then both possibilities are included. We let $L^1(\rho)$ stand for either $L^1(R; \rho)$ or $L^1(R^+; \rho)$, and $V(\rho)$ for either $V(R; \rho)$ or $V(R^+; \rho)$. Accordingly, Π is defined by either (2.5) or (2.7). In the special case $\rho \equiv 1$ we abbreviate $L^1(R; \rho)$, $L^1(R^+; \rho)$, $V(R; \rho)$ and $V(R^+; \rho)$ by $L^1(R)$, $L^1(R^+)$, $V(R)$ and $V(R^+)$.

I. Gelfand, D. Raikov and G. Shilov [3, p. 82] call a function φ defined on $\overline{\Pi}$ locally analytic (with respect to the elements of $V(\rho)$), if, at each point z_0 of $\overline{\Pi}$, φ has an expansion of the form

$$\varphi(z) = \sum_{l_1, \ldots, l_k} \alpha_{l_1, \ldots, l_k} [\hat{a}_1(z) - \hat{a}_1(z_0)]^{l_1} \cdots [\hat{a}_k(z) - \hat{a}_k(z_0)]^{l_k}, \tag{2.8}$$

valid in a neighborhood of z_0. Here the $\alpha_{l_1, \ldots, l_k}$ are constants, and the a_j $(1 \leqslant j \leqslant k)$ are elements of $V(\rho)$. The word "neighborhood" means an open subset of $\overline{\Pi}$ rather than an open subset of the compactified complex plane. It is supposed that the multiple power series

$$\eta(\xi_1, \ldots, \xi_k) = \sum_{l_1, \ldots, l_k} \alpha_{l_1, \ldots, l_k} \xi_1^{l_1} \cdots \xi_k^{l_k}$$

converges in a neighborhood of zero in $\mathbf{C}^k$. The elements $a_1, \ldots, a_k$ as well as their number k may depend on z_0. Observe that without loss of generality one may assume $a_j \in L^1(\rho)$ instead of $a_j \in V(\rho)$ (replace $\hat{a}_j(z)$ by $\hat{a}_j(z) - \hat{a}_j(\infty)$). We shall refer to the preceding concept as "local analyticity [3]" to distinguish it from a related concept defined below and referred to as "local analyticity".

Theorem 1 in [3, p. 82] applied to the ring $V(\rho)$ can be stated as follows (see [3, p. 291] for the history of this theorem):

THEOREM A. *Let φ be locally analytic* [3] *on $\overline{\Pi}$. Then φ is the Laplace transform of an element in $V(\rho)$.*

Of course, the preceding definition of local analyticity may be rephrased without an explicit reference to power series. Namely, the function η is an analytic function of k complex variables in a neighborhood of zero, and so

$$\psi(\xi_1, \ldots, \xi_k) \equiv \eta(\xi_1 - \hat{a}_1(z_0), \ldots, \xi_k - \hat{a}_k(z_0))$$

is analytic in a neighborhood of $(\hat{a}_1(z_0), \ldots, \hat{a}_k(z_0))$. Thus, φ is locally analytic [3] at z_0 if it is of the form

$$\varphi(z) = \psi(\hat{a}_1(z), \ldots, \hat{a}_k(z)) \tag{2.9}$$

in a neighborhood of z_0, where ψ is analytic at $(\hat{a}_1(z_0),\ldots,\hat{a}_k(z_0))$, and $a_j \in L^1(\rho)$ $(1 \leqslant j \leqslant k)$. The number k and the functions ψ and a_j may vary from point to point.

For our purposes the representation (2.9) is not sufficiently general since we want to let ψ in (2.9) depend explicitly on z, and not only implicitly through $\hat{a}_1(z),\ldots,\hat{a}_k(z)$. In addition, in some cases we want to replace the functions a_j by measures.

Let $M(\rho)$ be the set of locally finite, Borel measures μ on R or R^+ satisfying

$$\|\mu\| \equiv \int \rho(t)\, d|\mu|(t) < \infty,$$

where $|\mu|$ is the total variation measure of μ. Since every $a \in L^1(\rho)$ can be identified with an absolutely continuous measure, we have $L^1(\rho) \subset M(\rho)$. Define the convolution of $a \in L^1(\rho)$ and $\mu \in M(\rho)$ by

$$\mu * a(t) = a * \mu(t) = \int a(t-s)\, d\mu(s),$$

where the integration is over either R or $[0, t]$. If $a \in L^1(\rho)$ and $\mu \in M(\rho)$, then $\mu * a \in L^1(\rho)$. Again,

(2.10) $M(\rho) \subset M(e^{-rt})$ *for every* $r \in R$ *satisfying* $\rho_* \leqslant r \leqslant \rho^*$ $\left(\textit{or } r \geqslant \rho_*\right)$.

Also, for $z \in \Pi$, we let $\hat{\mu}(z)$ denote the bilateral Laplace-Stieltjes transform of $\mu \in M(\rho)$. When $\rho \equiv 1$ we write $M(R)$ or $M(R^+)$ instead of $M(\rho)$.

DEFINITION 2.1. *We call a function φ locally analytic at a point $z_0 \in \Pi$, if φ is defined in a neighborhood of z_0, and there exist measures $\mu_1,\ldots,\mu_k$ in $M(\rho)$ and a function $\psi(z, \xi_1,\ldots,\xi_k)$ analytic at $(z_0, \hat{\mu}_1(z_0),\ldots,\hat{\mu}_k(z_0))$ such that*

$$\varphi(z) = \psi(z, \hat{\mu}_1(z),\ldots,\hat{\mu}_k(z))$$

in a neighborhood of z_0. We say that φ is locally analytic at infinity if φ is defined in a neighborhood of infinity, and there exist functions $a_1,\ldots,a_n$ in $L^1(\rho)$, measures $\mu_1,\ldots,\mu_k$ in $M(\rho)$, and a function $\psi(z, \eta_1,\ldots,\eta_n, \xi_1,\ldots,\xi_k)$ analytic at $(0,0,\ldots,0)$ such that

$$\varphi(z) = \psi\left(z^{-1}, \hat{a}_1(z),\ldots,\hat{a}_n(z), \hat{\mu}_1(z)/z,\ldots,\hat{\mu}_k(z)/z\right)$$

in a neighborhood of infinity. Finally, we say that φ is locally analytic if it is locally analytic at each point of Π.

Here, as in the definition of local analyticity [3], neighborhood means an open subset of $\overline{\Pi}$. Also, it is understood that the sets $a_1,\ldots,a_k$ or $\mu_1,\ldots,\mu_k$ may be empty. For example, it suffices if at some point ψ is an analytic function of z alone. Observe that, since $L^1(\rho) \subset M(\rho)$, a measure μ_j may be replaced by a function a_j in Definition 2.1, but not conversely. Actually, the representations in Definition 2.1 are only important on the boundary of Π, since in the interior of Π local analyticity is equivalent to analyticity (as a function of the variable z).

LEMMA 2.2. *Every locally analytic function is analytic in the interior of Π. Conversely, every function analytic on $\overline{\Pi}$ is locally analytic.*

Lemma 2.2 follows directly from Definition 2.1 combined with the fact that Laplace transforms are analytic in the interior of their domains of convergence.

PROPOSITION 2.3. *Let φ be locally analytic. Then φ is locally analytic* [**3**]. *In particular, φ is the Laplace transform of an element of $V(\rho)$. If, in addition, $\varphi(\infty) = 0$, then φ is the Laplace transform of a function $a \in L^1(\rho)$.*

PROOF. Define $e(t) = \exp((\rho_* - 1)t)$ $(t \geq 0)$, $e(t) = 0$ $(t < 0)$. Then $e \in L^1(\rho)$ and $\hat{e}(z) = (z - \rho_* + 1)^{-1}$ $(z \in \Pi)$. In particular, $\hat{e}(z) \neq 0$ $(z \in \Pi)$.

Let $z_0 \in \Pi$, and let $\varphi(z) = \psi(z, \hat{\mu}_1(z), \ldots, \hat{\mu}_k(z))$ in a neighborhood of z_0 as in Definition 2.1. Define

$$\tilde{\psi}(\omega, \xi_1, \ldots, \xi_k) = \psi\big(1/\omega + \rho_* - 1, \xi_1/\omega, \ldots, \xi_k/\omega\big).$$

Then

$$\tilde{\psi}(\hat{e}(z), \hat{e}(z)\hat{\mu}_1(z), \ldots, \hat{e}(z)\hat{\mu}_k(z)) = \psi(z, \hat{\mu}_1(z), \ldots, \hat{\mu}_k(z))$$

in a neighborhood of z_0, and $\tilde{\psi}$ is analytic at $(\hat{e}(z_0), \hat{e}(z_0)\hat{\mu}_1(z_0), \ldots, \hat{e}(z_0)\hat{\mu}_k(z_0))$. Define $b_j = e * \mu_j$ $(1 \leq j \leq k)$. Then $b_j \in L^1(\rho)$ and $\hat{b}_j = \hat{e}\hat{\mu}_j$, so that $\varphi(z) = \tilde{\psi}(\hat{e}(z), \hat{b}_1(z), \ldots, \hat{b}_k(z))$ in a neighborhood of z_0. This means that φ is locally analytic [**3**] at z_0.

The corresponding argument at infinity is very similar. Let $\varphi(z) = \psi(z^{-1}, \hat{a}_1(z), \ldots, \hat{a}_n(z), \hat{\mu}_1(z)/z, \ldots, \hat{\mu}_k(z)/z)$ in a neighborhood of infinity as in Definition 2.1. Define

$$\tilde{\psi}(\omega, \eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_k) = \psi\left(\frac{\omega}{1 + (\rho_* - 1)\omega}, \eta_1, \ldots, \eta_n, \frac{\xi_1}{1 + (\rho_* - 1)\omega}, \ldots, \frac{\xi_k}{1 + (\rho_* - 1)\omega}\right).$$

Then $\tilde{\psi}$ is analytic at $(0, 0, \ldots, 0)$, and

$$\begin{aligned}\varphi(z) &= \psi(z^{-1}, \hat{a}_1(z), \ldots, \hat{a}_n(z), \hat{\mu}_1(z)/z, \ldots, \hat{\mu}_k(z)/z) \\ &= \tilde{\psi}(\hat{e}(z), \hat{a}_1(z), \ldots, \hat{a}_n(z), \hat{e}(z)\hat{\mu}_1(z), \ldots, \hat{e}(z)\hat{\mu}_k(z))\end{aligned}$$

in a neighborhood of infinity. As before, defining $b_j = e * \mu_j$, we observe that φ is locally analytic [**3**] at infinity.

We have shown that φ is locally analytic [**3**], and so Theorem A applies to yield that φ is the Laplace transform of an element in $V(\rho)$. This element is a measure in $M(\rho)$ of the form $\mu = a + \alpha\delta$, where $a \in L^1(\rho)$, $\alpha \in \mathbf{C}$, and δ is the unit point mass at zero. Since $\hat{\mu}(z) = \hat{a}(z) + \alpha \to \alpha$ as $z \to \infty$, we have that $\alpha = 0$ when $\varphi(\infty) = 0$. Thus, in this case, φ is the transform of a function $a \in L^1(\rho)$. $\square$

Proposition 2.3 contains the case of [**16**, Theorem 2] where [**16**, Condition (10)] holds (but certainly not the case where [**16**, Condition (6)] holds). Take $\rho(t) \equiv 1$ $(t \geq 0)$, and denote $L^1(R^+; \rho)$ by $L^1(R^+)$. Then $\rho_* = 0$ and $\Pi = \{z \mid \operatorname{Re} z \geq 0\}$. D. F. Shea and S. Wainger let φ be of the form $\varphi(z) = \psi(z, b/z + \hat{\beta}(z))$, where b is a constant and $\beta \in L^1(R^+)$. They assume that ψ is analytic on $\{(z, b/z + \hat{\beta}(z)) \mid z \in \Pi\}$ (including the point $(0, \infty)$ if $b \neq 0$), as well as at $(\infty, 0)$, and that $\varphi(\infty) = 0$. Define $\tilde{\psi}(z, \omega) = \psi(z, b/z + \omega)$. Then $\tilde{\psi}$ is analytic on $\{(z, \hat{\beta}(z)) \mid z \in \Pi\}$ as well as at $(\infty, 0)$, hence, φ is locally analytic. Proposition 2.3 applies and we get the same conclusion as in **[16]**, namely, that φ is the Laplace transform of a function $r \in L^1(R^+)$.

Proposition 2.3 also essentially contains the scalar version of [**16**, Theorem 3]. There ρ is a weight function on R^+, and $\varphi(z) = (z + \hat{\mu}(z))^{-1}$ $(z \in \Pi)$, where $\mu \in M(R^+; \rho)$ and $z + \hat{\mu}(z) \neq 0$ $(z \in \Pi)$. Clearly, φ is locally analytic on Π. Also, φ is locally analytic at infinity since it can be written in the form

$$\varphi(z) = \frac{1/z}{1 + \hat{\mu}(z)/z}.$$

Moreover, $\varphi(\infty) = 0$. Thus, by Proposition 2.3, φ is the Laplace transform of a function $r \in L^1(R^+; \rho)$. This is the key conclusion in [**16**, Theorem 3] from which the other conclusions can be deduced. The general case of [**16**, Theorem 3] (i.e., the vector-valued case) can be reduced to the scalar case.

3. Removing poles from a locally analytic function. Let φ be locally analytic. If $\varphi(z) \neq 0$ at some point $z \in \overline{\Pi}$, then the reciprocal φ^{-1} is also locally analytic at the same point. However, we are also interested in the local analytical properties of φ^{-1} at points where φ vanishes. In this connection the notion of a locally analytic zero of a locally analytic function is crucial.

In the interior of Π φ is analytic, and there it is easy to define what one means by the order of a zero. In particular, φ has only zeros of finite integral order in the interior of Π. On the boundary of Π the situation is much more complicated. The zeros need not be of integral nor of finite order. Even if they are of integral order, they need not be "locally analytic". For this reason we need the following definitions.

DEFINITION 3.1. *Let m be a positive integer. We call $z_0 \in \Pi$ a (locally analytic) zero of φ of order at least m if $(z - z_0)^{-m}\varphi(z)$ is locally analytic at z_0. If, in addition, $\lim_{z \to z_0, z \in \Pi}(z - z_0)^{-m}\varphi(z) \neq 0$, then we call z_0 a (locally analytic) zero of φ of order m. Similarly, if $z^m\varphi(z)$ is locally analytic at infinity, then we call infinity a (locally analytic) zero of φ of order at least m, and if, in addition, $\lim_{z \to \infty, z \in \Pi} z^m\varphi(z) \neq 0$, then we call infinity a (locally analytic) zero of φ of order m.*

More precisely, the function ψ defined by

$$\psi(z) = \begin{cases} (z - z_0)^{-m}\varphi(z), & z \neq z_0, \\ \lim\limits_{z \to z_0, z \in \Pi} (z - z_0)^{-m}\varphi(z), & z = z_0, \end{cases}$$

should be locally analytic at z_0, etc.

