TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 274, Number 2, December 1982

ON SOME OPEN PROBLEMS OF P. TURÁN CONCERNING BIRKHOFF INTERPOLATION

BY

A. K. VARMA

ABSTRACT. In 1974 P. Turán (see [6]) raised many interesting open problems in Approximation Theory some of which are on Birkhoff Interpolation. The object of this paper is to answer some of these problems (XXXVI-XXXIX). We obtain some new quadrature formulas where function values and second derivatives are only prescribed on the zeros of

$$\pi_n(x) = c_n \int_{-1}^{x} P_{n-1}(t)\, dt,$$

$P_n(x)$ being Legendre polynomial of degree n.

1. It is well known that in Hermite interpolation certain consecutive derivatives of the interpolating polynomial are required to agree with prescribed values at the nodes of interpolation. In 1906 G. D. Birkhoff [3] considered most general the interpolation problem where consecutive derivatives of the interpolation polynomial are not necessarily prescribed. Remarking on this paper of Birkhoff, J. Suranyi and P. Turán [5] mentioned that the paper is so general that one cannot expect better formulae than those of Hermite. On this account in a series of papers [1, 2, 5] Turán and his associates have initiated the problem of (0, 2) interpolation when the values and second derivatives of the interpolatory polynomials are prescribed at the zeros of

$$\pi_n(x) = (1 - x^2)P'_{n-1}(x) = -n(n-1)\int_{-1}^{x} P_{n-1}(t)\, dt \tag{1.1}$$

where $P_{n-1}(x)$ denotes the Legendre polynomial of degree $\leqslant n - 1$. For a detail study of (0, 2) interpolation we refer to the book of Lorentz and S. D. Riemenschneider [4] on Birkhoff interpolation.

In 1974 P. Turán has raised many interesting open problems in Approximation Theory some of which are related to Birkhoff interpolation (see [6, §3, pp. 47–56]). We now describe some of these problems.

If in the nth row of matrix A there are n interpolation points

$$1 \geqslant x_{1n} > x_{2n} > \cdots > x_{n-1,n} > x_{nn} \geqslant -1$$

Received by the editors March 2, 1981 and, in revised form, November 6, 1981.
1980 *Mathematics Subject Classification*. Primary 41A05; Secondary 41A55.

then A is called "very good" if for an arbitrary set of numbers y_{kn} and y''_{kn}, there is a uniquely determined polynomial $D_n(f;\, A) = D_n(f)$ of degree at most $2n-1$ for which

$$(1.2)\qquad \begin{aligned} D_n(f;\, A)_{x=x_{kn}} &= y_{kn} = f(x_{kn}),\\ \left(\frac{d^2}{dx^2}D_n(f;\, A)\right)_{x=x_{kn}} &= y''_{kn}, \end{aligned} \qquad k=1,2,\dots,n.$$

In that case, $D_n(f;\, A)$ can be uniquely written as

$$(1.3)\qquad D_n(f;\, A) = \sum_{i=1}^{n} f(x_{in}) r_{in}(x;\, A) + \sum_{i=1}^{n} y''_{in}\rho_{in}(x;\, A)$$

where

$$(1.4)\qquad \begin{aligned} r_{in}(x_{kn}) &= 1 \quad \text{if } i=k,\\ &= 0 \quad \text{if } i\neq k, \end{aligned} \qquad r''_{in}(x_{kn}) = 0, \quad k=1,2,\dots,n,$$

$$(1.5)\qquad \rho_{in}(x_{kn}) = 0, \quad \begin{aligned} \rho''_{in}(x_{kn}) &= 1 \quad \text{if } i=k,\\ &= 0 \quad \text{if } i\neq k, \end{aligned} \qquad k=1,2,\dots,n.$$

Problem XXXVI. What is the best class of functions for which the integrals of the polynomials

$$\sum_{i=1}^{n} f(x_{in}) r_{in}(x, \pi) \qquad (n \text{ even})$$

tend to $\int_{-1}^{1} f(x)\,dx$?

Concerning the above problem P. Turán remarks (pp. 54, 55 of **[6]**) that from his theorem with J. Balazs **[2]** gives, in the case of the special π matrix, convergence of the quadrature for functions $f(x)$ which are differentiable and whose derivative belongs to a Lipschitz class with arbitrary small exponent.

Problem XXXVII. Does there exist, for every "good" matrix A, a function $f_0(x) \in C[-1, 1]$ such that

$$\overline{\lim_{n\to\infty}} \left| \int_{-1}^{1} D_n^*(f;\, A)\,dx \right| = \infty$$

where

$$D_n^*(f;\, A) = \sum_{i=1}^{n} f(x_{in}) r_{in}(x, A).$$

Problem XXXVIII. Does there exist a matrix A satisfying

$$\int_{-1}^{1} r_{kn}(x;\, A)\,dx \geqslant 0, \qquad k=1,2,\dots,n,\ n>n_0?$$

Problem XXXIX. Determine the "good" matrices for which

$$\sum_{k=1}^{n} \left| \int_{-1}^{1} r_{kn}(x;\, A)\,dx \right|$$

is minimal.

