AN ELEMENTARY INTEGRAL SOLUTION OPERATOR
FOR THE CAUCHY-RIEMANN EQUATIONS
ON PSEUDOCONVEX DOMAINS IN C^n

BY

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ABSTRACT. An integral representation formula for $(0, q)$ forms is constructed on a strictly pseudoconvex domain D in C^n by using only the local geometry of the boundary of D. By combining this representation with elementary results about compact operators in Banach spaces, one obtains the solution of the Levi problem and, more importantly, an integral solution operator for $\bar{\partial}$ on D. The construction does not need any a priori knowledge of the solvability of $\bar{\partial}$ and thus allows us to establish fundamental global results by a direct and elementary method.

1. Introduction. The classical methods for solving the Cauchy-Riemann equations on pseudoconvex domains in C^n use either the solution of the Levi problem and the Oka-Cartan theory of holomorphically convex domains (typically including a fair amount of sheaf theory) or abstract methods from the theory of partial differential equations. By using the solution of $\bar{\partial}$, it has then been possible to construct *integral solution operators* (analogous to the Cauchy transform in C^1) on strictly pseudoconvex domains. These operators yield more refined estimates for solutions than those available through the classical methods.

The question arises whether one could develop integral solution operators and their numerous applications directly without making use of the classical methods for solving $\bar{\partial}$. In this paper we answer this question affirmatively by presenting an elementary construction of a solution operator for $\bar{\partial}$ on a strictly pseudoconvex domain D in C^n.

The construction involves two steps. First, one introduces an explicit integral operator E_q which extends $\bar{\partial}$-cohomology classes of type $(0, q)$, $q \geqslant 1$, from D to a larger domain. The operator E_q is obtained by generalizing the reproducing kernel for holomorphic functions given by Kerzman and Stein [6] to the case of $(0, q)$-forms. Combining the extension operator E_q with basic properties of compact operators in Banach spaces, one immediately obtains a simple solution of the Levi problem.[2] The

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[2]The author discovered this construction of the extension operator E_q shortly after the work of Kerzman and Stein. In the meantime, Harvey and Polking [3] obtained the solution of the Levi problem by a similar construction, and by using information about Cauchy principal values of certain singular integrals, together with the finiteness theorem of Schwartz for Fréchet spaces.

principle of this proof is really the same as Grauert's [2], however the details are quite different.

Via the extension operator E_q, one has thus reduced the problem of solving $\bar{\partial}u = f$ on D to the case where f is $\bar{\partial}$-closed on a strictly pseudoconvex neighborhood D_δ of $\overline{D}$. In the second step one solves this latter problem. Briefly, the idea is as follows. It is well known how to construct an integral solution operator in the case of convex domains. So, in the spirit of Oka [9], we embed a suitable neighborhood V of $\overline{D}$ into a convex domain U in a higher-dimensional $\mathbf{C}^N$, and pull back the canonical global support function for U to V. The main difficulty then is to obtain the required factorization (i.e. Hefer's lemma). It turns out that *finiteness* of $\bar{\partial}$-cohomology is sufficient for that.

By a result of Fornaess [1], one can even embed the domain D itself into a convex domain in some $\mathbf{C}^m$; however, the proof, as well as the applications to integral kernels in [1], uses the solution of $\bar{\partial}$. In comparison, the embedding result used here is almost trivial, once one has solved the Levi problem.

This paper deals only with strictly pseudoconvex domains in $\mathbf{C}^n$. But once one has solved $\bar{\partial}$ on such domains, one can easily prove the Runge property for sublevel sets of a strictly plurisubharmonic exhaustion function, following the classical arguments of Oka for polynomially convex sets. In this way one can then solve the $\bar{\partial}$-equation on arbitrary pseudoconvex domains in $\mathbf{C}^n$. Furthermore, these methods can be extended to complex manifolds, leading to the characterization of Stein manifolds in terms of strictly plurisubharmonic exhaustion functions. This requires, essentially, only the *finiteness* of the $\bar{\partial}$-cohomology. To show that the $\bar{\partial}$-cohomology is actually 0 in this case gets somewhat more complicated, due to the lack of a simple generalization of the Bochner-Martinelli-Koppelman formula to Stein manifolds.

In conclusion, I would like to thank Ingo Lieb for several stimulating discussions regarding the problem dealt with in this paper.