Definition 3.1 is straightforward, but it does not answer the following question: If φ has the representation $\varphi(z) = \psi(z, \hat{\mu}_1(z), \ldots, \hat{\mu}_k(z))$ in a neighborhood of $z_0 \in \Pi$ as in Definition 2.1, and if $\lim_{z \to z_0, z \in \Pi}(z - z_0)^{-m}\varphi(z) \neq 0$, then is it possible to show that this zero is a locally analytic zero of order m by simply examining the function ψ and the measures $\mu_1, \ldots, \mu_k$? The same question arises at infinity. We discuss this problem further in §4.

The inverse of a zero is a pole:

DEFINITION 3.2. *Let m be a positive integer. We call $z_0 \in \Pi$ a (locally analytic) pole of φ of order at most m if $(z - z_0)^m \varphi(z)$ is locally analytic at z_0. If, in addition, $\lim_{z \to z_0, z \in \Pi}(z - z_0)^m \varphi(z) \neq 0$, then we call z_0 a (locally analtyic) pole of order m. If $z^{-m}\varphi(z)$ is locally analytic at infinity, then we call infinity a (locally analtyic) pole of φ of order at most m, and if, in addition, $\lim_{z \to \infty, z \in \Pi} z^{-m}\varphi(z) \neq 0$, then we call infinity a (locally analytic) pole of order m.*

In the sequel we simply write "zero" and "pole" for "locally analytic zero" and "locally analytic pole".

LEMMA 3.3. *A function φ has a zero of order m at $z_0 \in \overline{\Pi}$ if and only if φ^{-1} has a pole of order m at z_0.*

This follows trivially from the definitions and the fact that $\varphi(z_0) \neq 0$ implies that φ is locally analytic at z_0 if and only if φ^{-1} is locally analytic at z_0.

Let φ be defined on $\overline{\Pi}$, except on a set Z consisting of finitely many points $Z = \{z_1, \ldots, z_N\}$. Assume that φ is locally analytic on $\overline{\Pi} \setminus Z$, and that φ has a pole of order at most m_j at z_j ($1 \leqslant j \leqslant N$). We want to be able to "remove" these poles. One way to do this is to divide them out.

THEOREM 3.4 (L^1-QUOTIENT THEOREM). (i) *Let φ be locally analytic on $\overline{\Pi}$ except on a finite set $Z = \{z_1, \ldots, z_N\} \subseteq \Pi$, and let φ have a pole of order at most m_j at z_j ($1 \leqslant j \leqslant N$). Let ρ_0 be any constant satisfying* $\operatorname{Re} \rho_0 < \rho_*$. *Then there exist a function $a \in L^1(\rho)$ and a constant $\alpha \in \mathbf{C}$ such that*

$$\varphi(z) = [\alpha + \hat{a}(z)] \prod_{j=1}^{N} \left(\frac{z - \rho_0}{z - z_j} \right)^{m_j} \qquad (z \in \overline{\Pi} \setminus Z).$$

The constant $\alpha = 0$ if and only if $\varphi(\infty) = 0$.

(ii) *Let φ be as above, except that φ also has a pole of order at most m_∞ at infinity, and let ρ_0 be as above. Then there exist a function $a \in L^1(\rho)$ and a constant $\alpha \in \mathbf{C}$ such that*

$$\varphi(z) = [\alpha + \hat{a}(z)](z - \rho_0)^{m_\infty} \prod_{j=1}^{N} \left(\frac{z - \rho_0}{z - z_j} \right)^{m_j} \qquad (z \in \Pi \setminus Z).$$

The pole at infinity is of order m_∞ if and only if $\alpha \neq 0$.

The proof of Theorem 3.4 is obvious. Namely, define

$$\psi(z) = \varphi(z) \prod_{j=1}^{N} \left(\frac{z - z_j}{z - \rho_0} \right)^{m_j}$$

in case (i),

$$\psi(z) = \varphi(z)(z - \rho_0)^{-m_\infty} \prod_{j=1}^{N} \left(\frac{z - z_j}{z - \rho_0} \right)^{m_j}$$

in case (ii), and apply Proposition 2.3 to the locally analytic function ψ.

We discuss the significance of the L^1-Quotient Theorem in §5.

If one wants to "subtract off" the poles rather than divide them out, then one needs further assumptions.

DEFINITION 3.5. *We call a function φ which is locally analytic at z_0 smooth of order zero at z_0. We say that φ is smooth of (integer) order $m \geqslant 1$ at z_0 if $\varphi = \eta + \psi$, where η is an analytic function of the variable z at z_0, and ψ is locally analytic with a zero of order at least m at z_0.*

THEOREM 3.6 (L^1-REMAINDER THEOREM). (i) *Let φ be locally analytic on $\overline{\Pi}$ except on a finite set $Z = \{z_1, \ldots, z_N\} \subset \Pi$, and let φ have a pole of order m_j at z_j $(1 \leqslant j \leqslant N)$. In addition, suppose that φ^{-1} is smooth of order $2m_j$ at z_j $(1 \leqslant j \leqslant N)$. Then there exist a function $a \in L^1(\rho)$ and constants α, $\beta_{l,j}$ $(1 \leqslant j \leqslant N,\ 1 \leqslant l \leqslant m_j)$, $\beta_{m_j,j} \neq 0$, such that*

$$\varphi(z) = \hat{a}(z) + \alpha + \sum_{l,j} \beta_{l,j}(z - z_j)^{-l} \qquad (z \in \overline{\Pi} \setminus Z).$$

If $\varphi(\infty) = 0$, then $\alpha = 0$.

(ii) *Let φ be as above except that φ also has a pole of order m_∞ at infinity and that φ^{-1} is smooth of order $2m_\infty$ at infinity. Then there exist a function $a \in L^1(\rho)$ and constants α_l $(0 \leqslant l \leqslant m_\infty)$ and $\beta_{l,j}$ $(1 \leqslant j \leqslant N,\ 1 \leqslant l \leqslant m_j)$, $\alpha_{m_\infty} \neq 0$, $\beta_{m_j,j} \neq 0$, such that*

$$\varphi(z) = \hat{a}(z) + \sum_{l} \alpha_l z^l + \sum_{l,j} \beta_{l,j}(z - z_j)^{-l} \qquad (z \in \Pi \setminus Z).$$

PROOF. (i) Fix j, $1 \leqslant j \leqslant N$. By Lemma 3.3, φ^{-1} has a zero of order m_j at z_j. Since φ^{-1} is also smooth of order $2m_j$ at z_j, it follows that there exist constants $\eta_{l,j}$ $(m_j \leqslant l \leqslant 2m_j - 1)$, $\eta_{m_j,j} \neq 0$, and a function $\tilde{\psi}_j$ locally analytic at z_j so that

$$\varphi^{-1}(z) = \sum_{l} \eta_{lj}(z - z_j)^l + (z - z_j)^{2m_j}\tilde{\psi}_j(z)$$

near z_j. (Here we have made the observation that the function $\eta(z)$ in Definition 3.5 may be taken to be a polynomial of degree at most $2m_j - 1$ since the remainder of η has a zero of order $2m_j$ and may be absorbed in the locally analytic function ψ in that definition.) It follows by elementary division that there exist constants $\beta_{l,j}$ $(1 \leqslant l \leqslant m_j)$, $\beta_{m_j,j} = \eta_{m_j,j}^{-1} \neq 0$, and a locally analytic function χ_j so that

$$\varphi(z) = \sum_{l} \beta_{l,j}(z - z_j)^{-l} + \chi_j(z)$$

for z near z_j. Clearly $\Sigma_l \beta_{l,j}(z - z_j)^{-l}$ is locally analytic on $\overline{\Pi} \setminus \{z_j\}$. Repeating the above procedure for each j, $1 \leqslant j \leqslant N$, we see that $\varphi(z) - \Sigma_{l,j}\beta_{l,j}(z - z_j)^{-l}$ is locally analytic on $\overline{\Pi}$, and part (i) of the theorem follows from Proposition 2.3.

(ii) To prove part (ii) we note that φ^{-1} has a zero of order m_∞ and is smooth of order $2m_\infty$ at infinity; hence, there exist constants η_l $(m_\infty \leqslant l \leqslant 2m_\infty - 1)$, $\eta_{m_\infty} \neq 0$, and a function $\tilde{\psi}_\infty$ locally analytic at infinity so that

$$\varphi^{-1}(z) = \sum_{l} \eta_l z^{-l} + z^{-2m_\infty}\tilde{\psi}_\infty(z)$$

for z near infinity. Again, division yields constants α_l $(1 \leqslant l \leqslant m_\infty)$, $\alpha_{m_\infty} = \eta_{m_\infty}^{-1} \neq 0$, and a function χ_∞ locally analytic at infinity so that

$$\varphi(z) = \sum_l \alpha_l z^l + \chi_\infty(z)$$

near infinity. Now, as in case (i), $\varphi(z) - \sum_l \alpha_l z^l - \sum_{l,j} \beta_{l,j}(z - z_j)^{-l}$ is locally analytic on $\overline{\Pi}$. Thus Proposition 2.3 yields that this expression is equal to $\hat{a}(z) + \alpha_0$, where α_0 is a constant and $a \in L^1(\rho)$. $\square$

4. On smoothness and the order of zeros. In §3 we first defined the concept of a zero of order at least m and then used this concept to define smoothness of order m. Observe that a point $z_0 \in \Pi$ is a zero of order m (at least m) of φ if and only if φ is smooth of order m at z_0 and $\lim_{z \to z_0, z \in \Pi}(z - z_0)^{-m}\varphi(z)$ exists and is nonzero (exists). It turns out that it is more convenient to regard smoothness rather than the order of a zero as the basic concept. However, in order to effectively use this concept, we need new ways of characterizing smoothness.

What complicates the use of Definition 3.5 of smoothness is the fact that a locally analytic function φ has infinitely many representations of the form required in Definition 2.1. For a specific representation it may not be apparent that φ is actually the sum of an analytic function of z and a locally analytic function with a zero of order at least m as required in Definition 3.5; hence, we do not know if our function φ is smooth of order m at a particular point.

As in Definition 2.1, take $z_0 \in \Pi$, $\mu_1, \ldots, \mu_k \in M(\rho)$, $\psi(z, \xi_1, \ldots, \xi_k)$ analytic at $(z_0, \hat{\mu}_1(z_0), \ldots, \hat{\mu}_k(z_0))$, and let $\varphi(z) = \psi(z, \hat{\mu}_1(z), \ldots, \hat{\mu}_k(z))$. In this section we show that smoothness at z_0 can arise in two ways. Either the dependence of ψ on $\hat{\mu}_1, \ldots, \hat{\mu}_k$ is of positive order, or $\hat{\mu}_1, \ldots, \hat{\mu}_k$ are smooth. Also, combinations of these two possibilities may occur.

DEFINITION 4.1. *Let $\psi(z, \xi_1, \ldots, \xi_k)$ be analytic at $(z_0, \eta_1, \ldots, \eta_k)$. We say that the dependence of ψ on z with respect to ξ_j at the point $(z_0, \eta_1, \ldots, \eta_k)$ is of order at least m if ψ is of the form*

$$\psi(z, \xi_1, \ldots, \xi_k) = \psi_1(z, \xi_1, \ldots, \xi_k) + (z - z_0)^m \psi_2(z, \xi_1, \ldots, \xi_k),$$

where ψ_1 and ψ_2 are analytic at $(z_0, \eta_1, \ldots, \eta_k)$ and ψ_1 is independent of ξ_j.

We shall refer to this concept simply as the dependence of ψ on ξ_j at the given point.

LEMMA 4.2. *Let $\varphi(z) = \psi(z, \hat{\mu}_1(z), \ldots, \hat{\mu}_k(z))$ in a neighborhood of $z_0 \in \Pi$, where $\mu_1, \ldots, \mu_k$ belong to $M(\rho)$ and ψ is analytic at $(z_0, \hat{\mu}_1(z_0), \ldots, \hat{\mu}_k(z_0))$. Moreover, suppose that for each j, $1 \leqslant j \leqslant k$, the dependence of ψ on ξ_j at $(z_0, \hat{\mu}_1(z_0), \ldots, \hat{\mu}_k(z_0))$ is of order at least m_j, where $0 \leqslant m_j \leqslant m$, and that $\hat{\mu}_j$ is smooth of order $m - m_j$ at z_0. Then φ is smooth of order m at z_0.*

We postpone the proof of Lemma 4.2 to the end of this section.

To illustrate Lemma 4.2 in the case where smoothness of φ is due to the fact that the dependence of ψ on the transforms $\hat{\mu}_j$ is of positive order, we once more discuss the example from [**16**] mentioned in §2. Let $\varphi(z) = \psi(z, b/z + \hat{\beta}(z))$ with $b \in \mathbf{C}$, $b \neq 0$, $\beta \in L^1(R^+)$ and ψ analytic at $(0, \infty)$. We claim that φ is smooth of order two at the origin. Without loss of generality, take $b = 1$ (rescale β and ψ). The analyticity of ψ at $(0, \infty)$ means that the function $\psi_1(z, \xi) \equiv \psi(z, \xi^{-1})$ is analytic at $(0,0)$; hence $\varphi(z) = \psi_1(z, z(1 + z\hat{\beta}(z))^{-1})$, where ψ_1 is analytic at $(0,0)$. Define $\tilde{\psi}(z, \xi) \equiv \xi(1 + z\xi)^{-1}$. Then $\tilde{\psi}$ is analytic at $(0, \hat{\beta}(0))$, and $\varphi(z) = \psi_1(z, z - z^2\tilde{\psi}(z, \hat{\beta}(z)))$ for z near 0. Expanding ψ_1 and $\tilde{\psi}$ in power series around $(0,0)$ and $(0, \hat{\beta}(0))$, respectively, yields that $\varphi(z)$ has the form $\varphi(z) = \alpha_0 + \alpha_1 z + z^2\psi_2(z, \hat{\beta}(z))$, where α_0 and α_1 are constants and ψ_2 is analytic at $(0, \hat{\beta}(0))$. Thus, φ is smooth of order two at $z = 0$, as we claimed. This observation (in a modified form) will be quite useful in §6.

As the statement of Lemma 4.2 shows, smoothness of φ may also be due to the smoothness of the transforms $\hat{\mu}_j$. Trivially, by Definition 3.5, the transform $\hat{\mu}$ of a measure $\mu \in M(\rho)$ is smooth of order zero on Π. Definition 3.5 also tells us when $\hat{\mu}$ is smooth of order m for $m \geqslant 1$ at a point $z_0 \in \Pi$. However, in this case the rather general Definition 3.5 may be made more concrete. Also, it may be rephrased in terms of certain iterated integrals of μ.