The answers to the above problems are given by the following theorems.

THEOREM 1. *Let the nth row of the matrix A be given by the zeros of $\pi_n(x)$ (as defined by* (1.1)) *and let $r_{kn}(x;\pi)$ be the fundamental functions of the first kind satisfying* (1.4). *Then for n even* ($n \geqslant 4$) *we have*

$$\int_{-1}^{1} r_{in}(x;\pi)\,dx = \int_{-1}^{1} r_{nn}(x;\pi)\,dx = \frac{3}{n(2n-1)} \tag{1.6}$$

and

$$\int_{-1}^{1} r_{kn}(x;\pi)\,dx = \frac{2(2n-3)}{n(n-2)(2n-1)P_{n-1}^2(x_{kn})}, \qquad k = 2,3,\ldots,n-1. \tag{1.7}$$

Moreover, we obtain the following quadrature formula

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx &= \frac{3}{n(2n-1)}(f(1)+f(-1)) \\ &\quad + \sum_{k=2}^{n-1} \frac{2(2n-3)f(x_{kn})}{n(n-2)(2n-1)P_{n-1}^2(x_{kn})} \\ &\equiv Q_n(f,\pi)\end{aligned} \tag{1.8}$$

exact for linear functions.

THEOREM 2. *Let the nth row of the matrix A be given by the zeros of $\pi_n(x)$ and let $\rho_{kn}(x;\pi)$ be the fundamental polynomials of the second kind satisfying* (1.5). *Then for n even* ($n \geqslant 4$) *we have*

$$\int_{-1}^{1} \rho_{1n}(x;\pi)\,dx = \int_{-1}^{1} \rho_{nn}(x;\pi)\,dx = 0 \tag{1.9}$$

and
(1.10)

$$\int_{-1}^{1} \rho_{kn}(x;\pi)\,dx = \frac{1-x_{kn}^2}{n(n-1)(n-2)(2n-1)P_{n-1}^2(x_{kn})}, \qquad k = 2,3,\ldots,n-1.$$

Further, if $f \in C^2[-1,1]$, we obtain the following quadrature formula

$$\int_{-1}^{1} f(x)\,dx = Q_n(f,\pi) + \sum_{k=2}^{n-1} \frac{(1-x_{kn}^2)f''(x_{kn})}{n(n-1)(n-2)(2n-1)P_{n-1}^2(x_{kn})} \tag{1.11}$$

exact for all polynomials up to degree $\leqslant 2n-1$, where $Q_n(f,\pi)$ are defined by (1.8).

THEOREM 3. *The best class of functions for which the integrals of the polynomials*

$$D_n^*(f,x,\pi) \equiv \sum_{k=1}^{n} f(x_{kn}) r_{kn}(x;\pi) \qquad (n\ \textit{even}) \tag{1.12}$$

tend to $\int_{-1}^{1} f(x)\,dx$ is $f \in C[-1,1]$. Also among the "good" matrices for which $\sum_{k=1}^{n} |\int_{-1}^{1} r_{kn}(x,A)\,dx|$ is minimal is given by π matrix (as defined by (1.1)) *and the minimal value of this functional is* 2.

Note. Concerning Theorem 3, see Remark 2 in §8.

2. From an earlier result of Suranyi and P. Turán [**5**] the problem of (0,2) interpolation based on the roots of $\pi_n(x)$ is uniquely solvable for even n. If

$$-1 = x_n < x_{n-1} < \cdots < x_2 < x_1 = 1 \tag{2.1}$$

are the zeros of $\pi_n(x)$ the explicit forms of the fundamental polynomials $r_k(x)$, $\rho_k(x)$ satisfying (1.4) and (1.5) respectively were obtained by J. Balazs and P. Turán [**1**]. Throughout this paper n will be assumed to be even.