2. The extension of $\bar{\partial}$-cohomology classes. Let D be a strictly pseudoconvex domain in $\mathbf{C}^n$ with boundary bD of class C^3, and let r be a C^3 defining function for D, defined on a neighborhood of $\overline{D}$, which is strictly plurisubharmonic in a neighborhood of bD. Define

$$F(z, \zeta) = 2 \sum_{j=1}^{n} \frac{\partial r}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \sum_{j, k=1}^{n} \frac{\partial^2 r}{\partial \zeta_j \partial \zeta_k}(\zeta)(\zeta_j - z_j)(\zeta_k - z_k).$$

The following result is a well-known consequence of Taylor's theorem and the strict plurisubharmonicity of r.

2.1 LEMMA. *There are positive constants ε and c, such that*
$$\operatorname{Re} F(z, \zeta) \geq -r(z) + c \, |z - \zeta|^2$$
for all (z, ζ) with $\zeta \in bD$ and $|z - \zeta| \leq \varepsilon$.

Choose $\chi \in C^\infty(\mathbf{C}^n \times \mathbf{C}^n)$ such that $0 \leq \chi \leq 1$, $\chi(z, \zeta) = 1$ for $|z - \zeta| \leq \varepsilon/2$ and $\chi(z, \zeta) = 0$ for $|z - \zeta| \geq \varepsilon$. For $j = 1, 2, \ldots, n$, define

$$P_j(z, \zeta) = \chi \left[2 \frac{\partial r}{\partial \zeta_j}(\zeta) - \sum_{k=1}^{n} \frac{\partial^2 r}{\partial \zeta_j \partial \zeta_k}(\zeta)(\zeta_k - z_k) \right] + (1 - \chi)(\bar{\zeta}_j - \bar{z}_j).$$

Then

$$\phi(z, \zeta) = \sum_{j=1}^{n} P_j(z, \zeta)(\zeta_j - z_j) = \chi \cdot F + (1 - \chi)|z - \zeta|^2$$

has the following properties for any fixed $\zeta \in bD$:

(2.2) $\qquad \operatorname{Re} \phi(z, \zeta) \geqslant -r(z) + c|z - \zeta|^2$ for $|z - \zeta| \leqslant \varepsilon/2$;

(2.3) $\qquad$ there is $\delta > 0$ (independent of ζ) such that $|\phi(z, \zeta)| \geqslant \delta$ for all z with $r(z) \leqslant \delta$ and $|z - \zeta| \geqslant \varepsilon/2$;

(2.4) $\quad \phi(\cdot, \zeta)$ and $P_j(\cdot, \zeta)$, $1 \leqslant j \leqslant n$, are holomorphic on $\{z \colon |z - \zeta| < \varepsilon/2\}$.

For $\lambda \in I = [0, 1]$ and $\zeta \in bD$, we define

$$w_j(z, \zeta, \lambda) = \lambda \frac{P_j(z, \zeta)}{\phi(z, \zeta)} + (1 - \lambda) \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^2}, \qquad 1 \leqslant j \leqslant n.$$

By (2.3), w_j has no singularities for $z \in D \cup \{z \colon r(z) \leqslant \delta, |z - \zeta| \geqslant \varepsilon/2\}$. Furthermore

$$\text{(2.5)} \qquad \sum_{j=1}^{n} w_j \cdot (\zeta_j - z_j) = 1.$$

Finally, set $W = \sum_{j=1}^{n} w_j \, d\zeta_j$.

For $q = 0, 1, \ldots, n - 1$, the Cauchy-Fantappié form $\Omega_q(W)$ associated to the $(1, 0)$ form W is defined by

$$\Omega_q(W) = \frac{(-1)^{q(q-1)/2}}{(2\pi i)^n} \binom{n-1}{q} W \wedge \left(\bigwedge_{n-q-1} \bar{\partial}_{\zeta, \lambda} W \right) \wedge \left(\bigwedge_{q} \bar{\partial}_z W \right).$$

We also set $\Omega_{-1} = \Omega_n = 0$. $\Omega_q(W)$ is a double differential form in $(\zeta, \lambda) \in bD \times I$ and $z \in D$ of degree $2n - q - 1$ in (ζ, λ) and type $(0, q)$ in z (so differentials in z commute with differentials in (ζ, λ)). For $\lambda = 0$, $\Omega_q(W)$ agrees with the Bochner-Martinelli-Koppelman kernel [7] of type $(0, q)$ which is defined on $\mathbf{C}^n \setminus \{z = \zeta\}$. The following results are well known (see, for example, [10] and the references given there).