LEMMA 4.3. *Let $\mu \in M(\rho)$, $z_0 \in \Pi$, and m be a positive integer. Then conditions* (i)–(iii) *below are equivalent.*

(i) *$\hat{\mu}$ is smooth of order m at z_0.*

(ii) $\hat{\mu}(z) = p(z) + (z - z_0)^m \hat{a}(z)$ $(z \in \Pi)$, *where p is a polynomial of degree at most $m - 1$, and $a \in L^1(\rho)$.*

(iii) *For each j, $1 \leqslant j \leqslant m$, the jth iterated integral $a_j(t)$ defined by*

$$(4.1)\qquad a_j(t) = \begin{cases} (-1)^j \displaystyle\int_t^\infty \int_{s_{j-1}}^\infty \cdots \int_{s_1}^\infty e^{z_0(t-s)}\, d\mu(s)\, ds_1 \cdots ds_{j-1}, & t > 0, \\ \displaystyle\int_{-\infty}^t \int_{-\infty}^{s_{j-1}} \cdots \int_{-\infty}^{s_1} e^{z_0(t-s)}\, d\mu(s)\, ds_1 \cdots ds_{j-1}, & t < 0. \end{cases}$$

belongs to $L^1(\rho)$.

We remark that the order of integration in (4.1) is important. All integrals in (4.1) converge absolutely, but for $j \geqslant 2$ the corresponding joint integrals need not do so. Of course, if they do converge absolutely, then

$$a_j(t) = \frac{1}{(j-1)!} \begin{cases} -\displaystyle\int_t^\infty (t-s)^{j-1} e^{z_0(t-s)}\, d\mu(s), & t > 0, \\ \displaystyle\int_{-\infty}^t (t-s)^{j-1} e^{z_0(t-s)}\, d\mu(s), & t < 0. \end{cases}$$

Also, if one is working on R^+, then one defines the $a_j(t)$ to be zero for $t < 0$. As we have noted before, $\hat{\mu}(z)$ is an analytic function in the interior of Π; hence, $\hat{\mu}$ is smooth of any order at interior points of Π. Thus, in applications, $a_j(t) \in L^1(\rho)$ is used as a sufficient condition to determine smoothness of $\hat{\mu}$ only at boundary points of Π. In §5 we discuss conditions on μ which have previously been used in the literature and which imply that $a_j \in L^1(\rho)$.

PROOF OF LEMMA 4.3. Clearly (ii) implies (i). Thus, it suffices to show that (i) implies (iii) and that (iii) implies (ii).

Before we continue the proof, let us observe that if one knows that $a_j \in L^1(\rho)$ for $1 \leqslant j \leqslant m-1$, then $a_1, \dots, a_m$ may be obtained recursively from a_1 by the formula

$$(4.2) \qquad a_j(t) = \begin{cases} -\int_t^\infty e^{z_0(t-s)} a_{j-1}(s)\, ds, & t > 0, \\ \int_{-\infty}^t e^{z_0(t-s)} a_{j-1}(s)\, ds, & t < 0. \end{cases}$$

The integrals in (4.2) converge absolutely because of (2.6). The integrals defining a_1 in (4.1), i.e.,

$$(4.3) \qquad a_1(t) = \begin{cases} -\int_t^\infty e^{z_0(t-s)}\, d\mu(s), & t > 0, \\ \int_{-\infty}^t e^{z_0(t-s)}\, d\mu(s), & t < 0, \end{cases}$$

converge absolutely because of (2.10).

Assume that $\hat{\mu}$ is smooth of order m at z_0. Define recursively for $z \in \overline{\Pi}$,

$$(4.4) \qquad \varphi_1(z) = \begin{cases} \dfrac{\hat{\mu}(z) - \hat{\mu}(z_0)}{z - z_0}, & z \neq z_0, \\ \hat{\mu}'(z_0), & z = z_0, \end{cases}$$

and for $2 \leqslant j \leqslant m$

$$(4.5) \qquad \varphi_j(z) = \begin{cases} \dfrac{\varphi_{j-1}(z) - \varphi_{j-1}(z_0)}{z - z_0}, & z \neq z_0, \\ \varphi'_{j-1}(z_0), & z = z_0. \end{cases}$$

Here the existence of $\hat{\mu}'(z_0)$ and $\varphi'_j(z_0)$ $(1 \leqslant j \leqslant m-1)$ is ensured by the fact that $\hat{\mu}$ is smooth of order m at z_0, and, for $1 \leqslant j \leqslant m$, φ_j is smooth of order $m-j$ at z_0. In particular, every φ_j is locally analytic at z_0. Clearly φ_j is also locally analytic everywhere else in $\overline{\Pi}$, and $\varphi_j(\infty) = 0$. Thus, by Proposition 2.3, there exist functions $b_j \in L^1(\rho)$, $1 \leqslant j \leqslant m$, such that $\hat{b}_j = \varphi_j$.

We claim that $a_j = b_j$, $1 \leqslant j \leqslant m$. By (4.3) and (4.4), the functions $e^{-z_0 t} a_1(t)$ and $e^{-z_0 t} b_1(t)$ have the same distribution derivative, namely the sum of $e^{-z_0 s} d\mu(s)$ and a point mass at the origin of size $-\hat{\mu}(z_0)$. Thus, the difference $c = e^{-z_0 t}[a_1(t) - b_1(t)]$ is a constant. Since, in addition, $e^{-z_0 t} a_1(t) \to 0$ as $t \to \infty$ and $e^{-z_0 t} b_1(t) \in L^1(\rho)$, it follows that $c = 0$, and that $a_1 = b_1 \in L^1(\rho)$. One completes the proof by using induction (i.e., one repeats the preceding argument with a_1, b_1 replaced by a_j, b_j, and μ replaced by $a_{j-1} = b_{j-1}$, $j = 2, \dots, m$). This shows that (i) implies (iii).

Next, suppose that $a_j \in L^1(\rho)$ for $1 \leqslant j \leqslant m$. Then, by (4.3), the distribution derivative of $e^{-z_0 t} a_1(t)$ equals the sum of $e^{-z_0 t} d\mu(s)$ and a point mass at the origin of size $-\hat{\mu}(z_0)$, and so $\hat{\mu}(z) = \hat{\mu}(z_0) + (z - z_0)\hat{a}_1(z)$. Similarly, by (4.2), for $2 \leqslant j \leqslant m$, $\hat{a}_{j-1}(z) = \hat{a}_{j-1}(z_0) + (z - z_0)\hat{a}_j(z)$. Thus,

$$\hat{\mu}(z) = \hat{\mu}(z_0) + (z - z_0)\hat{a}_1(z_0) + \cdots + (z - z_0)^{m-1}\hat{a}_{m-1}(z_0) + (z - z_0)^m \hat{a}_m(z).$$

In particular, we obtain (ii).

PROOF OF LEMMA 4.2. By the hypothesis and Lemma 4.3, each $\hat{\mu}_j$ may be written in the form

$$\hat{\mu}_j(z) = p_j(z) + (z - z_0)^{m-m_j}\hat{a}_j(z)$$

where p_j is a polynomial of degree at most $m - m_j - 1$ and $a_j \in L^1(\rho)$ (if $m_j = m$, then replace a_j by μ_j). Also, by Definition 4.1 and the hypothesis of Lemma 4.2, for each j, $1 \leqslant j \leqslant k$, ψ has the form

$$\psi(z, \xi_1, \ldots, \xi_k) = \psi_{1j}(z, \xi_1, \ldots, \xi_k) + (z - z_0)^{m_j}\psi_{2j}(z, \xi_1, \ldots, \xi_k), \tag{4.6}$$

with ψ_{1j} and ψ_{2j} analytic at $(z_0, \hat{\mu}_1(z_0), \ldots, \hat{\mu}_k(z_0))$ and ψ_{1j} independent of ξ_j. Define

$$\tilde{\psi}(z, \xi_1, \ldots, \xi_k) = \psi\left(z, p_1(z) + (z - z_0)^{m-m_1}\xi_1, \ldots, p_k(z) + (z - z_0)^{m-m_k}\xi_k\right).$$

Then $\tilde{\psi}$ is analytic at $(z_0, \hat{a}_1(z_0), \ldots, \hat{a}_k(z_0))$ and $\varphi(z) = \tilde{\psi}(z, \hat{a}_1(z), \ldots, \hat{a}_k(z))$ for z in a neighborhood of z_0. Moreover, it is easy to see using (4.6) and the definition of $\tilde{\psi}$ that

$$\left(\frac{\partial}{\partial z}\right)^p \left(\frac{\partial}{\partial \xi_j}\right)^q \tilde{\psi}(z_0, \hat{a}_1(z_0), \ldots, \hat{a}_k(z_0)) = 0$$

for $1 \leqslant j \leqslant k$, $q \geqslant 1$ and $0 \leqslant p \leqslant m - 1$. This means that $\tilde{\psi}$ is of the form

$$\tilde{\psi}(z, \xi_1, \ldots, \xi_k) = \tilde{\psi}_1(z) + (z - z_0)^m \tilde{\psi}_2(z, \xi_1, \ldots, \xi_k),$$

where $\tilde{\psi}_1$ is analytic at z_0 and $\tilde{\psi}_2$ is analytic at $(z_0, \hat{a}_1(z_0), \ldots, \hat{a}_k(z_0))$. Thus φ is smooth of order m at z_0. $\square$

5. Some applications. In this section we show how the results in §§2–4 can be used to obtain and extend some theorems of several earlier papers. In addition, we give the new Propositions 5.1 and 5.2 which provide necessary and sufficient conditions for the solutions of (1.1) and (1.2) to be bounded or integrable. Proposition 5.3 relates Propositions 5.1 and 5.2 to the theorems discussed earlier in this section.

First, consider a result due to K. B. Hannsgen [**8**, Theorem 1.1]. Take $\rho(t) \equiv 1$ on R^+. Then $L^1(\rho)$ is $L^1(R^+)$, $\rho_* = 0$, and $\Pi = \{z \in \mathbf{C} \mid \operatorname{Re} z \geqslant 0\}$. Hannsgen studies a function

$$f(z) = \varphi(z, \hat{a}_1(z), \ldots, \hat{a}_k(z))/\psi(z, \hat{a}_1(z), \ldots, \hat{a}_k(z))$$

where the $\hat{a}_j(z)$ ($1 \leqslant j \leqslant k$) are transforms of functions $a_j(t)$, and where φ and ψ are analytic on $\{(z, \hat{a}_1(z), \ldots, \hat{a}_k(z)) \mid z \in \Pi \setminus \{0\}\}$. Although the $a_j(t)$ are not necessarily integrable, Hannsgen's hypothesis that each $a_j(t)$ belongs to the class $H(M, D_j)$ (see [**8**, p. 103] for the definition of $H(M, D_j)$) implies that each a_j may be written in the form $a_j = a_{j1} + a_{j2}$, where $a_{j1} \in L^1(R^+)$ with $a_{j1}(t) \equiv 0$ ($t \geqslant 1$) and $\hat{a}_{j2}(z) = p_j(1/z) + z^{-D_j}\hat{\mu}_j(z)$ ($\operatorname{Re} z \geqslant 0$, $z \neq 0$). Here D_j is a nonnegative integer, p_j is a polynomial of degree at most D_j, and μ_j is a measure on R^+ satisfying

$$\int_0^\infty t^M \, d|\mu_j|(t) < \infty, \tag{5.1}$$

where $M \geqslant 1$ is a fixed integer such that $D_j \leqslant M + 1$ for each j; in particular, $d\mu_j(t) = da_{j2}^{(D_j-1)}(t)$, where $da_{j2}^{(-1)}(t) \equiv a_{j2}(t)\,dt$.

Define

$$\tilde{\varphi}(z, \eta_1, \ldots, \eta_k, \xi_1, \ldots, \xi_k) = \varphi(z, \eta_1 + p_1(1/z) + z^{-D_1}\xi_1, \ldots, \eta_k + p_k(1/z) + z^{-D_k}\xi_k),$$

and let $\tilde{\psi}$ be the function obtained from ψ by the same change of variables. Also, define

$$f_1(z) = \varphi(z, \hat{a}_1(z), \ldots, \hat{a}_k(z)) = \tilde{\varphi}(z, \hat{a}_{11}(z), \ldots, \hat{a}_{k1}(z), \hat{\mu}_1(z), \ldots, \hat{\mu}_k(z)),$$
$$f_2(z) = \psi(z, \hat{a}_1(z), \ldots, \hat{a}_k(z)) = \tilde{\psi}(z, \hat{a}_{11}(z), \ldots, \hat{a}_{k1}(z), \hat{\mu}_1(z), \ldots, \hat{\mu}_k(z)).$$

Then $f(z) = f_1(z)/f_2(z)$, and f_1 and f_2 are locally analytic on $\Pi \setminus \{0\}$. Moreover, since the Mth moment of each $a_{j1}(t)$ is finite and (5.1) holds, it follows from Lemmas 4.2 and 4.3 that both f_1 and f_2 are smooth of order M on $\Pi \setminus \{0\}$.

Hannsgen assumes that $f_2(z) \neq 0$ on $\Pi \setminus \{0\}$ except at a finite number of points z_l, $1 \leqslant l \leqslant L$, and that at each point z_l f_1 and f_2 satisfy

$$\lim_{z \to z_l, z \in \Pi} (z - z_l)^{-m_l} f_2(z) \neq 0, \qquad \lim_{z \to z_l, z \in \Pi} (z - z_l)^{-m_l} f_1(z) \quad \text{exists},$$

where the m_l are constants with $1 \leqslant m_l \leqslant M$. Since f_1 and f_2 are smooth of order M, it follows that f_1 has a zero of order at least m_l at z_l, and f_2 has a zero of order m_l at z_l. Thus, $1/f_2$ has a pole of order m_l at z_l, and $f(z) = f_1(z)/f_2(z)$ is locally analytic at each z_l. Elsewhere in $\Pi \setminus \{0\}$ f_1 and f_2 are locally analytic with $f_2(z) \neq 0$; thus, f is locally analytic in all of $\Pi \setminus \{0\}$. Finally, Hannsgen assumes that $f(z)$ is locally analytic at infinity with $f(\infty) = 0$. Thus, it follows from his assumptions and our results in §§2–4 that the function $f(z)$ is locally analytic on $\overline{\Pi}$ except possibly at $z = 0$. If, in addition, one knows that $f(z)$ is locally analytic at the origin, then Proposition 2.3 yields Hannsgen's conclusion that $f(z)$ is the Laplace transform of an integrable function on R^+. The question of whether $f(z)$ is locally analytic at the origin is far from trivial, and we discuss it further in §7.

The result described above is proved in [**8**] for $M \geqslant 2$ and $0 \leqslant D_j \leqslant M + 1$, that is, for kernels $a_j(t)$ belonging to Hannsgen's classes $H(M, D)$, $M \geqslant 2$, $0 \leqslant D \leqslant M + 1$. We remark that in $\Pi \setminus \{0\}$ the argument given here applies equally well to kernels $a_j(t)$ in his classes $H(1,0)$, $H(1,1)$ and $H(1,2)$. Thus, Hannsgen's theorem can be extended to these classes if $f(z)$ is locally analytic at the origin.

In [**8**, Corollary 2.1] Hannsgen applies the preceding result to obtain an L^1-Remainder Theorem for the (matrix) resolvent of a system of linear Volterra integrodifferential equations. Again, this result could be deduced directly from our Theorem 3.6, provided that the transform of the matrix resolvent is locally analytic at the origin (see, also, §7).