THEOREM A (J. BALAZS AND P. TURÁN). *One has*

$$\rho_1(x) = \frac{-\pi_n(x)}{n^2(n-1)^2}\left[1 + \tfrac{1}{3}P_{n-1}(x)\right], \tag{2.2}$$

$$\rho_n(x) = \frac{\pi_n(x)}{n^2(n-1)^2}\left[1 - \tfrac{1}{3}P_{n-1}(x)\right], \tag{2.3}$$

and for $1 < i < n$,

(2.4)
$$\rho_i(x) = \frac{\pi_n(x)(1-x_i^2)}{2(\pi_n'(x_i))^2}\left\{\int_{-1}^{x}\frac{P_{n-1}'(t)}{t-x_i}\,dt - \left(\frac{1}{6}I_i + \frac{2}{3}\frac{x_i}{1-x_i^2}\right)P_{n-1}(x) - \left(\frac{1}{2}I_i + \frac{2}{1-x_i^2}\right)\right\}$$

where I_i *stands for the integral*

$$I_i = \int_{-1}^{1}\frac{P_{n-1}'(t)}{t-x_i}\,dt, \qquad 1 < i < n. \tag{2.5}$$

$$r_1(x) = \frac{3+x}{4}l_1^2(x) - \frac{(1-x^2)}{4}l_1(x)l_1'(x) + \left(\frac{5}{16} + \frac{1}{8n(n-1)}\right)\pi_n(x)\left(1 + \frac{1}{3}P_{n-1}(x)\right), \tag{2.6}$$

$$r_n(x) = \frac{3-x}{4}l_n^2(x) + \frac{(1-x^2)}{4}l_n(x)l_n'(x) + \left(\frac{5}{16} + \frac{1}{8n(n-1)}\right)\pi_n(x)\left(1 - \frac{1}{3}P_{n-1}(x)\right), \tag{2.7}$$

(2.8)
$$r_i(x) = l_i^2(x) + \frac{\pi_n(x)}{\pi_n'(x_i)}\left\{\int_x^1\frac{l_i'(t)}{t-x_i}\,dt + \frac{P_{n-1}(x)-3}{6}I_i'\right\}, \qquad 2 \leqslant i \leqslant n-1.$$

Here

$$l_i(x) = \frac{\pi_n(x)}{(x-x_i)\pi_n'(x_i)}, \qquad i = 1,2,\ldots,n, \tag{2.9}$$

and

$$I_i' = \int_{-1}^{1} \frac{l_i'(t)}{t - x_i}\,dt. \tag{2.10}$$

THEOREM B. (J. BALAZS AND P. TURÁN). *Let* $f(x) \in C^1[-1,1]$ *and* $f'(x) \in \operatorname{Lip}\alpha$, $\alpha > 0$. *If* $|y_{in}''| \leq \delta_n$, $n^{-1}\delta_n \to 0$ *then* $D_n(f, \pi)$ *as defined in* (1.3) *converges uniformly to* f *on* $[-1,1]$.

3. Properties of Legendre polynomials. We shall use some well-known facts about Legendre polynomials. A good reference of these properties see J. Balazs and P. Turán [1] and Lorentz and Riemenschneider [4]. One has the identities

$$\pi_n(x) = (1-x^2)P_{n-1}'(x) = \frac{n(n-1)}{2n-1}\left(P_{n-2}(x) - P_n(x)\right), \tag{3.1}$$

$$\pi_n'(x) = -n(n-1)P_{n-1}(x), \tag{3.2}$$

$$(1-x^2)P_{n-1}''(x) - 2xP_{n-1}'(x) + n(n-1)P_{n-1}(x) = 0, \tag{3.3}$$

$$(1-x^2)\pi_n''(x) + n(n-1)\pi_n(x) = 0, \tag{3.4}$$

$$(n-1)P_{n-1}(x) + P_{n-2}'(x) = xP_{n-1}'(x). \tag{3.5}$$

We shall make use of the following integrals which are easy to verify:

$$\int_{-1}^{1}(1-x^2)\left(P_{n-1}'(x)\right)^2 dx = \frac{2n(n-1)}{2n-1}, \tag{3.6}$$

$$\int_{-1}^{1}\left(P_{n-1}'(x)\right)^2 dx = n(n-1), \tag{3.7}$$

$$\int_{-1}^{1}x^2\left(P_{n-1}'(x)\right)^2 dx = \frac{(2n-3)n(n-1)}{2n-1}, \tag{3.8}$$

$$\int_{-1}^{1}(1-x^2)P_{k-1}'(x)\,dx = 0, \qquad k = 3,4,\ldots, \tag{3.9}$$
$$= \frac{4}{3} \quad \text{if } k = 2.$$

From integration by parts we obtain

$$\int_{-1}^{1}xP_{n-1}(x)P_{n-1}'(x)\,dx = \frac{2(n-1)}{2n-1}. \tag{3.10}$$

The polynomials $P_n(x)$ are orthogonal on $[-1,1]$ with weight function $\omega(x) \equiv 1$. We express this fact by

$$\int_{-1}^{1}P_n(x)P_m(x)\,dx = 0, \qquad n \neq m, \tag{3.11}$$
$$= \frac{2}{2n+1}, \qquad n = m.$$

From (3.1) and (3.11) it easily follows that

$$\int_{-1}^{1}\pi_n(x)P_{n-1}(x)\,dx = 0 \tag{3.12}$$

and

$$\int_{-1}^{1} \pi_n(x) q_{n-3}(x)\,dx = 0 \tag{3.13}$$

for arbitrary polynomial $q_{n-3}(x)$ of degree $\leqslant n-3$. It is well known that $P'_{n-1}(x)$ form an orthogonal system on $[-1, 1]$ with weight $\omega(x) = 1 - x^2$. This can be expressed by

$$\int_{-1}^{1} P'_{n-1}(x) q_{n-3}(x)(1-x^2)\,dx = 0 \tag{3.14}$$

for arbitrary polynomial $q_{n-3}(x)$ of degree $\leqslant n-3$.