2.6 LEMMA. $\bar{\partial}_{\zeta, \lambda} \Omega_q(W) = (-1)^q \bar{\partial}_z \Omega_{q-1}(W)$ for any $W = \sum_{j=1}^{n} w_j \, d\zeta_j$ which satisfies (2.5).

2.7 PROPOSITION. Let $0 \leqslant q \leqslant n$, $\Omega_q = \Omega_q(W)$, as defined above, and $f \in C_{0,q}^1(\overline{D})$. Then, for $z \in D$,

$$f(z) = \int_{bD \times \{1\}} f(\zeta) \wedge \Omega_q(\zeta, z) + \bar{\partial}_z T_q f(z) + T_{q+1} \bar{\partial} f(z),$$

where

$$T_q f = \int_{bD \times I} f \wedge \Omega_{q-1} - \int_{D \times \{0\}} f \wedge \Omega_{q-1}.$$

Let $\mu(\zeta) = (\zeta, 1)$. For $f \in C_{0,q}^0(bD)$, we define $E_q f = \int_{bD} f \wedge \mu^* \Omega_q(W)$.

2.8 LEMMA. *For $q \geqslant 1$ and $f \in C^0_{0,q}(bD)$, $E_q f$ is defined and of class C^∞ on $\overline{D}_\delta = \{z: r(z) \leqslant \delta\}$. Moreover, for each multi-index α, there is a constant $c_\alpha < \infty$, such that*

$$\operatorname{Sup}_{D_\delta} \left| D^\alpha E_q f(z) \right| \leqslant c_\alpha |f|_{L^\infty(bD)} \quad \text{for all } f \in C^0_{0,q}(bD).$$

PROOF. By (2.4), the components of $W(\zeta, z, 1)$ are holomorphic on $\{z: |z - \zeta| < \varepsilon/2\}$. Hence, for $z \in \overline{D}_\delta$ fixed $\mu^* \Omega_q(W) = 0$ for $|\zeta - z| < \varepsilon/2$, and, by (2.3), any derivative with respect to z of $\mu^* \Omega_q(W)$ is uniformly bounded for $z \in \overline{D}_\delta$ and $\zeta \in bD$ with $|\zeta - z| \geqslant \varepsilon/2$.

2.9 LEMMA. *If $f \in C^1_{0,q}(\overline{D})$, $q \geqslant 1$, and $\bar{\partial} f = 0$ on D, then $\bar{\partial}_z(E_q f)(z) = 0$ on D_δ.*

PROOF. For $z \in D$, the statement follows directly from (2.7); by continuity it thus holds also for $z \in \overline{D}$. Fix $z_0 \in D_\delta \setminus \overline{D}$. Since Ω_q and Ω_{q+1} are smooth for $\zeta \in bD$ and z in the neighborhood of z_0, by (2.6) and Stokes' theorem, one obtains

$$\bar{\partial}_z(E_q f) = (-1)^{q+1} \int_{bD} f \wedge \bar{\partial}_\zeta \left(\mu^* \Omega_{q+1}\right) = \int_{bD} \bar{\partial} f \wedge \mu^* \Omega_{q+1} = 0$$

in a neighborhood of z_0.   Q.E.D.

Combining (2.7) and (2.9) one sees that $E_q f$ defines an extension of the $\bar{\partial}$-cohomology class in $H^q(D, \mathcal{O})$ of $f \in C^1_{0,q}(\overline{D})$ with $\bar{\partial} f = 0$ to $H^q(D_\delta, \mathcal{O})$. We will need the following information about regularity properties of T_q.

2.10 PROPOSITION. *For any $k = 0, 1, 2, \ldots$ and $q \geqslant 1$, if $f \in C^0_{0,q}(\overline{D}) \cap C^k_{0,q}(D)$, one has $T_q f \in C^k_{0,q-1}(D)$. Moreover, there is a constant $C < \infty$, independent of f, such that*

$$\operatorname{Sup}_{D} \left| T_q f(z) \right| \leqslant C |f|_{L^\infty(\overline{D})}.$$

PROOF. Since $\phi(z, \zeta) = F(z, \zeta)$ for $|z - \zeta| \leqslant \varepsilon/2$, one can estimate the kernels, as in the classical case, using (2.2) (Henkin [5], for $n = 2$ and $q = 1$; Lieb [8] for the general case).