Next, consider some results due to G. S. Jordan and R. L. Wheeler [**13**]. Let ρ be a weight function on R^+ satisfying the additional normalizing condition $\rho_* = 0$ (as observed in [**13**, Remark 2.2], the assumption $\rho_* = 0$ causes no loss of generality in the following discussion). (The weight in [**13**] is, in addition, nondecreasing, but we make no explicit use of this assumption below.) Jordan and Wheeler study the integral and integrodifferential Volterra equations (1.1), (1.2) and (1.7). The kernels a and μ in these equations belong to $L^1(R^+, \rho)$ and $M(R^+; \rho)$, respectively.

Observe that (1.1) may be regarded as a special case of (1.7). In this section we treat equations (1.1), (1.2) and (1.5), (1.6) and obtain [**13**, Theorems 2.1, 2.2 and 3.2 and Corollary 2.1]. Our discussion of equation (1.7) and Theorem 3.1 of [**13**] which involves the ring $M(R;\rho)$ and our notion of pseudo local analyticity with respect to this ring is postponed until §8.

We let φ_1 and φ_2 be the Laplace transforms of the resolvents r_1 and r_2 of (1.1) and (1.2), defined in (1.5) and (1.6), i.e.

$$\varphi_1(z)=\frac{\hat{a}(z)}{1+\hat{a}(z)}, \tag{5.2}$$

$$\varphi_2(z)=\frac{1}{z+\hat{\mu}(z)}. \tag{5.3}$$

In [**13**] the conditions on a and μ imply $a\in L^1(R^+;\rho)$ and $\mu\in M(R^+;\rho)$. Because of the normalizing condition $\rho_*=0$, we have $\Pi=\{z\in\mathbf{C}\mid \operatorname{Re} z\geqslant 0\}$. It is assumed that $1+\hat{a}(0)\neq 0$ and $\hat{\mu}(0)\neq 0$. Thus, φ_1 and φ_2 are locally analytic at the origin. It is also easy to check that φ_1 and φ_2 are locally analytic at infinity (write φ_2 in the form $\varphi_2(z)=z^{-1}(1+\hat{\mu}(z)/z)^{-1}$). Jordan and Wheeler use a combination of moment conditions on derivatives of a and μ which imply that for a fixed positive integer M, $\hat{a}$ and $\hat{\mu}$ are smooth of order $2M$ away from the origin. Thus, by Lemma 4.2, $[\varphi_1(z)]^{-1}$ is smooth of order $2M$ whenever $z\neq 0$ and $\hat{a}(z)\neq 0$, and $[\varphi_2(z)]^{-1}$ is smooth of order $2M$ for $z\neq 0$. In $\operatorname{Re} z>0$ both $\hat{a}(z)$ and $\hat{\mu}(z)$ are analytic, and hence $[\varphi_1(z)]^{-1}$ and $[\varphi_2(z)]^{-1}$ are smooth of any order in $\operatorname{Re} z>0$ (as long as $\hat{a}(z)\neq 0$). The zeros of the denominators in (5.2) and (5.3) are required to be different from zero and finite in number, and the orders of the zeros on the line $\operatorname{Re} z=0$ are restricted to at most M. Thus, our Theorem 3.6 shows that each resolvent many be written as a finite sum of products of polynomials and exponentials, plus a remainder term in $L^1(R^+;\rho)$. This implies Theorems 2.1 and 2.2 (as well as Corollary 2.1) in [**13**].

In [**13**, Theorem 3.2] equation (1.2) is studied with the same hypothesis on μ as above, and with a hypothesis on f which implies that $f\in L^1(R^+;\rho)$, and that $\hat{f}$ is smooth of order M away from the origin. Again, $\hat{f}$ is smooth of any order in $\operatorname{Re} z>0$. The transform of the solution x of (1.2) satisfies $\hat{x}(z)=\varphi_2(z)(x_0+\hat{f}(z))$. By Theorem 3.6 we may write this as

$$\hat{x}(z)=x_0\hat{a}(z)+\hat{f}(z)\hat{a}(z)+\sum_{l,j}\beta_{l,j}(z-z_j)^{-l}\big(x_0+\hat{f}(z)\big), \tag{5.4}$$

where $a\in L^1(\rho)$, $z_j\neq 0$ $(1\leqslant j\leqslant n)$ are the zeros of $z+\hat{\mu}(z)$, with order p_j, respectively, and $\beta_{l,j}$ $(1\leqslant j\leqslant n,\ 1\leqslant l\leqslant p_j)$ are constants. Now apply Lemma 4.3 with μ replaced by f, z_0 replaced by z_j $(1\leqslant j\leqslant n)$, and m replaced by l to get representations for $\hat{f}(z)$ which when used in (5.4) give a sum of the type

$$\hat{x}(z)=\hat{b}(z)+\sum_{l}\gamma_{l,j}(z-z_j)^{-l},$$

where $b\in L^1(R^+;\rho)$. This yields the conclusion of [**13**, Theorem 3.2], namely, that x is a finite sum of products of polynomials and exponentials, plus a remainder term in $L^1(R^+;\rho)$. (One can also show that the derivative of the remainder belongs to $L^1(R^+;\rho)$.)

If we weaken the hypotheses on a and μ so that a and μ are smooth of order M rather than smooth of order $2M$ away from the origin, then the L^1-Remainder Theorem can no longer be applied. However, it is still possible to apply the L^1-Quotient Theorem. The conclusion of Theorem 3.4 is weaker than the conclusion of Theorem 3.6, but it is strong enough to settle, for example, the question of what type of bounded (or integrable) perturbations f in (1.1) and (1.2) give rise to bounded (or integrable) solutions x, as we shall see below.

PROPOSITION 5.1. *Let $a \in L^1(R^+;\rho)$, and let the function $1+\hat{a}(z)$ have finitely many zeros $z_1,\ldots,z_n$ in Π of (integral) order $p_1,\ldots,p_n$. Moreover, suppose that x and f are locally bounded and satisfy* (1.1). *For $1 \leqslant j \leqslant n$, $1 \leqslant l \leqslant p_j$, define*

$$F_{l,j}(t) = \frac{1}{(l-1)!}\int_0^t (t-s)^{l-1} e^{z_j(t-s)} f(s)\,ds, \qquad t>0. \tag{5.5}$$

Then $x = O(\rho(t)^{-1})$ as $t \to \infty$ if and only if $f = O(\rho(t)^{-1})$ and $F_{l,j}(t) = O(\rho(t)^{-1})$ for $1 \leqslant j \leqslant n$, $1 \leqslant l \leqslant p_j$ as $t \to \infty$. The same statement is true with $O(\rho(t)^{-1})$ replaced by $o(\rho(t)^{-1})$. Finally, $x \in L^1(R^+;\rho)$ if and only if $f \in L^1(R^+;\rho)$ and $F_{l,j} \in L^1(R^+;\rho)$ for $1 \leqslant j \leqslant n$, $1 \leqslant l \leqslant p_j$.

A similar result holds for equation (1.2). There we shall have to require that x' satisfies the same growth condition as x. However, in general x' exists only almost everywhere, and therefore we use a slightly different notation. We let the space $L^\infty(R^+;\rho)$ consist of those locally bounded functions y on R^+ which satisfy $\operatorname{ess\,sup}_{t\in R^+}|y(t)|\rho(t) < \infty$. If, in addition,

$$\lim_{t\to\infty}\ \operatorname*{ess\,sup}_{s\geqslant t}|y(s)|\rho(s) = 0,$$

then we write $y \in L_0^\infty(R^+;\rho)$.

PROPOSITION 5.2. *Let $\mu \in M(R^+;\rho)$, and let the function $z+\hat{\mu}(z)$ have finitely many zeros $z_1,\ldots,z_n$ in Π of order $p_1,\ldots,p_n$. Moreover, suppose that x is locally absolutely continuous and f is locally bounded, and that* (1.2) *holds almost everywhere. For $1 \leqslant j \leqslant n$, $1 \leqslant l \leqslant p_j$, define*

$$F_{l,j}(t) = \frac{1}{(l-1)!}e^{z_j t}\left[x_0 t^{l-1} + \int_0^t (t-s)^{l-1} e^{-z_j s} f(s)\,ds\right], \qquad t>0.$$

Then x, $x' \in L^\infty(R^+;\rho)$ if and only if $f \in L^\infty(R^+;\rho)$ and $F_{l,j} \in L^\infty(R^+;\rho)$ for $1 \leqslant j \leqslant n$, $1 \leqslant l \leqslant p_j$. The same statement is true with $L^\infty(R^+;\rho)$ replaced by $L_0^\infty(R^+;\rho)$ or by $L^1(R^+;\rho)$.

The proofs of Propositions 5.1 and 5.2 are completely similar. We leave the proof of Proposition 5.1 to the reader.

PROOF OF PROPOSITION 5.2. Assume that x, $x' \in L^\infty(R^+;\rho)$. Then $x * \mu \in L^\infty(R^+;\rho)$, and by (1.2), also $f \in L^\infty(R^+;\rho)$.

Transform (1.2) to get

$$(z+\hat{\mu}(z))\hat{x}(z) = x_0 + \hat{f}(z) \qquad \big(\operatorname{Re} z > \rho_*\big). \tag{5.6}$$

Let z_j be a zero of order p_j of $z + \hat{\mu}(z)$, and let $1 \leqslant l \leqslant p_j$. Define $\varphi(z) = (z + \hat{\mu}(z))/(z - z_j)^l$. Then φ is locally analytic on Π. It is also locally analytic at infinity, because we can write φ in the form

$$\varphi(z) = z^{1-l}\frac{1 + \hat{\mu}(z)/z}{(1 - z_j/z)^l}.$$

Thus, by Proposition 2.3, $\varphi(z) = \alpha + \hat{b}(z)$ for some constant α and some function $b \in L^1(R^+; \rho)$ (if $l > 1$, then $\alpha = 0$). Substitute this into (5.6) to get

$$\frac{x_0 + \hat{f}(z)}{(z - z_j)^l} = \alpha\hat{x}(z) + \hat{b}(z)\hat{x}(z) \qquad (\operatorname{Re} z > \rho_*).$$

By elementary Laplace transform theory, this implies that $F_{l,j} = \alpha x + b * x$. Thus $F_{l,j} \in L^\infty(R^+; \rho)$, as we claimed.

Conversely, suppose that $f \in L^\infty(R^+; \rho)$ and $F_{l,j} \in L^\infty(R^+; \rho)$ for $1 \leqslant j \leqslant n$, $1 \leqslant l \leqslant p_j$. Then it is possible to show that the Laplace transform of x converges in some half-plane $\operatorname{Re} z > \gamma$, and that (5.6) holds for $\operatorname{Re} z > \gamma$. Divide (5.6) by $z + \hat{\mu}(z)$, and use Theorem 3.4 to get

$$\hat{x}(z) = \hat{b}(z)\prod_{j=1}^{n}\left(\frac{z - \rho_0}{z - z_j}\right)^{p_j}(x_0 + \hat{f}(z)), \tag{5.7}$$

where $b \in L^1(R^+; \rho)$, and ρ_0 is a constant satisfying $\operatorname{Re}\rho_0 < \rho_*$. Expand the product in (5.7) to get

$$\begin{aligned}\hat{x}(z) &= \hat{b}(z)\Big(1 + \sum_{l,j}\beta_{l,j}(z - z_j)^{-l}\Big)(x_0 + \hat{f}(z))\\ &= \hat{b}(z)\Big(x_0 + \hat{f}(z) + \sum_{l,j}\beta_{l,j}\hat{F}_{l,j}(z)\Big)\end{aligned}$$

for some constants $\beta_{l,j}$, $1 \leqslant j \leqslant n$, $1 \leqslant l \leqslant p_j$. Thus,

$$x = x_0 b + b * f + \sum_{l,j}\beta_{l,j} b * F_{l,j},$$

and this implies that $x \in L^\infty(R^+; \rho)$. (Note that $b \in L_0^\infty(R^+; \rho)$ since both b and b' belong to $L^1(R^+; \rho)$ and (2.1) holds for $s, t \in R^+$.) Once this is known it follows from (1.2) that also $x' \in L^\infty(R^+; \rho)$. This completes the proof of the first claim in Proposition 5.2. To get the two remaining claims one argues in exactly the same way as above, but replaces $L^\infty(R^+; \rho)$ throughout by $L_0^\infty(R^+; \rho)$ or $L^1(R^+; \rho)$. □

The versions of Propositions 5.1 and 5.2 one gets by taking $\rho \equiv 1$ are essentially contained in [**17**, §§5–6].

In [**14**, Theorem 8(a)] R. K. Miller proves a system version of Proposition 5.2 (without weights). He uses a remainder theorem instead of a quotient theorem, and therefore his approach requires more smoothness of the transform of the kernel than our approach does.

We also observe that it is possible to have a trade off between the order of smoothness of the transform of the kernel and the orders of the zeros of $\hat{f}(z)$. In particular, we have

PROPOSITION 5.3. *Let $a \in L^1(R^+\,;\rho)$, and let the function $1+\hat{a}(z)$ have finitely many zeros $z_1,\dots,z_n$ in Π of (integral) order $p_1,\dots,p_n$. In addition, suppose that for each j, $1 \leqslant j \leqslant n$, there exists an integer s_j, $0 \leqslant s_j \leqslant p_j$, so that $1+\hat{a}(z)$ is smooth of order $p_j + s_j$ at z_j. Let f and x be locally integrable and satisfy (1.1). Then $f \in L^1(R^+\,;\rho)$ and for each j, $1 \leqslant j \leqslant n$, $\hat{f}(z)$ is smooth of order p_j and has a zero of order at least $p_j - s_j$ at z_j if and only if for each j, $1 \leqslant j \leqslant n$, there exists a polynomial q_j of degree at most $s_j - 1$ so that*

$$x(t) - \sum_{j=1}^{n} q_j(t)e^{z_j t} \in L^1(R^+\,;\rho). \tag{5.8}$$

Here we have used the conventions that $q_j(t) \equiv 0$ when $s_j = 0$, and that the requirement that $\hat{f}(z)$ have a zero of order at least zero when $s_j = p_j$ is vacuously satisfied. Since the transform of the function $F_{l,j}(t)$ defined in (5.5) satisfies $\hat{F}_{l,j}(z) = \hat{f}(z)(z-z_j)^{-l}$ for $\operatorname{Re} z \geqslant \operatorname{Re} z_j$, it is easy to see that $\hat{f}$ has a zero of order at least $p_j - s_j$ at z_j if and only if $F_{l,j} \in L^1(R^+\,;\rho)$ for $1 \leqslant l \leqslant p_j - s_j$. Clearly, when $s_j = 0$ ($1 \leqslant j \leqslant n$), Proposition 5.3 reduces to the $L^1(R^+\,;\rho)$ version of Proposition 5.1, whereas the conclusion that (5.8) holds in the case where $s_j = p_j$ ($1 \leqslant j \leqslant n$) is essentially the L^1-remainder result discussed earlier in this section.

The proof of Proposition 5.3 is similar to the proofs of other results in this section and will not be given. Of course, there is a corresponding result for equation (1.2). In addition, one can obtain results similar to Proposition 5.3 but with the integrability conditions replaced by O or o growth conditions as in Propositons 5.1 and 5.2. For early versions of $O(\rho(t)^{-1})$ and $o(\rho(t)^{-1})$ remainder results, see **[10]** where the necessary smoothness of the transforms is a consequence of moment hypotheses. The precise formulation of O and o analogues of Proposition 5.3 in the present setting is left to the interested reader.