4. Preliminaries. Since x_k's $(k = 2, 3, \ldots, n-1)$ are zeros of $P'_{n-1}(x)$, therefore $P'_{n-1}(x)/(x - x_k)$ is indeed a polynomial of degree $n-3$. Now on using (3.14) we obtain

$$\int_{-1}^{1} (1-x^2) \frac{P'_{n-1}(x)}{x - x_k} P'_{n-1}(x)\,dx = 0, \qquad k = 2, 3, \ldots, n-1. \tag{4.1}$$

From [**4**, Chapter XII] it follows that for $n \geqslant 3$ (Christoffel-Darboux formula for $P'_{n-1}(x)$)

$$\frac{P'_{n-1}(t)P'_{n-2}(x) - P'_{n-1}(x)P'_{n-2}(x)}{(n-1)(t-x)} = \sum_{i=2}^{n-1} \frac{(2i-1)}{i(i-1)} P'_{i-1}(t)P'_{i-1}(x). \tag{4.2}$$

Putting $x = x_k$ in (4.2), using (3.9) and $P'_{n-1}(x_k) = 0$ we obtain

$$\int_{-1}^{1} (1-t^2) \frac{P'_{n-1}(t)}{t - x_k}\,dt = \frac{2(n-1)}{P'_{n-2}(x_k)}, \qquad k = 2, 3, \ldots, n-1. \tag{4.3}$$

It is easy to verify that there exists a polynomial $q_{n-4}(x)$ of degree $\leqslant n-4$ such that $(k = 2, 3, \ldots, n-1)$

$$\left(\frac{P'_{n-1}(x)}{(x-x_k)P''_{n-1}(x_k)} \right)^2 - \frac{P'_{n-1}(x)}{(x-x_k)P''_{n-1}(x_k)} = P'_{n-1}(x) q_{n-4}(x).$$

Therefore on using (3.14) we have

$$\int_{-1}^{1} \left(\frac{P'_{n-1}(x)}{(x-x_k)P''_{n-1}(x_k)} \right)^2 (1-x^2)\,dx = \int_{-1}^{1} \frac{P'_{n-1}(x)(1-x^2)\,dx}{(x-x_k)P''_{n-1}(x_k)}. \tag{4.4}$$

As a special case of (4.2) ($x = x_k$ and observing that $P'_{n-1}(x_k) = 0$, $k = 2, 3, \ldots, n-1$) we obtain

$$\frac{P'_{n-1}(t)}{t - x_k} = \frac{2n-3}{n-2} P'_{n-2}(t) + \lambda_{n-4}(t) \tag{4.5}$$

where $\lambda_{n-4}(t)$ is a polynomial of degree $\leqslant n-4$. Therefore from (2.9) and (4.5) we have

$$l_k(x) = \frac{(2n-3)(1-x^2)P'_{n-2}(x)}{(n-2)\pi'_n(x_k)} + \mu_{n-2}(x)$$

where $\mu_{n-2}(x)$ is a polynomial of degree $\leqslant n-2$. Thus on using (3.3) we obtain

$$l_k'(x) = -\frac{(n-1)(2n-3)}{\pi_n'(x_k)} P_{n-2}(x) + g_{n-3}(x) \tag{4.6}$$

where $g_{n-3}(x)$ is some polynomial of degree $\leqslant n-3$. Lastly from (4.5) we have

$$\int_x^1 \frac{P_{n-1}'(t)}{t-x_k}\,dt = -\frac{(2n-3)}{n-2} P_{n-2}(x) + q_{n-3}(x) \tag{4.7}$$

where $q_{n-3}(x)$ is a polynomial of degree $\leqslant n-3$.

5. Investigation of $\int_{-1}^1 r_k(x)\,dx$. First we prove the following lemmas.

LEMMA 5.1. *We have*

$$\int_{-1}^1 l_k^2(x)\,dx = \frac{4}{n(2n-1)P_{n-1}^2(x_k)}, \qquad k = 1,2,\ldots,n, \tag{5.1}$$

where $l_k(x)$ *is defined by* (2.9).