2.11 REMARK. T_q is even smoothing of order $\frac{1}{2}$ in the following sense:

$$| T_q f |_{\operatorname{Lip}_{1/2}(D)} \leqslant C |f|_{L^\infty(\overline{D})}.$$

For a proof see the estimations in [10].

3. The Levi problem. We consider the vectorspace

$$Z^1_q(\overline{D}) = \left\{ f \in C^1_{0,q}(\overline{D}), \bar{\partial} f = 0 \right\}$$

and the C^1 norm on $Z^1_q(\overline{D})$ given by

$$\left\| \sum f_J d\bar{z}^J \right\|_1 = \sum_J \sum_{|\alpha| \leqslant 1} \sup_{\overline{D}} |D^\alpha f_J(z)|.$$

Z^1_q with this norm is a Banach space. By Lemmas 2.8 and 2.9 we obtain

3.1 LEMMA. *For $q \geqslant 1$ the restriction of E_q to $Z^1_q(\overline{D})$ defines a compact linear operator $E_q: Z^1_q \to Z^1_q$.*

3.2 THEOREM. *For $q \geq 1$ the operator $L_q = \bar{\partial} T_q$ defines a bounded linear transformation L_q: $Z_q^1(\bar{D}) \to Z_q^1(\bar{D})$ whose range has finite codimension.*

PROOF. By Proposition 2.7, $L_q = I - E_q$ on Z_q^1, where $I =$ identity. Since E_q is compact, the theorem follows from standard facts about compact operators in Banach spaces.

3.3 THEOREM. *Let $D \subset\subset \mathbf{C}^n$ be strictly pseudoconvex with C^3-boundary. Then D is holomorphically convex.*

PROOF. Fix $\zeta \in bD$. By (2.2)–(2.4), $h(z) = \phi(z, \zeta) \in C^\infty(\bar{D})$, $h(\zeta) = 0$, $h(z) \neq 0$ for $z \in \bar{D} \setminus \{\zeta\}$, and h is holomorphic for $|z - \zeta| < \varepsilon/2$. Therefore, $f_j = \bar{\partial}(1/h^j) \in Z_1^1(\bar{D})$ for $j = 1, 2, \ldots$. By Theorem 3.2 there exist constants $c_1, \ldots, c_l \in \mathbf{C}$, $c_l \neq 0$, and a $(0, 1)$ form $g \in Z_1^1(\bar{D})$ such that $\Sigma_{j=1}^l c_j f_j = L_1 g = \bar{\partial} T_1 g$. By (2.10), $T_1 g \in C^1(D)$ and $\sup_D |T_1 g(z)| < \infty$. Therefore $H = \Sigma_{j=1}^l c_j/h^j - T_1 g$ is holomorphic on D and $\lim_{z \to \zeta} |H(z)| = \infty$.   Q.E.D.

4. Factorization. A compact set $K \subset \mathbf{C}^n$ is called pseudoconvex if K is the intersection of open pseudoconvex sets in $\mathbf{C}^n$. Equivalently, K is pseudoconvex if and only if K has a neighborhood basis of strictly pseudoconvex open sets with C^∞ boundary. Note that the closure of a strictly pseudoconvex open set is pseudoconvex.

4.1 PROPOSITION. *Let $K \subset \mathbf{C}^n$ be a compact pseudoconvex set. Let $K_1 = K \cap \{z: z_1 = 0\}$. Then every $f \in \mathcal{O}(K_1)$ has an extension $Ef \in \mathcal{O}(K)$ such that $Ef = f$ in a neighborhood of K_1 in $\{z \in \mathbf{C}^n: z_1 = 0\}$.*

PROOF. Choose a neighborhood W of K such that f is holomorphic on $W_1 = W \cap \{z: z_1 = 0\}$. Let $\pi: \mathbf{C}^n \to \mathbf{C}^n$ be defined by $\pi(z) = (0, z_2, \ldots, z_n)$. Since W_1 and $W \setminus \pi^{-1}(W_1)$ are disjoint and closed (relative to W), there is $\chi \in C^\infty(W)$ such that $\chi \equiv 1$ in a neighborhood of W_1 and $\chi \equiv 0$ in a neighborhood of $W - \pi^{-1}(W_1)$. For $j = 1, 2, \ldots, \alpha_j = \bar{\partial} \chi(f \circ \pi)/z_1^j$ is a well-defined, $\bar{\partial}$-closed $C_{0,1}^\infty$ form on W. Choose D strictly pseudoconvex with smooth boundary such that $K \subset D \subset\subset W$. Then $\alpha_j \in Z_1^1(\bar{D})$ for $j = 1, 2, \ldots$ and, by Theorem 3.2, there are $C_1, \ldots, C_l \in \mathbf{C}$, $C_l \neq 0$, and $g \in Z_1^1(\bar{D})$ such that $\Sigma_{j=1}^l C_j \alpha_j = \bar{\partial} T_1 g$. Dividing by a nonzero constant, we may assume $C_l = 1$. Thus