6. Increasing the weight through positive order dependence. Our next results are based on the following idea: If both $\hat{a}(z)$ and $(d/dz)^m\hat{a}(z)$ are locally analytic on $\overline{\Pi}$, then both $a(t) \in L^1(\rho)$ and $t^m a(t) \in L^1(\rho)$ and, hence, $a \in L^1((1+|t|^m)\rho(t))$.

Suppose that in a neighborhood of $z_0 \in \Pi$, $\varphi(z) = \psi(z, \hat{\mu}_1(z),\dots,\hat{\mu}_k(z))$ with ψ and $\mu_1,\dots,\mu_k$ as in Definition 2.1. Then

$$\frac{d\varphi}{dz} = \frac{\partial\psi}{\partial z} + \sum_{j=1}^{k} \frac{\partial\psi}{\partial\xi_j}\frac{d\hat{\mu}_j}{dz},$$

and this expression is locally analytic provided not only $\hat{\mu}_j(z)$ but also the derivatives $d\hat{\mu}_j(z)/dz$ are transforms of measures in $M(\rho)$. This requirement is equivalent to

$$\int (1+|t|)\rho(t)\,d|\mu_j|(t) < \infty, \tag{6.1}$$

or, in the absolutely continuous case,

$$\int (1+|t|)\rho(t)|a_j(t)|\,dt < \infty. \tag{6.2}$$

Thus, if all the measures and functions used in the representations of φ at different points satisfy (6.1) or (6.2), then both $\varphi(z)$ and $d\varphi(z)/dz$ are locally analytic. If moreover $\varphi(\infty) = 0$, then φ is the transform of a function $a(t)$ satisfying

$$\int (1+|t|)\rho(t)|a(t)|\,dt < \infty.$$

However, this result is not new in the sense that we could have obtained it directly from the fact that the measures and functions in the representations for φ belong to $M(\rho_1)$ and $L^1(\rho_1)$, respectively, where $\rho_1(t) \equiv (1+|t|)\rho(t)$.

The situation is different if the dependence of the analytic function ψ on some of its arguments is of order at least m as in Definition 4.1. Here one can allow different orders of dependence on the different arguments, but for simplicity we treat only the case when the dependence of $\psi(z, \xi_1, \ldots, \xi_k)$ on ξ_j $(1 \leqslant j \leqslant k)$ at a point $(z_0, \hat{\mu}_1(z_0), \ldots, \hat{\mu}_k(z_0))$ is at least a common number m. Then one can write ψ in the form

$$\psi(z, \xi_1, \ldots, \xi_k) = \psi_1(z) + (z - z_0)^m \psi_2(z, \xi_1, \ldots, \xi_k) \tag{6.3}$$

(cf. the proof of Lemma 4.2). To see what the final result will look like, let us argue formally as follows. Differentiating (6.3) m times, one gets an analytic function of z and of $(z - z_0)^l \hat{\mu}_j^{(l)}(z)$ $(0 \leqslant l \leqslant m,\ 1 \leqslant j \leqslant k)$. The inverse transform of $(z - z_0)^l \hat{\mu}_j^{(l)}$ equals $(d/dt - z_0)^l(-t)^l d\mu_j(t)$, so if we want $(d/dz)^m \varphi(z)$ to be locally analytic at z_0, then we must require something like $(d/dt - z_0)^l t^l d\mu_j(t) \in M(\rho)$ $(0 \leqslant l \leqslant m,$ $1 \leqslant j \leqslant k)$.

Actually, a slightly weaker condition suffices. For every p, $0 \leqslant p \leqslant l$, the function $(z - z_0)^l \hat{\mu}_j^{(l)}(z)$ may be regarded as an analytic product of $(z - z_0)^{l-p}$ and $(z - z_0)^p \hat{\mu}_j^{(l)}(z)$. It follows that we may fix arbitrary integers $p_{j,l}$ satisfying $0 \leqslant p_{j,l} \leqslant l$, and regard $(d/dz)^m \varphi(z)$ as an analytic function of z and of $(z - z_0)^{p_{j,l}} \hat{\mu}_j^{(l)}(z)$ $(0 \leqslant l \leqslant m,\ 1 \leqslant j \leqslant k)$. Thus, we arrive at the following condition, which has to be satisfied by each measure μ_j:

$$\begin{cases} \textit{For each } l = 0, 1, \ldots, m \textit{ there exists an integer } p,\ 0 \leqslant p \leqslant l, \\ \textit{such that } (d/dt - z_0)^p t^l d\mu(t) \in M(\rho). \end{cases} \tag{6.4}$$

Here $(d/dt - z_0)^p$ stands for the differentiation operator

$$\left(\frac{d}{dt} - z_0\right)^p = \sum_{q=0}^{p} \binom{p}{q} (-z_0)^{p-q} \left(\frac{d}{dt}\right)^q = e^{z_0 t} \left(\frac{d}{dt}\right)^p e^{-z_0 t}; \tag{6.5}$$

i.e., one takes the sum of derivatives of $t^l d\mu(t)$ indicated in (6.5), or equivalently, one differentiates $e^{-z_0 t} t^l d\mu(t)$ p times, and then multiplies by $e^{z_0 t}$. For $p > 0$, interpret the differentiations in the distribution sense, or equivalently, suppose that μ is induced by a sufficiently smooth function (i.e., one whose $(p-1)$th derivative is locally of bounded variation), compute the first $p-1$ derivatives in the usual way, and interpret the pth derivative as a measure derivative.

Before we continue, let us remark that if $z_0 = 0$, and if one chooses either $p = l$ or $p = 0$ in (6.4), then one gets

$$\left(\frac{d}{dt}\right)^l t^l d\mu(t) \in M(\rho) \qquad (0 \leqslant l \leqslant m) \tag{6.6}$$

and

$$(1 + |t|)^m d\mu(t) \in M(\rho), \tag{6.7}$$

respectively. Similar conditions occur in §5, and these particular ones are used later in this section.

At points different from z_0 negative powers of $(z - z_0)$ also are analytic, so for these points the restriction $p \leqslant l$ in (6.4) may be deleted. The same is true at infinity, where we also need the following absolutely continuous version of (6.4), which is applied to the functions a_q, $1 \leqslant q \leqslant n$, in Definition 2.1.

$$\begin{cases} \text{For each } l = 0, 1, \ldots, m \text{ there exists an integer } p \geqslant 0 \\ \text{such that } (d/dt - z_0)^p t^l a(t) \in L^1(\rho). \end{cases} \tag{6.8}$$

Here the same conventions are used as in (6.4).

THEOREM 6.1. *Let $\varphi(z)$ be locally analytic on $\overline{\Pi}$ with $\varphi(\infty) = 0$. In addition, suppose that at the point $z_0 \in \Pi$ φ has a representation $\varphi(z) = \psi(z, \hat{\mu}_1(z), \ldots, \hat{\mu}_k(z))$, where the order of dependence of $\psi(z, \xi_1, \ldots, \xi_k)$ on each of $\xi_1, \ldots, \xi_k$ at $(z_0, \hat{\mu}_1(z_0), \ldots, \hat{\mu}_k(z_0))$ is at least m, and where the measures μ_j satisfy* (6.4). *At points of $\overline{\Pi}$ different from z_0, suppose that φ has representations involving only functions a_q and measures μ_j satisfying* (6.8) *and* (6.4) (*without the restriction $p \leqslant l$*), *respectively. Then φ is the transform of a function $a \in L^1(\rho_1)$, where $\rho_1(t) = (1 + |t|^m)\rho(t)$.*

Before we prove Theorem 6.1, let us apply it to the problems which motivate it. In [**12**] Jordan and Wheeler study the resolvent equations

$$r_1(t) + \int_0^t a(t-s) r_1(s)\, ds = a(t), \qquad t \in R^+,$$

and

$$r_2'(t) + \int_0^t a(t-s) r_2(s)\, ds = 0, \qquad t \in R^+,\ r_2(0) = 1.$$

As we observed in §5, if $a \in L^1(R^+; \rho)$ and $1 + \hat{a}(z) \neq 0$ or $z + \hat{a}(z) \neq 0$ for $\operatorname{Re} z \geqslant \rho_*$, then r_1 or r_2, respectively, belongs to $L^1(R^+; \rho)$. However, Jordan and Wheeler observe (roughly) that if $\lim_{t\to\infty} a(t) \neq 0$, then r_1 and r_2 converge to zero faster than if $a \in L^1(R^+; \rho)$. In this case $a \notin L^1(R^+; \rho)$, so we cannot use directly the forms (5.2) and (5.3) for the transforms of the resolvents if we want to apply Theorem 6.1.

Assume, as in [**12**], that $\rho_* = 0$, and suppose that

$$a(t) = a_0(t) + \int_0^t d\mu_1(s) + \cdots + \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}\, d\mu_m(s),$$

where the function a_0 and the measures $\mu_1, \ldots, \mu_m$ satisfy (6.8) (with $p \leq l$) and (6.4) with $z_0 = 0$, respectively. Moreover, suppose that $\lim_{t\to\infty} t^{-m}\int_0^t a(s)\,ds \neq 0$, or equivalently, $\hat{\mu}_m(0) \neq 0$. Write the transform $\varphi_1(z)$ of $r_1(t)$ in the form

$$\varphi_1(z) = \frac{\hat{a}(z)}{1 + \hat{a}(z)} = 1 - z^m/\left[\hat{\mu}_m(z) + z\hat{\mu}_{m-1}(z) + \cdots + z^m(\hat{a}_0(z) + 1)\right].$$

Clearly, at the point $(0, \hat{a}_0(0), \hat{\mu}_1(0), \ldots, \hat{\mu}_m(0))$ the dependence on the transforms used in this expression is of order at least m. Thus, the hypothesis of Theorem 6.1 is satisfied at $z_0 = 0$. At points of $\overline{\Pi}$ different from zero, suppose that $1 + \hat{a}(z) \neq 0$, and write

$$\varphi_1(z) = 1 - 1/\left[\hat{a}_0(z) + 1 + z^{-1}\hat{\mu}_1(z) + \cdots + z^{-m}\hat{\mu}_m(z)\right].$$

It is clear that this expression is locally analytic, and that a_0 and the μ_j satisfy the hypothesis of Theorem 6.1 at these points. Thus, $r_1(t) \in L^1(R^+; (1 + t^m)\rho(t))$.

To get [**12**, Theorem 1] from the preceding argument, take $m = 1$, let a_0 be Jordan and Wheeler's a_1 and note that (6.7) holds, and let $d\mu_1(s)$ be their $a_2'(s)ds$ and note that (6.6) is satisfied. Theorem 2 of [**12**] may be obtained in the same way if one replaces the function $\varphi_1(z)$ by $\varphi_2(z) = (z + \hat{a}(z))^{-1}$, and assumes that $z + \hat{a}(z) \neq 0$ for $\operatorname{Re} z \geq 0$. Theorem 3 of [**12**] follows essentially from the discussion in §5.

We now return to the proof of Theorem 6.1. We begin by proving the identity

$$(6.9)\quad (z - z_0)^p\hat{\mu}^{(l)}(z) = \left[\left(\frac{d}{dt} - z_0\right)^p(-t)^l d\mu(t)\right]\hat{\ }(z), \qquad z \in \Pi \setminus \{z_0\}.$$

LEMMA 6.2. *Let $\mu \in M(\rho)$, and $(d/dt - z_0)^p t^l d\mu(t) \in M(\rho)$, where p and l are nonnegative integers, and $z_0 \in \Pi$. Then $\hat{\mu}$ is l times continuously differentiable on $\Pi \setminus \{z_0\}$, and* (6.9) *holds.*

Here we define the derivatives in the obvious way; namely, if φ is defined on Π and $z_1 \in \Pi$, then

$$(6.10)\qquad \frac{d}{dz}\varphi(z)|_{z=z_1} = \lim_{z\to z_1, z\in\Pi}\frac{1}{z - z_1}(\varphi(z) - \varphi(z_1)).$$

In particular, in the interior of Π we get the usual complex derivative.

PROOF OF LEMMA 6.2. Choose $r \in R$, $\rho_* \leq r \leq \rho^*$. Then, by (2.10), $e^{-rt}d\mu(t)$ is a finite measure, whose Fourier transform η_r satisfies $\eta_r(\omega) \equiv \hat{\mu}(r + i\omega)$ $(\omega \in R)$. Thus $t^l e^{-rt}d\mu(t)$ induces a tempered distribution, whose Fourier transform equals $i^l\eta_r^{(l)}$, where the differentiation should be interpreted in the distribution sense. Define μ_r to be the distribution

$$\mu_r = \left(\frac{d}{dt} + r - z_0\right)^p t^l e^{-rt}d\mu(t)$$
$$= \sum_{q=0}^{p}\binom{p}{q}(r - z_0)^{p-q}\left(\frac{d}{dt}\right)^q t^l e^{-rt}d\mu(t).$$

Then the distribution Fourier transform of μ_r equals the distribution

$$\sum_{q=0}^{p}\binom{p}{q}(r - z_0)^{p-q}(i\omega)^q i^l\eta_r^{(l)}(\omega) = (i\omega + r - z_0)^p i^l\mu_r^{(l)}(\omega).$$

However, since $(d/dt + r - z_0)^p = e^{(z_0 - r)t}(d/dt)^p e^{(r-z_0)t}$, we observe that

$$\mu_r = e^{-rt}\left(\frac{d}{dt} - z_0\right)^p t^l d\mu(t),$$

and so by the hypothesis and (2.10), μ_r is a finite measure. Let ν denote the measure $(d/dt - z_0)^p t^l d\mu(t)$; then the Fourier transform of μ_r equals $\hat{\nu}(r + i\omega)$ $(\omega \in R)$. Thus,

$$\eta_r^{(l)}(\omega) = (-i)^l (i\omega + r - z_0)^{-p} \hat{\nu}(r + i\omega) \tag{6.11}$$

for $-\infty < \omega < \infty$ if $r \neq \operatorname{Re} z_0$, and for $\omega \neq \operatorname{Im} z_0$ if $r = \operatorname{Re} z_0$. The right-hand side of (6.11) is a continuous function of ω, as long as $r + i\omega \neq z_0$, and this implies that η_r is l times continuously differentiable for $r + i\omega \neq z_0$ (see [**1**, p. 106]). Thus, (6.11) is also true in the classical sense for $r + i\omega \neq z_0$. Substitute $\eta_r(\omega) = \hat{\mu}(r + i\omega)$ in (6.11) to get

$$(r + i\omega - z_0)^p \left(-i\frac{d}{d\omega}\right)^l \hat{\mu}(r + i\omega) = (-1)^l \hat{\nu}(r + i\omega) \tag{6.12}$$

for $r + i\omega \in \Pi \setminus \{z_0\}$.