PROOF. Since

$$1 - x^2 = 1 - x_k^2 - 2x_k(x-x_k) + (x-x_k)^2$$

it follows from (2.9) and (4.1) that

$$\begin{aligned}
\int_{-1}^1 l_k^2(x)\,dx &= \int_{-1}^1 \frac{(1-x^2)^2(P_{n-1}'(x))^2}{(x-x_2)^2(\pi_n'(x_k))^2}\,dx \\
&= \int_{-1}^1 \frac{(1-x^2)\left[1-x_k^2-2x_k(x-x_k)+(x-x_k)^2\right](P_{n-1}'(x))^2\,dx}{(x-x_k)^2(\pi_n'(x_k))^2} \\
&= \frac{(1-x_k^2)}{(\pi_n'(x_k))^2}\int_{-1}^1 (1-x^2)\left(\frac{P_{n-1}'(x)}{x-x_k}\right)^2 dx \\
&\quad - \frac{1}{(\pi_n'(x_k))^2}\int_{-1}^1 (1-x^2)(P_{n-1}'(x))^2\,dx.
\end{aligned}$$

Next, on using (3.6), (4.3), and (4.4) we obtain ($k = 2,3,\ldots,n-1$)

$$\int_{-1}^1 l_k^2(x)\,dx = \frac{(1-x_k^2)P_{n-1}''(x_k)}{(\pi_n'(x_k))^2}\,\frac{2(n-1)}{P_{n-2}'(x_k)} - \frac{2n(n-1)}{(2n-1)(\pi_n'(x_k))^2}.$$

Now, on using (3.5) and (3.2) we obtain

$$\begin{aligned}
\int_{-1}^1 l_k^2(x)\,dx &= \frac{2}{n(n-1)P_{n-1}^2(x_k)} - \frac{2}{(2n-1)n(n-1)P_{n-1}^2(x_k)} \\
&= \frac{4}{n(2n-1)P_{n-1}^2(x_k)}, \qquad k = 2,3,\ldots,n-1.
\end{aligned}$$

The proof of (5.1) for $k = 1$ follows from (2.9), (3.7) and (3.8). Here we have

$$\int_{-1}^{1} l_1^2(x)\,dx = \frac{1}{n^2(n-1)^2}\int_{-1}^{1}(1+x)^2(P'_{n-1}(x))^2\,dx$$
$$= \frac{1}{n^2(n-1)^2}\int_{-1}^{1}(1+x^2)(P'_{n-1}(x))^2\,dx$$
$$= \frac{1}{n^2(n-1)^2}\left[n(n-1)+\frac{(2n-3)n(n-1)}{(2n-1)}\right] = \frac{4}{n(2n-1)}.$$

This proves (5.1) for $k = 1$. Proof of (5.1) for $k = n$ is similar, so we omit the details.

Next we prove the following

LEMMA 5.2. *We have* ($k = 2, 3, \ldots, n-1$)

$$\int_{-1}^{1}\pi_n(x)\left(\int_x^1 \frac{l'_k(t)}{t-x_k}\,dt\right)dx = \frac{2\pi'_n(x_k)}{n(n-2)(2n-1)P_{n-1}^2(x_k)}. \tag{5.2}$$

PROOF. From (2.9) and (3.4) we obtain

$$(1-t^2)(t-x_k)l''_k(t) + 2(1-t^2)l'_k(t) + n(n-1)(t-x_k)l_k(t) = 0. \tag{5.3}$$

We may express (5.3) as

$$-\frac{2l'_k(t)}{t-x_k} = l''_k(t) + \frac{n(n-1)P'_{n-1}(t)}{(t-x_k)\pi'_n(x_k)}.$$

Therefore, we have

$$-2\int_x^1 \frac{l'_k(t)}{t-x_k}\,dt = l'_k(1) - l'_k(x) + \frac{n(n-1)}{\pi'_n(x_k)}\int_x^1 \frac{P'_{n-1}(t)}{t-x_k}\,dt. \tag{5.4}$$

On using (4.6) and (4.7) we obtain

$$-2\int_x^1 \frac{l'_k(t)}{t-x_k}\,dt = \frac{(2n-3)(n-1)P_{n-2}(x)}{\pi'_n(x_k)}$$
$$-\frac{n(n-1)(2n-3)P_{n-2}(x)}{(n-2)\pi'_n(x_k)} + \lambda_{n-3}(x)$$
$$= -\frac{2(2n-3)(n-1)P_{n-2}(x)}{(n-2)\pi'_n(x_k)} + \lambda_{n-3}(x),$$

where $\lambda_{n-3}(x)$ is a polynomial of degree $\leq n-3$. From above it follows that

$$\frac{\pi_n(x)}{\pi'_n(x_k)}\int_x^1 \frac{l'_k(t)}{t-x_k}\,dt = \frac{(2n-3)\pi_n(x)P_{n-2}(x)}{n^2(n-1)(n-2)P_{n-1}^2(x_k)} + \frac{\pi_n(x)}{\pi'_n(x_k)}\lambda_{n-3}(x).$$