$$\bar{\partial}\left(z_1^l T_1 g\right) = \sum_{j=1}^l C_j z_1^{l-j} \bar{\partial} \chi(f \circ \pi)$$

and, hence, $Ef = \Sigma_{j=1}^l C_j z_1^{l-j} \chi f \circ \pi - z_1^l T_1 g$ is holomorphic on D and satisfies $Ef(0, z_2, \ldots, z_n) = f(0, z_2, \ldots, z_n)$.   Q.E.D.

One can now repeat, with minor modifications, the arguments of Hefer [4] to prove the following result.

4.2 THEOREM. *Let K be a pseudoconvex compact set in $\mathbf{C}^n$. Given $f \in \mathcal{O}(K)$, there are $Q_j \in \mathcal{O}(K \times K)$, $1 \leq j \leq n$, such that*

$$f(w) - f(z) = \sum_{j=1}^n Q_j(z, w)(w_j - z_j) \quad \text{for } (z, w) \in K \times K.$$

5. The solution of $\bar{\partial}$ on strictly pseudoconvex domains. We first solve the $\bar{\partial}$-equation on holomorphically convex compact subsets of a strictly pseudoconvex domain.

5.1 THEOREM. *Let $D \subset\subset \mathbf{C}^n$ be strictly pseudoconvex and let $K \subset D$ be compact with $\hat{K}_{\mathcal{O}(D)} = K$. There are a neighborhood V of K and linear integral operators R_q: $C^0_{0,q}(D) \to C^0_{0,q-1}(V)$, $q = 1, 2, \ldots, n$, such that:*

   (i) *for any $k = 0, 1, 2, \ldots$, if $f \in C^k_{0,q}(D)$, the $R_q f \in C^k_{0,q-1}(V)$;*
   (ii) *if $f \in C^1_{0,q}(D)$ and $\bar{\partial} f = 0$, then $\bar{\partial}(R_q f) = f$ on V.*

PROOF. Let $\omega \subset\subset D$ be a strictly pseudoconvex neighborhood of K. There are finitely many functions $h_1, \ldots, h_N \in \mathcal{O}(D)$ such that the analytic polyhedron $A = \{z \in \omega: |h_k(z)| < 1, 1 \leqslant k \leqslant N\}$ satisfies $K \subset A \subset\subset \omega$. Let Δ^N be the open unit polydisc in $\mathbf{C}^N$. The map $H = (h_1, \ldots, h_N): A \to \Delta^N$ is proper and, by adding additional functions to the h_j's if necessary, one can also assume that H is regular and one-to-one. Since Δ^N is convex, there is a convex neighborhood $U \subset\subset \Delta^N$ of $H(K)$ which can be taken to have smooth boundary transversal to the submanifold $H(A)$ of Δ^N. Let $\rho \in C^\infty(\Delta^N)$ be a defining function for U. For $t \in \mathbf{C}^N$ and $\eta \in \Delta^N$, define

$$G(t, \eta) = \sum_{k=1}^N \frac{\partial \rho}{\partial \eta_k}(\eta)(\eta_k - t_k).$$

Because of the convexity of U, $G(t, \eta) \neq 0$ for $(t, \eta) \in U \times bU$. $V = H^{-1}(U) \subset\subset A$ is a neighborhood of K with smooth boundary, and $\phi(z, \zeta) = G(H(z), H(\zeta)) \neq 0$ for $(z, \zeta) \in V \times bV$. Since $h_k \in \mathcal{O}(\bar{\omega})$, Theorem 4.2 gives functions $Q_{jk} \in \mathcal{O}(\bar{\omega} \times \bar{\omega})$, $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant N$, such that