This is almost the same as (6.9). The only difference is that in (6.12) one uses derivatives in the imaginary direction rather than derivatives in the sense of (6.10). Of course, if $\rho_* = \rho^*$, then (6.9) and (6.12) are equivalent, and the proof is complete. If $\rho_* < \rho^*$, then one may use a general complex function argument to strengthen (6.12) to (6.9). If z_0 is an interior point of Π, then $\hat{\mu}$ and $\hat{\nu}$ are analytic at z_0, and (6.12) then shows that $\hat{\nu}$ has a zero of order at least p at z_0. In particular, the function

$$\psi(z) = (-1)^l (z - z_0)^{-p} \hat{\nu}(z) \tag{6.13}$$

is continuous on Π, and analytic on the interior of Π. If z_0 is a boundary point, then it is still true that the function ψ defined in (6.13) is continuous on $\Pi \setminus \{z_0\}$, and analytic on the interior of Π. The interior of Π is simply connected, and this implies that the function ψ has an lth order integral function $\psi^{(-l)}$, whose lth derivative equals ψ. By (6.12), (6.13), the analyticity of $\hat{\mu}$ and $\psi^{(-l)}$ in the interior of Π, and their continuity on $\Pi \setminus \{z_0\}$, we obtain $\psi^{(-l)}(z) = \hat{\mu}(z) + p(z)$ for $z \in \Pi \setminus \{z_0\}$, where $p(z)$ is a polynomial of degree at most $l - 1$. Thus, $\hat{\mu}$ is l times continuously differentiable on $\Pi \setminus \{z_0\}$ (in the sense of (6.10)), and (6.9) holds as claimed. $\square$

PROOF OF THEOREM 6.1. By Proposition 2.3, $\varphi = \hat{a}$ for some $a \in L^1(\rho)$.

Let $z \in \Pi \setminus \{z_0\}$. Then by the hypothesis and Lemma 6.2, φ is m times continuously differentiable in a neighborhood of z, and $\varphi^{(m)}$ is locally analytic at z. At infinity φ has a representation of the form

$$\varphi(z) = \psi\big(z^{-1}, \hat{a}_1(z), \ldots, \hat{a}_n(z), \hat{\mu}_1(z)/z, \ldots, \hat{\mu}_k(z)/z\big),$$

where ψ is analytic at $(0, 0, \ldots, 0)$, and $a_1, \ldots, a_n, \mu_1, \ldots, \mu_k$ satisfy (6.8) and (6.4) (without the restriction $p \leqslant l$). By Lemma 6.2, one may differentiate φ m times, and one gets an analytic function of z^{-1}, $\hat{a}_q^{(l)}(z)$ $(0 \leqslant l \leqslant m,\ 1 \leqslant q \leqslant n)$, and of $\hat{\mu}_j^{(l)}(z)/z$ $(0 \leqslant l \leqslant m,\ 1 \leqslant j \leqslant k)$. Moreover, still by Lemma 6.2, $\varphi^{(m)}$ is locally analytic at infinity.

At z_0 φ has a representation of the form (6.3), valid in a neighborhood U of z_0. Again, by Lemma 6.2 and the hypothesis, φ is m times continuously differentiable on $U \setminus \{z_0\}$. Starting from (6.3) one can show inductively that $\varphi^{(l)}$ $(0 \leq l \leq m)$ is of the form

$$\varphi^{(l)} = \tilde{\psi}_1 + (z - z_0)^{m-l}\tilde{\psi}_2 \tag{6.14}$$

in $U \setminus \{z_0\}$, where $\tilde{\psi}_1$ is analytic at z_0, and $\tilde{\psi}_2$ is an analytic function of z and of $(z - z_0)^r \hat{\mu}_j^{(r)}(z)$ $(0 \leq r \leq l,\ 1 \leq j \leq k)$. In particular, by (6.4) and Lemma 6.2 (recall the discussion preceding (6.4)), $\tilde{\psi}_2$ is continuous at z_0, and so it follows inductively from (6.14) that φ is also m times continuously differentiable at z_0. Moreover, $\varphi^{(m)}$ is locally analytic at z_0. Define $\psi(z) = \varphi^{(m)}(z)$ $(z \in \Pi)$. Then ψ is locally analytic, and $\psi(\infty) = 0$. By Proposition 2.3, there exists a function $b \in L^1(\rho)$ such that $\psi = \hat{b}$.

To complete the proof it suffices to show that $b(t) = (-1)^m a(t)$.

Choose an arbitrary $r \in [\rho_*, \rho^*]$ (e.g. $r = \rho_*$ will do). Then by (2.6), the function $e^{-rt}a(t)$ is integrable. Thus, $(-t)^m e^{-rt}a(t)$ induces a tempered distribution, whose Fourier transform equals $\varphi^{(m)}(z)$, restricted to the line $\operatorname{Re} z = r$. On the other hand, this is also the Fourier transform of $e^{-rt}b(t)$. Thus, $(-t)^m a(t) = b(t)$. $\square$

7. On the use of nonintegrable functions in locally analytic representations. In this section we extend our results to include the situation where some of the functions or measures used in the representation for $\varphi(z)$ do not belong to $L^1(\rho)$ or $M(\rho)$.

We use the standard conventions $1/\infty = 0$, $1/0 = \infty$, $a + \infty = \infty$ $(a \neq \infty)$ and $a \cdot \infty = \infty$ $(a \neq 0)$. The expressions $\infty + \infty$, ∞/∞ and $0 \cdot \infty$ are undefined. As usual, we say that $\psi(\xi_1, \xi_2, \ldots, \xi_k)$ is analytic at $(\eta_1, \eta_2, \ldots, \eta_k)$, where some of the components η_j may be ∞, provided that we can write $\psi(\xi_1, \xi_2, \ldots, \xi_k) = \tilde{\psi}(\xi_1^{\pm 1}, \xi_2^{\pm 1}, \ldots, \xi_k^{\pm 1})$ in a neighborhood of $(\eta_1, \eta_2, \ldots, \eta_k)$ where $\tilde{\psi}$ is analytic at $(\xi_1^{\pm 1}, \xi_2^{\pm 1}, \ldots, \xi_k^{\pm 1})$. Here the exponent -1 is used at precisely those components for which $\eta_j = \infty$.

DEFINITION 7.1. *Let θ be a function defined on $\overline{\Pi}$ possibly assuming the value ∞. We say that θ is extended locally analytic at $z_0 \in \overline{\Pi}$ if θ is locally analytic (in the sense of Definition 2.1) at z_0 when $\theta(z_0) \neq \infty$, and $1/\theta$ is locally analytic at z_0 when $\theta(z_0) = \infty$. We call θ extended locally analytic when it is extended locally analytic at each point of $\overline{\Pi}$.*

Clearly, the function $\theta(z) = z$ is extended locally analytic. We remark that an equivalent definition would result if we were to replace $\theta(z_0) = \infty$ in Definition 7.1 by $\theta(z_0) \neq 0$.

Our first result gives a sufficient condition for a function φ to be extended locally analytic.

THEOREM 7.2. *Let φ be defined on $\overline{\Pi}$ possibly assuming the value ∞. Suppose that at each point $z_0 \in \overline{\Pi}$, there exist extended locally analytic functions $\theta_1, \theta_2, \ldots, \theta_k$, and a function $\psi(\xi_1, \xi_2, \ldots, \xi_k)$ analytic at $(\theta_1(z_0), \theta_2(z_0), \ldots, \theta_k(z_0))$ so that in a neighborhood of z_0,*

$$\varphi(z) = \psi(\theta_1(z), \theta_2(z), \ldots, \theta_k(z)) \tag{7.1}$$

if $\varphi(z_0) \neq \infty$, *while*

$$1/\varphi(z) = \psi(\theta_1(z), \theta_2(z), \ldots, \theta_k(z)) \tag{7.2}$$

if $\varphi(z_0) = \infty$. *Then* φ *is extended locally analytic. If, in addition,* φ *is complex-valued on* $\overline{\Pi}$, *then* φ *is locally analytic on* $\overline{\Pi}$, *and* φ *is the transform of an element of* $V(\rho)$. *If also* $\varphi(\infty) = 0$, *then* φ *is the transform of a function in* $L^1(\rho)$.

As usual, the number k and the functions ψ and θ_j may vary from point to point in Theorem 7.2.

The proof of the first part of Theorem 7.2 is a trivial consequence of the definitions of local and extended local analyticity together with the fact that compositions of analytic mappings are analytic. Of course, the last part of Theorem 7.2 follows from Proposition 2.3.

A first indication of the usefulness of Theorem 7.2 is the fact that it may be applied to obtain necessary and sufficient conditions for the resolvent r_1 or r_2, defined by (1.5) and (1.6), to belong to $L^1(R^+; \rho)$ in the case where a or μ does not necessarily belong to $L^1(R^+; \rho)$ or $M(R^+; \rho)$.

Specifically, let ρ be a weight on R^+, and let a be a locally integrable function on R^+ such that $e^{-st}a(t) \in L^1(R^+)$ for each real $s > \rho_*$. In addition, assume that for each $y \in R$,

$$\hat{a}(\rho_* + iy) \equiv \lim_{s \to \rho_*^+} \hat{a}(s + iy)$$

exists as a value in the extended complex plane. Finally, assume that $\hat{a}(z) \to 0$ as $z \to \infty$ in Π. The class of such kernels a will be denoted by $S(R^+; \rho_*)$. We have

PROPOSITION 7.3. *Let* $a \in S(R^+; \rho_*)$. *Then* $r_1 \in L^1(R^+; \rho)$ *if and only if* $\hat{a}(z)$ *is extended locally analytic on* $\overline{\Pi}$ *and* $1 + \hat{a}(z) \neq 0$ *for* $z \in \Pi$.

We remark that if $a \in S(R^+; \rho_*)$ and $r_1 \in L(R^+; \rho)$, then $a \notin L^1(R^+; \rho)$ if and only if $\hat{a}(z) = \infty$ for some z satisfying $\operatorname{Re} z = \rho_*$. ($\hat{a}(z)$ is always finite for z in the interior of Π whenever $a \in S(R^+; \rho_*)$.) On the other hand, under the same assumption, $\hat{a}(z) = \infty$ for some z satisfying $\operatorname{Re} z = \rho_*$ if and only if $a \notin L^1(R^+; e^{-\rho_* t})$. Thus, those functions $a \notin L^1(R^+; \rho)$ whose resolvents belong to $L^1(R^+; \rho)$ cannot be integrable even with respect to the weight $e^{-\rho_* t}$.

To prove Proposition 7.3 recall that the function φ_1 defined in (5.2) is the Laplace transform of r_1. Thus, if a is extended locally analytic on $\overline{\Pi}$ and $1 + \hat{a}(z) \neq 0$ for $z \in \Pi$, Theorem 7.2 with $\varphi_1(z) = \psi(\hat{a}(z)) \equiv \hat{a}(z)[1 + \hat{a}(z)]^{-1}$ yields that φ_1 is the Laplace transform of a function in $L^1(R^+; \rho)$; hence, $r_1 \in L^1(R^+; \rho)$ by uniqueness of Laplace transforms. Conversely, if $r_1 \in L^1(R^+; \rho)$, we first observe that $1 + \hat{a}(z) \neq 0$ for $z \in \Pi$ by taking transforms in equation (1.5). Next, note that $\varphi_1(z_0) = 1$ at precisely those points $z_0 \in \Pi$ where $\hat{a}(z_0) = \infty$. Then by writing $\hat{a}(z) = \varphi_1(z)[1 - \varphi_1(z)]^{-1}$ in a neighborhood of $z_0 \in \overline{\Pi}$ if $\hat{a}(z_0) \neq \infty$, and $1/\hat{a}(z) = [1 - \varphi_1(z)]/\varphi_1(z)$ in a neighborhood of $z_0 \in \Pi$ if $\hat{a}(z_0) = \infty$, we can apply Theorem 7.2 to deduce also that $\hat{a}$ is extended locally analytic on $\overline{\Pi}$.

We next turn to the corresponding result for the differential resolvent r_2. Let $T(R^+;\rho_*)$ consist of those locally finite Borel measures μ on R^+ such that $e^{-st}d\mu(t) \in M(R^+)$ for each $s > \rho_*$, and which also have the properties that

$$\hat{\mu}\left(\rho_* + iy\right) \equiv \lim_{s \to \rho_*^+} \hat{\mu}(s + iy)$$

exists in the extended complex plane for each $y \in R$, and that $\hat{\mu}(z) \to \hat{\mu}(\infty)$ (finite) as $z \to \infty$ in Π.

The result for r_2 is somewhat different from Proposition 7.3. It is true that if $\hat{\mu}$ is extended locally analytic on $\overline{\Pi}$ and $z + \hat{\mu}(z) \neq 0$ for $z \in \Pi$, then both r_2 and r_2' belong to $L^1(R^+;\rho)$; to obtain this conclusion simply apply Theorem 7.2 to the Laplace transforms φ_2 and φ_3 of r_2 and r_2' given by

$$\varphi_2(z) = [z + \hat{\mu}(z)]^{-1}, \quad \varphi_3(z) = -\hat{\mu}(z)[z + \hat{\mu}(z)]^{-1},$$

respectively. On the other hand, if $r_2 \in L^1(R^+;\rho)$, then $z + \hat{\mu}(z) \neq 0$ for $z \in \Pi$, and, by Theorem 7.2 again, $\hat{\mu}$ is extended locally analytic on Π. But even if r_2' also belongs to $L^1(R^+;\rho)$, one cannot use Theorem 7.2 to deduce that $\hat{\mu}$ is extended locally analytic at infinity. However, the condition $r_2' \in L^1(R^+;\rho)$ does imply that $\varphi_3(\infty) = 0$, and then it follows by Theorem 7.2 that $\hat{\mu}(z)/z = -\varphi_3(z)[1 + \varphi_3(z)]^{-1}$ is locally analytic at infinity. Moreover, by writing $\varphi_2(z) = z^{-1}[1 + \hat{\mu}(z)/z]^{-1}$ and $\varphi_3(z) = -[\hat{\mu}(z)/z][1 + \hat{\mu}(z)/z]^{-1}$ in a neighborhood of $z_0 = \infty$, we see that, at $z_0 = \infty$, the condition that $\hat{\mu}(z)$ be locally analytic can be weakened to $\hat{\mu}(z)/z$ is locally analytic, and the conclusion that r_2 and r_2' both belong to $L^1(R^+;\rho)$ still follows. Thus, we have

PROPOSITION 7.4. *Let $\mu \in T(R^+;\rho_*)$. Then r_2 and r_2' both belong to $L^1(R^+;\rho)$ if and only if $\hat{\mu}(z)$ is extended locally analytic on Π, $\hat{\mu}(z)/z$ is locally analytic at infinity, and $z + \hat{\mu}(z) \neq 0$ for $z \in \Pi$.*

To further illustrate the usefulness of Theorem 7.2 we briefly discuss how our methods can be used to easily obtain a recent result of G. Gripenberg [**5**] on the integrability of the matrix resolvent R, defined by $R(t) + R * A(t) = A(t)$, $t \in R^+$, of a system of Volterra integral equations. Here $A = (a_{jk})$ and $R = (r_{jk})$ are $n \times n$ matrices of locally integrable functions.