Now, on using (3.13), (3.1) and (3.11) we obtain (5.2). This proves Lemma 5.2. Next, we will need the following

LEMMA 5.3. *We have*

$$\int_{-1}^{1}\frac{3+x}{4}l_1^2(x)\,dx = \frac{8n-9}{2n(n-1)(2n-1)}, \tag{5.5}$$

$$\int_{-1}^{1} \frac{(1-x^2)}{4} l_1(x) l_1'(x)\,dx = \frac{2n-3}{2n(n-1)(2n-1)}, \tag{5.6}$$

and

$$\int_{-1}^{1} \pi_n(x)\left(1+\tfrac{1}{3}P_{n-1}(x)\right)dx = 0. \tag{5.7}$$

PROOF. From (2.9), (3.7) and (3.8) we have

$$\begin{aligned}\int_{-1}^{1} \frac{3+x}{4} l_1^2(x)\,dx &= \frac{1}{4n^2(n-1)^2}\int_{-1}^{1}(3+x)(1+x)^2(P_{n-1}'(x))^2\,dx\\ &= \frac{1}{4n^2(n-1)^2}\int_{-1}^{1}(3+5x^2)(P_{n-1}'(x))^2\,dx\\ &= \frac{1}{4n^2(n-1)^2}\left[3n(n-1)+\frac{5n(n-1)(2n-3)}{2n-1}\right]\end{aligned}$$

from which (5.5) follows. For the proof of (5.6) we use (2.9), (3.3), (3.7), (3.8) and (3.10). First we express

$$\begin{aligned}&\int_{-1}^{1}(1-x^2) l_1(x) l_1'(x)\,dx\\ &\quad= \frac{1}{n^2(n-1)^2}\int_{-1}^{1}(1-x^2)\left\{(1+x)(P_{n-1}'(x))^2+(1+x)^2P_{n-1}'(x)P_{n-1}''(x)\right\}dx\\ &\quad= \frac{1}{n^2(n-1)^2}\int_{-1}^{1}(1-x^2)\left\{(P_{n-1}'(x))^2+2xP_{n-1}'(x)P_{n-1}''(x)\right\}dx\\ &\quad= \frac{1}{n^2(n-1)^2}\left[\int_{-1}^{1}(1+3x^2)(P_{n-1}'(x))^2-2n(n-1)\int_{-1}^{1}xP_{n-1}(x)P_{n-1}'(x)\,dx\right]\\ &\quad= \frac{1}{n^2(n-1)^2}\left[n(n-1)+\frac{3n(n-1)(2n-3)}{2n-1}-\frac{4n(n-1)^2}{2n-1}\right].\end{aligned}$$

From above (5.6) also follows. (5.7) is an immediate consequence of (3.1) and (3.11). This proves the lemma. Now the main result of this section is given by the following

LEMMA 5.4. *We have*

$$\int_{-1}^{1} r_k(x)\,dx = \frac{2(2n-3)}{n(n-2)(2n-1)P_{n-1}^2(x_k)}, \qquad k=2,3,\dots,n-1, \tag{5.8}$$

$$\int_{-1}^{1} r_1(x)\,dx = \frac{3}{n(2n-1)} = \int_{-1}^{1} r_n(x)\,dx \tag{5.9}$$

where $r_k(x)$ are defined by (2.6)–(2.8).

PROOF. (5.8) is an immediate consequence of (2.8), (5.1), (5.2) and (3.12). On using (2.6) and Lemma 5.3 we get the first part of (5.9). Proof of the second part of (5.9) is similar and so we omit the details.

6. Investigation of $\int_{-1}^{1} \rho_k(x)\,dx$. Here we shall prove the following

LEMMA 6.1. *For $k = 1, 2, \ldots, n$ we have*

$$\int_{-1}^{1} \rho_k(x)\,dx = \frac{1 - x_k^2}{n(n-1)(n-2)(2n-1)P_{n-1}^2(x_k)} \tag{6.1}$$

where $\rho_k(x)$ are defined by (2.1)–(2.4).