$$h_k(\zeta) - h_k(z) = \sum_{j=1}^n Q_{jk}(z, \zeta)(\zeta_j - z_j);$$

therefore, the functions

$$P_j(z, \zeta) = \sum_{k=1}^N \frac{\partial \rho}{\partial \eta_k}(H(\zeta)) Q_{jk}(z, \zeta)$$

are holomorphic in $z \in \omega$, and

$$\sum_{j=1}^n P_j(z, \zeta)(\zeta_j - z_j) = \phi(z, \zeta).$$

Now set

$$w_j(z, \zeta, \lambda) = \lambda \frac{P_j(z, \zeta)}{\phi(z, \zeta)} + (1 - \lambda) \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^2}$$

on $V \times bV \times [0, 1]$, and $W = \sum_{j=1}^n w_j \, d\zeta_j$. Finally, define the kernels $\Omega_q(W)$, $q \geqslant 0$, as in §2. Observe that for $q \geqslant 1$, $\Omega_q(W) = 0$ for $(\zeta, \lambda) \in bV \times \{1\}$, as W is holomorphic in z on V for such (ζ, λ). Therefore, by Proposition 2.7, the operator

(5.2) $$R_q f = \int_{bV \times I} f \wedge \Omega_{q-1}(W) - \int_{V \times \{0\}} f \wedge \Omega_{q-1}(W)$$

solves the $\bar{\partial}$-equation on V and hence on $K \subset V$. This proves (ii) of the theorem; (i) follows from 2.10 or directly from the above representation of $R_q f$.

5.3 REMARK. It is immediate from (5.2) and the definition of $\Omega_{q-1}(W)$ that R_q has the following smoothing property:

$$|R_q f|_{\mathrm{Lip}_\alpha(K)} \leqslant C_\alpha |f|_{L^\infty(\bar{V})} \quad \text{for any } \alpha < 1.$$

We now combine Theorem 5.1 with the extension operator E_q of §2. Let $D \subset\subset \mathbf{C}^n$ be strictly pseudoconvex with C^3-boundary. For $q \geqslant 1$ construct E_q: $C^0_{0,q}(bD) \to C_{0,q}(\bar{D}_\delta)$ and T_q: $C^0_{0,q}(\bar{D}) \to C^0_{0,q-1}(D)$ as in §2. Note that D_δ can be chosen strictly pseudoconvex, and hence holomorphically convex, by Theorem 3.3. Let $K = \hat{\bar{D}}_{\theta(D_\delta)}$.[3] Now apply Theorem 5.1 to K and D_δ, obtaining the operator R_q: $C^0_{0,q}(D_\delta) \to C^0_{0,q-1}(K)$. Set $T_q + R_q E_q = S_q$; then S_q: $C^0_{0,q}(\bar{D}) \to C^0_{0,q-1}(D)$. The main properties of S_q are summarized in the following theorem.

5.4 THEOREM. (i) For any $k = 0, 1, 2, \ldots,$ $S_q f \in C^k_{0,q-1}(D)$ for $f \in C^0_{0,q}(\bar{D}) \cap C^k_{0,q}(D)$.

(ii) If $f \in C^0_{0,q}(\bar{D}) \cap C^1_{0,q}(D)$ and $\bar{\partial} f = 0$, then $\bar{\partial}(S_q f) = f$ on D.

(iii) $|S_q f|_{\mathrm{Lip}_{1/2}(D)} \leqslant C |f|_{L^\infty(\bar{D})}$ for $f \in C^0_{0,q}(\bar{D})$.

PROOF. (i) follows from 2.10, 2.8 and 5.1(i); (ii) follows from 2.7, 2.9 and 5.1(ii); finally, (iii) follows from 2.8, 2.11 and 5.3.

5.5 COROLLARY. Let $K \subset \mathbf{C}^n$ be a pseudoconvex compact set[4] and $q \geqslant 1$. Then for every $\bar{\partial}$-closed $C^\infty_{0,q}$ form f on a neighborhood of K there is a $C^\infty_{0,q-1}$ form u such that $\bar{\partial} u = f$ on a neighborhood of K.

In terms of sheaf cohomology this can be reformulated as follows.

5.6 COROLLARY. Let $K \subset \mathbf{C}^n$ be a pseudoconvex compact set. Then $H^q(K, \mathcal{O}) = 0$ for $q \geqslant 1$.

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[3] In fact $\hat{\bar{D}}_{\theta(D_\delta)} = \bar{D}$, but this is not needed here.

[4] See the beginning of §4.

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