In [**5**, Theorem 1] Gripenberg assumes that $\rho(t) \equiv 1$, and that there exists an element $\sigma \in S_n$ where $S_n \equiv$ symmetric group on $\{1,\dots,n\}$ so that

$$a_{jk} \in L^1(R^+) \quad \text{for } k \neq \sigma(j),\ 1 \leqslant j, k \leqslant n. \tag{7.3}$$

Under this assumption he gives necessary and sufficient conditions for $R(t)$ to belong to $L^1(R^+)$, i.e., for $r_{jk} \in L^1(R^+)$ for $1 \leqslant j, k \leqslant n$. For the precise statement of this theorem, we refer the reader to [**5**, Theorem 1], but, roughly, the necessary conditions include hypotheses which insure that the components a_{jk} of A that do not belong to $L^1(R^+)$ have Laplace transforms that, in the terminology used here, are extended locally analytic on $\overline{\Pi} = \{z \mid \operatorname{Re} z \geqslant 0\} \cup \{\infty\}$.

The following elementary lemma lays bare the reason for the assumption (7.3) if one uses local analyticity to investigate the integrability of $R(t)$. In [**5**] Gripenberg gives examples which show the complications that arise when assumption (7.3) is dropped.

LEMMA 7.5. *Let $H = (\eta_{jk})$ denote an n by n matrix of constants in the extended complex plane. Define the matrix-valued function $\Psi = (\psi_{jk})$ by*

$$\Psi(\Xi) = \Xi \operatorname{adj}(I + \Xi)/\det(I + \Xi)$$

where $\Xi = (\xi_{jk})$. For each component ψ_{jk} to be an analytic function of $(\xi_{11}, \dots, \xi_{nn})$ at $(\eta_{11}, \dots, \eta_{nn})$, it is necessary and sufficient that (i) *there exists $\sigma \in S_n$ so that $\eta_{jk} \neq \infty$ for $k \neq \sigma(j)$, and* (ii) *$|\det(I + \Xi)| \prod_{j \in J} |\xi_{j\sigma(j)}|^{-1}$ is bounded away from zero for $(\xi_{11}, \dots, \xi_{nn})$ in a neighborhood of $(\eta_{11}, \dots, \eta_{nn})$ where $J \equiv \{j \colon \eta_{j\sigma(j)} = \infty\}$.*

The proof of Lemma 7.5 is an elementary consequence of the fact that

$$\det(I + \Xi) = \sum_{\tau \in S_n} (\operatorname{sign} \tau)\big(\delta_{1\tau(1)} + \xi_{1\tau(1)}\big) \cdots \big(\delta_{n\tau(n)} + \xi_{n\tau(n)}\big),$$

and the definition of analyticity of the functions ψ_{jk} at $(\eta_{11}, \dots, \eta_{nn})$ when some of the coordinates are infinite (see the beginning of this section).

Propositions 7.3 and 7.4 show that extended local analyticity of $\hat{a}$ or $\hat{\mu}$ is the key question to be decided when using transform methods to determine whether the resolvent r_1 or r_2 defined by (1.5) or (1.6), respectively, belongs to $L^1(R^+; \rho)$ in the case where a or μ does not belong to $L^1(R^+; \rho)$ or $M(R^+; \rho)$. Unfortunately, it is difficult to determine whether a given function or measure has an extended locally analytic transform.

In the case where $\rho(t) \equiv 1$ for $t \in R^+$ and $\Pi = \{z \mid \operatorname{Re} z \geqslant 0\}$, we have the deep result due to Shea and Wainger [**16**, Theorem 2, Condition (6)]. They assume that $\varphi(z) = \psi(z, \hat{a}(z))$ with $a(t) = b(t) + \beta(t)$ where $b \in L^1_{\mathrm{loc}}(R^+)$, $b \notin L^1(R^+)$ is nonnegative, nonincreasing and convex on $(0, \infty)$, and $\beta \in L^1(R^+)$. (Actually [**16**, Theorem 2, Condition (6)] also requires that $t\beta(t) \in L^1(R^+)$; this assumption is removed in [**11**].) The function ψ is analytic on $\{(z, \hat{a}(z)) \mid z \in \overline{\Pi}\}$ with $\psi(\infty, 0) = 0$, and the conclusion asserts that φ is the transform of a function in $L^1(R^+)$. Since $1 + \hat{b}(z) \neq 0$ for $z \in \Pi$, Shea and Wainger's result may be applied in the special case where $\psi(z, \omega) = \omega(1 + \omega)^{-1}$ and $\beta(t) \equiv 0$ to yield that the resolvent r_1 of $a = b$ belongs to $L^1(R^+)$. Thus, by Proposition 7.3, $\hat{b}$ is extended locally analytic on $\overline{\Pi}$, and, in particular, $1/\hat{b}$ is locally analytic at $z = 0$. Of course, the general case of [**16**, Theorem 2, Condition (6)] now follows from this result by our Theorem 7.2 with $\tilde{\psi}(z, \hat{b}(z), \hat{\beta}(z)) \equiv \psi(z, \hat{b}(z) + \hat{\beta}(z))$.

Shea and Wainger's proof of the fact that the resolvent r_1 of a nonnegative, nonincreasing and convex kernel is integrable is based on a classical theorem of Hardy and Littlewood concerning Fourier transforms of functions in the Hardy space H^1 of the upper half-plane $\operatorname{Im} z \geqslant 0$. Rephrasing this part of Shea and Wainger's proof in the terminology of this paper, we get

PROPOSITION 7.6. *Let φ be continuous on $\{z \mid \operatorname{Re} z \geqslant 0\}$ and analytic in $\{z \mid \operatorname{Re} z > 0\}$, and suppose that φ restricted to the line $\operatorname{Re} z = 0$ is locally absolutely continuous. If* *there exists an integer $k \geqslant 0$ such that*

$$\tag{7.4} \begin{gathered} \varphi(z) = O\big(|z|^k\big) \qquad (z \to \infty;\ \operatorname{Re} z \geqslant 0), \\ \int_{-\infty}^{\infty} (1 + |\omega|)^{-k} |\varphi'(i\omega)| \, d\omega < \infty, \end{gathered}$$

then φ is locally analytic on $\{z \mid \operatorname{Re} z \geqslant 0\}$ with respect to $V(R^+)$. If (7.4) *holds with $k = 0$, then φ is also locally analytic at infinity.*

To get the local analyticity on $\{z \mid \operatorname{Re} z \geqslant 0\}$, define $\psi(z) = (1 + z)^{-(k+2)}\varphi(z)$, and use the argument in [**16**, pp. 323–324] (with $f_1(z) = \psi(-iz)$) to show that ψ is the Laplace transform of a function $a \in L^1(R^+)$. Thus $\varphi(z) = (1 + z)^{k+2}\hat{a}(z)$ is locally analytic on $\{z \mid \operatorname{Re} z \geqslant 0\}$ with respect to $V(R^+)$. The proof of the local analyticity at infinity when $k = 0$ is very similar. Let a be the inverse Fourier transform of $-\varphi'(i\omega)$, and use the argument in [**16**] to show that $a(t) = tb(t)$, where $b \in L^1(R^+)$. Now $\hat{b}(z)$ and $\varphi(z)$ have the same derivative in $\{z \mid \operatorname{Re} z > 0\}$, so their difference there, and hence in $\{z \mid \operatorname{Re} z \geqslant 0\}$ by continuity, is a constant. Thus, $\varphi(z) = \varphi(\infty) + \hat{b}(z)$, and φ is locally analytic at infinity.

Proposition 7.6 enables us to complete the discussion of Hannsgen's [**8**, Theorem 1.1] begun in §5. Hannsgen assumes that his function f, restricted to the line $\operatorname{Re} z = 0$, is locally absolutely continuous at the origin, and his proof shows that Proposition 7.6 may be applied. In particular, f is locally analytic at the origin. Observe that by replacing Hannsgen's application of the Hardy-Littlewood theorem by Proposition 7.6 one can drop Hannsgen's additional assumption $f'(i\omega) = O(|\omega|^{-k})$ $(\omega \to 0, \omega \in R$; k being a nonnegative integer).

Some other conditions which imply that $1/\hat{b}$ is locally analytic at $z = 0$ (with $\rho(t) \equiv 1$) when $b \notin L^1(R^+)$ may be found in [**5**, Theorem 4].

Sharp results similar to [**16**, Theorem 2, Condition (6)] and Proposition 7.6 for weighted spaces are not available at this time. However, a number of papers [**7**, **18**, **4**] and most recently [**6**] give O estimates for the rates of decay as $t \to \infty$ of the resolvents r_1 or r_2 of certain classes of nonintegrable kernels. Clearly these estimates can be used to show that the resolvents of such a kernel belong to an appropriate weighted space $L^1(R^+; \rho)$. In particular, in [**6**, Theorem 2] it is shown that the integral resolvent r_1 satisfies

$$\tag{7.5} r_1(t) = O\left(t^{-2}\int_0^t \left(1 + \int_0^s a(u)\, du\right)^{-1} ds\right) \qquad (t \to \infty)$$

whenever $a \in L^1_{\mathrm{loc}}(R^+)$ is nonnegative, nonincreasing and convex on $(0, \infty)$, with $-a'$ convex on $(0, \infty)$, $a(t) \to 0$ as $t \to \infty$ and $a \notin L^1(R^+)$. We remark that this result of Gripenberg gives the sharp estimate that $r_1(t) = O(t^{-(1+m)})$ $(t \to \infty)$ in the case where $a(t) = t^{m-1}$, $0 < m < 1$ (actually, $\lim_{t\to\infty} t^{1+m} r_1(t)$ exists and is non-zero). On the other hand, when $a(t) = (1 + t)^{-1}$, then (7.5) becomes $r_1(t) = O((t \log t)^{-1})$; this estimate is not sharp as it is not difficult to show that for this a, $r_1(t) = O(t^{-1}(\log t)^{-2})$.

We conclude this section with a brief discussion of the case where a locally analytic function φ has fractional order zeros on the boundary of Π. As we observed in §3, while the zeros of φ in the interior of Π are of finite integral order, the zeros of φ on the boundary of Π need not be of integral or even finite order.

We consider only the case where $\rho(t) \equiv 1$ on R^+; thus, $L^1(\rho)$ is $L^1(R^+)$, $\rho_* = 0$ and $\Pi = \{z \mid \operatorname{Re} z \geqslant 0\}$. For any $m \in R$ and $\omega \in \Pi$, ω^m denotes the principal mth power of ω, namely, $\omega^m = |\omega|^m \exp(\mathrm{i}m \arg \omega)$ where $-\pi/2 \leqslant \arg \omega \leqslant \pi/2$. Let Γ denote the gamma function $\Gamma(m) \equiv \int_0^\infty t^{m-1}e^{-t}\,dt$ $(m > 0)$. As we have observed, the resolvent r_1 of $a(t) = t^{m-1}/\Gamma(m)$ $(0 < m < 1)$ satisfies $t^{(1+m)}r_1(t) \to \beta \neq 0$ $(t \to \infty)$. In particular, $r_1 \in L^1(R^+)$ and Proposition 7.3 yields that $\hat{a}(z) = z^{-m}$ or, equivalently, z^m is extended locally analytic with respect to (w.r.t.) $V(R^+)$. Clearly, if $\operatorname{Re} z_0 = 0$, then $e^{z_0 t}a(t)$ has Laplace transform $(z - z_0)^{-m}$ and resolvent $e^{z_0 t}r_1(t) \in L^1(R^+)$; thus, Proposition 7.3 again yields that $(z - z_0)^m$ is extended locally analytic w.r.t $V(R^+)$. Finally, since $(z - z_0)^k$ is always extended locally analytic for any integer k, $(z - z_0)^{m+k}$ is extended locally analytic w.r.t. $V(R^+)$. Thus, we have

LEMMA 7.7. *Let z_0 satisfy $\operatorname{Re} z_0 = 0$ and m be a real number. Then $(z - z_0)^m$ is extended locally analytic w.r.t. $V(R^+)$.*

We pause to remark that, in contrast to integral powers of z, fractional powers of z need not be extended locally analytic w.r.t. weighted spaces. In particular, as Proposition 7.3 and the discussion preceding Lemma 7.7 show, z^m $(0 < m < 1)$ is not locally analytic w.r.t. $V(R^+; \rho)$ at zero whenever $\rho(t)$ is a weight on R^+ with $\rho_* = 0$ and $\int_0^\infty (1 + t)^{-(m+1)}\rho(t)\,dt = \infty$. Also, it is easy to see that if $\liminf_{t\to\infty} t^{-m}\rho(t) > 0$ for some positive integer m, and $\operatorname{Re} z_0 = 0$, then $(z - z_0)^\gamma$ is not extended locally analytic w.r.t. $V(R^+; \rho)$ at z_0 for $-m < \gamma < m$ and γ noninteger (because $(z - z_0)^\gamma$ is not smooth of order m at z_0 w.r.t. $V(R^+)$ for fractional values of γ, $0 < \gamma < m$).

Let z_0 satisfy $\operatorname{Re} z_0 = 0$, φ be locally analytic w.r.t. $V(R^+)$ at z_0, and m be any positive number. We define z_0 to be *a zero of φ of order at least m* or a *zero of φ of order m* just as in Definition 3.1, the only difference being that the requirement that m be a positive integer is now omitted. Similarly, a *singularity* (pole) *of order at most m* at z_0 satisfying $\operatorname{Re} z_0 = 0$ or a *singularity of order m* at z_0 is defined as in Definition 3.2 where again the requirement that m be a positive integer has been relaxed to m being a positive number. Clearly, Lemma 3.3 still holds for fractional order zeros and singularities on $\operatorname{Re} z = 0$.

Using the above definitions of fractional order zeros and singularities on $\operatorname{Re} z = 0$, we easily get the following extension of the criterion in Proposition 5.1 giving the nature of the integrable perturbations in equation (1.1) which give rise to integrable solutions x.

PROPOSITION 7.8. *Let $a \in L^1(R^+)$, and let the function $1 + \hat{a}(z)$ have finitely many zeros $z_1, \ldots, z_n$ in Π of order m_j (where m_j is a positive integer when $\operatorname{Re} z_j > 0$). Suppose that x and f are locally integrable and satisfy equation* (1.1). *For $1 \leqslant j \leqslant n$ and $0 < l \leqslant m_j$, define*

$$F_{l,j} = \frac{1}{\Gamma(l)} \int_0^t (t - s)^{l-1} e^{z_j(t-s)} f(s)\,ds, \qquad t > 0. \tag{7.6}$$

Then $x \in L^1(R^+)$ *if and only if* f *and* $F_{l,j}$ *belong to* $L^1(R^+)$ *for* $1 \leqslant j \leqslant n$ *and* $0 < l \leqslant m_j$. *Here* l *is restricted to integral values when* $\operatorname{Re} z_j > 0$.

The corresponding result for equation (1.2) is also valid.

PROOF OF PROPOSITION 7.8. The proof that $x \in L^1(R^+)$ implies that f and $F_{l,j}$ belong to $L^1(R^+)$ for $1 \leqslant j \leqslant n$, $0 < l \leqslant m_j$ (with l integral when $\operatorname{Re} z_j > 0$) is completely similar to the proof of the corresponding portion of Proposition 5.2 and is left to the reader.