PROOF. From (2.2), (3.12), (3.1) and (3.11)

$$\int_{-1}^{1} \rho_1(x)\,dx = c_1 \int_{-1}^{1} \pi_n(x)\left(1 + c_2 P_{n-1}(x)\right) dx = 0. \tag{6.2}$$

Similarly

$$\int_{-1}^{1} \rho_n(x)\,dx = 0. \tag{6.3}$$

For the proof of (6.1) for $k = 2, 3, \ldots, n-1$ we use (2.4), (3.12), (3.13) and (4.7). Then we may express

$$\begin{aligned}
\int_{-1}^{1} \rho_k(x)\,dx &= \frac{(1 - x_k^2)}{2(\pi_n'(x_k))^2} \int_{-1}^{1} \pi_n(x) \int_x^1 \frac{P_{n-1}'(t)}{t - x_k}\,dt \\
&= \frac{(1 - x_k^2)}{2(\pi_n'(x_k))^2} \int_{-1}^{1} \pi_n(x) \frac{(2n-3)}{2n-3} P_{n-2}(x)\,dx \\
&= \frac{(2n-3)(1 - x_k^2)}{2(n-2)(\pi_n'(x_k))^2} \frac{2n(n-1)}{(2n-3)(2n-1)}.
\end{aligned}$$

On using

$$\pi_n'(x_k) = -n(n-1)P_{n-1}(x_k)$$

we obtain (6.1) for $k = 2, 3, \ldots, n-1$. On combining (6.2), (6.3) and the above result we obtain (6.1).

7. Proof of Theorems 1, 2, 3. Theorem 1 and Theorem 2 are a direct consequence of (1.3), (5.8), (5.9) and (6.1). Proof of Theorem 3 can be completed with the help of a well-known theorem of Korovkin about positive functionals. First, we note that according to Theorem B of J. Balazs and P. Turán **[2]**

$$\sum_{k=1}^{n} r_{kn}(x) f(x_{kn}) \rightrightarrows f(x) \quad \text{on } [-1, 1]$$

if $f = 1$ or $f = x$ or $f = x^2$. Hence the positive functional $Q_n(f) \to \int_{-1}^{1} f\,dx$ for $f = 1, x, x^2$. Hence applying Korovkin's theorem it follows that this convergence has place for all $f \in C[-1, 1]$.

8. Remarks.

REMARK 1. Suppose instead of choosing π matrix we choose P matrix (whose nth row consists of n zeros and a Legendre polynomial of degree n). In this case the quadrature formula for (0, 2) interpolation coincides with the well-known Gaussian quadrature formula based on Legendre knots. This can be seen very easily without even knowing the explicit forms of the fundamental functions $r_{kn}(x, P)$ and $\rho_{kn}(x, P)$ (see (1.2)–(1.5)). For $-1 < x_{nn} < \cdots < x_{1n} < 1$ denote the zeros of $P_n(x)$. Set

$$\lambda_{kn}(x) = \frac{P_n(x)}{(x - x_{kn})P_n'(x_{kn})}, \qquad k = 1, 2, \ldots, n,$$

the fundamental functions of Lagrange interpolation satisfying

$$\begin{aligned} \lambda_{kn}(x_{jn}) &= 1, \qquad j = k, j = 1, 2, \ldots, n. \\ &= 0, \qquad j \neq k, \end{aligned}$$

In view of (1.4) we may write $r_{kn}(x, P) = \lambda_{kn}(x) + P_n(x)q_{n-1}(x)$, where $q_{n-1}(x)$ is a polynomial of degree $\leqslant n - 1$. Therefore

$$\begin{aligned} \int_{-1}^{1} r_{kn}(x, P)\, dx &= \int_{-1}^{1} \lambda_{kn}(x)\, dx + \int_{-1}^{1} P_n(x) q_{n-1}(x)\, dx \\ &= \int_{-1}^{1} \lambda_{kn}(x)\, dx, \qquad k = 1, 2, \ldots, n. \end{aligned}$$

Here in the last part we use (3.11). From (1.5) it follows that $\rho_{kn}(x, P) = P_n(x)t_{n-1}(x)$ where $t_{n-1}(x)$ is a polynomial of degree $\leqslant n - 1$. Therefore

$$\int_{-1}^{1} \rho_{kn}(x, P)\, dx = \int_{-1}^{1} P_n(x) t_{n-1}(x)\, dx = 0$$

(in view of (3.11)). Thus we obtain in this case

$$\begin{aligned} \int_{-1}^{1} f(x)\, dx &\approx \sum_{k=1}^{n} f(x_{kn}) \int_{-1}^{1} \lambda_{kn}(x)\, dx \\ &= 2 \sum_{k=1}^{n} \frac{f(x_{kn})}{(1 - x_k^2)(P_n'(x_{kn}))^2}. \end{aligned}$$

But this is precisely the Gaussian quadrature formula based on Legendre knots.

REMARK 2. Our next remark is about Theorem 3 of this paper. Here we proved that $Q_n(f)$ as defined by (1.8) converges to $\int_{-1}^{1} f(x)\, dx$ under the assumption that $f \in C[-1, 1]$. However, it is possible to prove still more: if f is a bounded Riemann integrable function on $[-1, 1]$ then $Q_n(f)$ still converges to $\int_{-1}^{1} f(x)\, dx$. Proof of this fact can be given on the same lines as for convergence of the Gaussian rule [see Davis and Rabinowitz's book on *Numerical integration*, pp. 48–52].

We omit the details.

REMARK 3. It is well known [Erdös and Turán] that the Lagrange interpolation process is more effective with respect to convergence in mean than with respect to uniform convergnce. On the suggestion of P. Turán, P. Vertesi [7] studied the following: what is the largest class of functions for which

$$\lim_{n \to \infty} \int_{-1}^{1} \left[\sum_{k=1}^{n} f(x_{kn}) r_{kn}(x, \pi) - f(x) \right]^2 dx = 0?$$

It turns out that this class is essentially the same as given by Theorem B of J. Balazs and P. Turán [**2**].

REMARK 4. Here we state a new g.f. analogous to (1.11). Let

$$1 = t_{1n} > t_{2n} > \cdots > t_{n+1,n} > t_{n+2,n} = -1 \tag{8.1}$$

be the zeros of $(1 - t^2)P_n(t)$, where $P_n(t)$ being the nth Legendre polynomial of degree n. Let $f \in C^2[-1, 1]$, then we have

$$\int_{-1}^{1} f(x)\,dx = \sum_{k=1}^{n+2} a_{kn} f(t_{kn}) + \sum_{k=2}^{n+1} b_{kn} f''(t_{kn}) \tag{8.2}$$

exact for all polynomials of degree $\leq 2n + 1$. Here

$$\begin{cases} a_{1n} = a_{n+2,n} = \dfrac{-1}{(n-1)(2n+1)}, & \\ a_{kn} = \dfrac{2n(2n-1)}{(2n+1)(n-1)\left(1 - t_{kn}^2\right)\left(P_n'(t_k)\right)^2}, & k = 2, 3, \ldots, n-1, \\ b_{kn} = \dfrac{1}{(n-1)(2n+1)\left(P_n'(t_k)\right)^2}, & k = 2, 3, \ldots, n+1. \end{cases} \tag{8.3}$$

Finally, we would like to state the following q.f. based on Tchebycheff nodes of the first kind. Let

$$1 = x_{1n} > x_{2n} > \cdots > x_{n+1,n} > x_{n+2,n} = -1 \tag{8.4}$$

be the zeros of $(1 - x^2)T_n(x)$, where $T_n(x) = \cos n\theta$, $\cos\theta = x$. Then following q.f.

$$\int_{-1}^{1} f(x) \frac{1}{\left(1 - x^2\right)^{1/2}}\,dx = \sum_{k=1}^{n+2} \alpha_{kn} f(x_{kn}) + \sum_{k=2}^{n+1} \beta_{kn} f''(x_{kn}) \tag{8.5}$$

exact for all polynomials of degree $\leq 2n + 1$. Here

$$\begin{cases} \alpha_{1n} \equiv \alpha_{n+2,n} = \dfrac{-\pi}{4(2n-1)}, & \\ \alpha_{kn} = \dfrac{\pi}{n}\left[1 + \dfrac{1}{2n(2n-1)\left(1 - x_{kn}^2\right)}\right], & k = 2, 3, \ldots, n+1, \\ \beta_{kn} = \dfrac{\pi}{2n^2(2n-1)}\left(1 - x_{kn}^2\right), & k = 2, 3, \ldots, n+1. \end{cases} \tag{8.6}$$

For the proof of such q.f. we will return elsewhere.

The author is very grateful to Professor Ron DeVore for the invitation to visit the University of South Carolina during January 1981–May 1981 and making my stay very pleasant. The author is also grateful to the referee for valuable suggestions.

REFERENCES

1. J. Balazs and P. Turán, *Notes on interpolation*. II, Acta Math. Acad. Sci. Hungar. **8** (1957), 201–215.

2. ______, *Notes on interpolation*. III, Acta Math. Acad. Sci. Hungar. **9** (1958), 195–213.

3. G. D. Birkhoff, *General mean value and remainder theorems with applications to mechanical differentiation and integration*, Trans. Amer. Math. Soc. **7** (1906), 107–136.

4. G. G. Lorentz and S. D. Riemenschneider, *Birkhoff interpolation*.

5. J. Suranyi and P. Turán, *Notes on interpolation*. I, Acta. Math. Acad. Sci. Hungar. **6** (1955), 67–80.

6. P. Turán, *On some open problems of approximation theory*, J. Approx. Theory **29** (1980), 23–85.

7. P. O. H. Vertesi, *A problem of P. Turán on the mean convergence of lacunary interpolation*, Acta Math. Acad. Sci. Hungar. **26** (1975), 153–162.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32601