Conversely, assume that f and $F_{m_j,j}$ $(1 \leqslant j \leqslant n)$ belong to $L^1(R^+)$. Define

$$\varphi(z) = \frac{\hat{f}(z)}{1 + \hat{a}(z)} \qquad \big(z \in \overline{\Pi} \setminus \{z_1, \ldots, z_n\}\big).$$

Then φ is locally analytic on $\overline{\Pi} \setminus \{z_1, \ldots, z_n\}$ and $\varphi(\infty) = 0$. Moreover, in a neighborhood of z_j $(1 \leqslant j \leqslant n)$, we may write

$$\varphi(z) = \hat{F}_{m_j,j}(z)(z - z_j)^{m_j}(1 + \hat{a}(z))^{-1} = \hat{F}_{m_j,j}(z)\psi_j(z)^{-1},$$

where, by the definition of a zero of order m_j, $\psi_j(z) \equiv (z - z_j)^{-m_j}(1 + \hat{a}(z))$ is locally analytic and nonzero at $z = z_j$. Thus, φ is locally analytic at z_j $(1 \leqslant j \leqslant n)$; hence, by Proposition 2.3, $\varphi(z) = \hat{b}(z)$ for some $b \in L^1(R^+)$. Since the Laplace transform of x converges and $(1 + \hat{a}(z))\hat{x}(z) = \hat{f}(z)$ holds in some half-plane $\operatorname{Re} z > \gamma$, we see that $\hat{x}(z) = \varphi(z) = \hat{b}(z)$ for $\operatorname{Re} z$ sufficiently large; hence, $x = b \in L^1(R^+)$. $\square$

8. Remarks on local analyticity on $M(\rho)$. In this section we consider the weighted space of measures $M(\rho)$, and state and prove an analogue of Proposition 2.3 appropriate to this setting. We then indicate how this result can be used to analyze equation (1.7).

Let $\mu \in M(\rho)$ and decompose μ into its absolutely continuous, discrete, and singular parts $\mu = \mu_a + \mu_d + \mu_s$. Denote by $\alpha(\mu)$ the spectral radius of μ, that is,

$$\alpha(\mu) = \lim_{n \to \infty} \|\mu^{n*}\|^{1/n}$$

where μ^{n*} is the n-fold convolution of μ with itself. Define

$$K(\mu) = \big\{w \in \mathbf{C} \,\big|\, |w - \hat{\mu}_d(z)| \leqslant \alpha(\mu_s) \text{ for some } z \in \Pi\big\}^{-}. \tag{8.1}$$

Here the bar denotes closure.

Let $\mathfrak{M}(\rho)$ denote the maximal ideal space of $M(\rho)$ and recall that if $\mu \in M(\rho)$ and $M \in \mathfrak{M}(\rho)$, then the Gelfand transform $\mu(M)$ satisfies $|\mu(M)| \leqslant \alpha(\mu)$ [**3**, p. 36]. Also, let $\mathfrak{M}_a(\rho) \subseteq \mathfrak{M}(\rho)$ denote the collection of all absolutely continuous maximal ideals. Then $\mathfrak{M}_a(\rho)$ can be identified with Π, and if $z \in \Pi$ and M_z is the associated element of $\mathfrak{M}_a(\rho)$, then $\mu(M_z) = \hat{\mu}(z)$ [**2**, p. 58; **3**, p. 176; **9**, p. 148]. Note that $K(\mu)$ is compact and nonempty, and that the distance from $\hat{\mu}(z)$ to $K(\mu)$ tends to zero as $|z| \to \infty$ in Π since $\hat{\mu}_a(z) \to 0$ $(|z| \to \infty)$ and $|\hat{\mu}_s(z)| \leqslant \alpha(\mu_s)$. Thus, we can make the following

DEFINITION 8.1. *We call a function* φ *defined on* Π *pseudo locally analytic* $(M(\rho))$ *if at each point* $z_0 \in \Pi$ *it is locally analytic in the sense of Definition* 2.1, *and if there* *exist measures* $\mu_1,\dots,\mu_k \in M(\rho)$ *and a function* $\psi(z,\xi_1,\dots,\xi_k)$ *analytic in a neighborhood of* $(0, K(\mu_1),\dots,K(\mu_k))$ *such that* $\varphi(z) = \psi(z^{-1}, \hat{\mu}_1(z),\dots,\hat{\mu}_k(z))$ *in a deleted neighborhood of infinity in* Π.

We remark that, in contrast to Definition 2.1, Definition 8.1 does not require that $\varphi(z)$ tend to a limit as $z \to \infty$. An example illustrating this is provided by setting $\rho(t) \equiv 1$ and $\varphi(z) = \hat{\delta}_x(z)$ where δ_x is the unit point mass measure at a point $x \neq 0$.

Theorem A, §2, specializes Theorem 1 of [**3**, p. 82] to the ring $V(\rho)$, and our proof of Proposition 2.3 reduces to showing that a function locally analytic (Definition 2.1) is locally analytic [**2**] (i.e. in the sense of Definition 1 [**3**, p. 82] using the ring $V(\rho)$). In the same manner, suppose we can show that every function φ defined on Π which is pseudo locally analytic $(M(\rho))$ can be defined on the entire maximal ideal space $\mathfrak{M}(\rho)$ in such a manner that the extension is locally analytic in the sense of Definition 1 [**3**, p. 82] using the ring $M(\rho)$. Then we can apply Theorem 1 of [**3**, p. 82] specialized to the ring $M(\rho)$ to get

PROPOSITION 8.2. *Let* φ *be pseudo locally analytic* $(M(\rho))$. *Then there exists a unique measure* $\mu \in M(\rho)$ *such that* $\varphi(z) = \hat{\mu}(z)$ $(z \in \Pi)$.

The proof given below is adapted from [**2**, pp. 63–65].

PROOF. That μ is unique, if it exists, follows from the fact that $\hat{\mu}(z)$ $(z \in \Pi)$ determines μ uniquely [**2**, p. 64; **3**, pp. 176–177; **9**, p. 149].

At every point $z \in \Pi$, Definitions 8.1 and 2.1 are equivalent and $\mu(M_z) = \hat{\mu}(z)$. Thus the proof that $\varphi(M_z) = \varphi(z)$ is locally analytic at each $M_z \in \mathfrak{M}_a(\rho)$ is a simplification of the first part of the proof of Proposition 2.3 and is omitted.

We still have to show that we can define φ on $\mathfrak{M}(\rho) \setminus \mathfrak{M}_a(\rho)$ in such a way that it becomes locally analytic there. As before, set $e(t) = \exp((\rho_* - 1)t)$ $(t \geq 0)$, $e(t) = 0$ $(t < 0)$, and note that since e induces an absolutely continuous measure in $M(\rho)$, $e(M) = 0$ for $M \in \mathfrak{M}(\rho) \setminus \mathfrak{M}_a(\rho)$ [**2**, p. 64; **3**, p. 178; **9**, pp. 148–149]. By Definition 8.1 we have $\varphi(z) = \psi(z^{-1}, \hat{\mu}_1(z),\dots,\hat{\mu}_k(z))$ for all z in a deleted neighborhood U_∞ of infinity in Π. Define

$$\tilde{\psi}(w,\xi_1,\dots,\xi_k) = \psi\left(\frac{w}{1 + (\rho_* - 1)w}, \xi_1,\dots,\xi_k\right),$$

and observe that $\tilde{\psi}$ is analytic in a neighborhood of $(0, K(\mu_1),\dots,K(\mu_k))$, and that

$$\varphi(z) = \tilde{\psi}(\hat{e}(z), \hat{\mu}_1(z),\dots,\hat{\mu}_k(z)) \qquad (z \in U_\infty).$$

Changing the notation slightly, if M is the maximal ideal corresponding to a point $z \in U_\infty$, then we have

$$\varphi(M) = \tilde{\psi}(e(M), \mu_1(M),\dots,\mu_k(M)). \tag{8.2}$$

We claim

$$\mu_j(M) \in K(\mu_j) \qquad (j = 1,\dots,k) \tag{8.3}$$

for every $M \in \mathfrak{M}(\rho) \setminus \mathfrak{M}_a(\rho)$. Assume this for the moment. Then we can use (8.2) (with $e(M) = 0$) as a definition of $\varphi(M)$ for every $M \in \mathfrak{M}(\rho) \setminus \mathfrak{M}_a(\rho)$. This means that (8.2) holds for every $M \in U$, where $U = U_\infty \cup \{\mathfrak{M}(\rho) \setminus \mathfrak{M}_a(\rho)\}$. The complement of U is compact, hence closed, so U is an open subset of $\mathfrak{M}(\rho)$. In particular, U is a neighborhood of every $M \in \mathfrak{M}(\rho) \setminus \mathfrak{M}_a(\rho)$. In other words, φ is also locally analytic at every $M \in \mathfrak{M}(\rho) \setminus \mathfrak{M}_a(\rho)$, and Proposition 8.2 follows from [**3**, p. 82].

We still have to show that (8.3) holds. Write $\mu_j = \mu_{ja} + \mu_{jd} + \mu_{js}$. Then, for $M \in \mathfrak{M}(\rho) \setminus \mathfrak{M}_a(\rho)$, $\mu_{ja}(M) = 0$, $|\mu_{js}(M)| \leqslant \alpha(\mu_{js})$, and $\mu_{jd}(M)$ belongs to the closure of the set $\{\hat{\mu}_{jd}(z) \mid z \in \Pi\}$ [**2**, pp. 61–62; **3**, p. 172; **9**, pp. 147, 150]. This implies that (8.3) holds. $\square$

Using the fact that $\alpha(\mu_s) \leqslant \|\mu_s\|$, one obtains a sharpened version of Satz 7 of [**2**] (with $\|\mu_s\|$ replaced by $\alpha(\mu_s)$ in the statement of that result) as an immediate corollary of Proposition 8.2.

We remark that we may replace $\alpha(\mu_s)$ in (8.1) by

$$\beta(\mu_s) = \inf_{a \in L^1(\rho)} \alpha(\mu_s + a) \tag{8.4}$$

and use these modified $K(\mu_j)$'s in Definition 8.1. Since $|\mu_s(M)| \leqslant \alpha(\mu_s + a)$ for every $M \in \mathfrak{M}(\rho) \setminus \mathfrak{M}_a(\rho)$ and $a \in L^1(\rho)$, this leads to a somewhat sharper verison of Proposition 8.2.

Since at points $z_0 \in \Pi$ Definitions 8.1 and 2.1 are identical, we can treat the notions of zeros, poles and smoothness at such points, as given in Definitions 3.1, 3.2 and 3.5, in the present context. The obvious analogues (replace $[\alpha + \hat{a}(z)]$ with $\hat{\mu}(z)$ in the conclusions) of the first parts of the quotient and remainder Theorems 3.4 and 3.6 clearly hold.

We conclude with a discussion of how the theory presented here can be used to obtain and extend Theorem 3.1 of [**13**] dealing with equation (1.7). Let $\rho(t)$ be a weight on R^+ satisfying the normalizing assumption $\rho_* = 0$ (which, as we noted in §5, causes no loss of generality). In addition we now make the assumption that $\rho(t)$ is nondecreasing on R^+. Jordan and Wheeler study locally integrable solutions x of (1.7) when μ and f belong to $M(R^+; \rho)$ and $L^1(R^+; \rho)$, respectively, and $\hat{\mu}(z)$ has a finite number of zeros $z_1, \ldots, z_n$ in $\{z \mid \operatorname{Re} z \geqslant 0\}$ none of which is equal to zero. The zeros of $\hat{\mu}(z)$ on $\operatorname{Re} z = 0$ are all assumed to have order at most M for some fixed positive integer M. Also, each of μ and f is assumed to satisfy a combination of moment and monotonicity conditions which imply, by Lemma 4.3, that $\hat{\mu}(z)$ and $\hat{f}(z)$ are smooth of orders $2M$ and M, respectively, on $\operatorname{Re} z = 0$, $z \neq 0$.

Finally, Jordan and Wheeler assume that the discrete and singular parts of μ satisfy $\inf_{-\infty < y < \infty} |\hat{\mu}_d(iy)| > \|\mu_s\|$, and that $[\hat{\mu}(z)]^{-1}$ is bounded on $\{z \mid \operatorname{Re} z \geqslant 0\}$ except near the zeros of $\hat{\mu}(z)$.

Since ρ is nondecreasing on R^+, it can be extended to be a weight on all of R by defining $\rho(t) \equiv 1$, $t < 0$, [**10**, p. 598], and then μ (extended to a measure on R by defining it as the zero measure on $(-\infty, 0)$) belongs to $M(R; \rho)$. We note that for this extension $\rho^* = \rho_* = 0$ and that $\Pi = \{z \mid \operatorname{Re} z = 0\}$. Thus, $\varphi(z) = [\hat{\mu}(z)]^{-1}$ $(z \in \Pi)$ is such that we can apply the analogue of Theorem 3.6 in this setting to get $\nu_0 \in M(R; \rho)$ and constants $\beta_{l,j}$ determined by the zeros of $\hat{\mu}(z)$ on $\operatorname{Re} z = 0$ such that

$$\hat{\nu}_0(z) = [\hat{\mu}(z)]^{-1} - \sum_{l,j} \beta_{l,j}(z - z_j)^{-l}, \qquad z \in \Pi.$$

We remark that only the z_j on $\operatorname{Re} z = 0$ occur in the above sum, and that $1 \leqslant l \leqslant M$ here. Now, since $[\hat{\mu}(z)]^{-1}$ is bounded on $\{z \mid \operatorname{Re} z \geqslant 0\}$ away from the zeros of $\hat{\mu}(z)$, we can use a modification of a function theory argument due to Paley and Wiener [**15**] (see [**10**, pp. 600–601] and replace P_0 there with ν_0) to obtain $\nu \in M(R^+ ; \rho)$ and additional constants $\beta_{l,j}$ determined by the principal parts of $[\hat{\mu}(z)]^{-1}$ at the poles z_j satisfying $\operatorname{Re} z_j > 0$ so that

$$\hat{\nu}(z) = [\hat{\mu}(z)]^{-1} - \sum_{l,j} \beta_{l,j}(z - z_j)^{-l}, \qquad \operatorname{Re} z \geqslant 0. \tag{8.5}$$

Of course, the sum in (8.5) is taken over all zeros z_j of $\hat{\mu}(z)$ in $\operatorname{Re} z \geqslant 0$. Finally, since $\hat{x}(z)$ exists for all z with $\operatorname{Re} z$ sufficiently large, we can take Laplace transforms in (1.7), and use (8.5) together with the fact that $\hat{f}(z)$ is smooth (w.r.t. $V(R^+ ; \rho)$) of order at least M on $\operatorname{Re} z = 0$, $z \neq 0$, to conclude by Proposition 2.3 that $x(t)$ may be expressed as a finite sum of products of polynomials and exponentials, plus a remainder term in $L^1(R^+ ; \rho)$.

Analogues of Propositions 5.1 and 5.3 for equation (1.7) also hold; we leave their formulations and proofs to the reader.

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