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ARITHMETIC EQUIVALENT OF ESSENTIAL SIMPLICITY OF ZETA ZEROS

BY

JULIA MUELLER

ABSTRACT. Let $R(x)$ and $S(t)$ be the remainder terms in the prime number theorem and the Riemann-von Mangoldt formula respectively, that is $\psi(x) = x + R(x)$ and $N(t) = (1/2\pi)\int_0^t \log(\tau/2\pi)\,d\tau + S(t) + 7/8 + O(1/t)$. We are interested in the following integrals: $J(T,\beta) = \int_1^{T^\beta}(R(x + x/T) - R(x))^2 dx/x^2$ and $I(T,\alpha) = \int_1^T (S(t + \alpha/L) - S(t))^2\,dt$, where $L = (2\pi)^{-1}\log T$. Furthermore, denote by $N(T,\alpha)(N^*(T))$ the number of pairs of zeros $\frac{1}{2} + i\gamma$, $\frac{1}{2} + i\gamma'$ with $0 < \gamma \leq T$ and $0 < (\gamma' - \gamma)L \leq \alpha$ $((\gamma' - \gamma)L = 0)$—i.e., off-diagonal and diagonal pairs.

THEOREM. *Assume the Riemann hypothesis. The following three hypotheses* (A), (B) *and* (C_1, C_2) *are equivalent: for* $\beta \to \infty$ *and* $\alpha \to 0$ *as* $T \to \infty$ *we have*

(A) $J(T,\beta) \sim \beta T^{-1}\log^2 T$,

(B) $I(T,\alpha) \sim \alpha T$ *and*

(C_1) $N^*(T) \sim TL$, (C_2) $N(T,\alpha) = o(TL)$. Hypothesis (C_1, C_2) is called the *essential simplicity hypothesis.*

1. Introduction. Let $R(x)$ be the remainder term in the prime number theorem when the theorem is stated as

$$\psi(x) = x + R(x).$$

Let $S(T)$ be the remainder term in the Riemann-von Mangoldt zero counting formula

$$N(T) = \frac{1}{2\pi}\int_0^T \log\frac{t}{2\pi}\,dt + S(T) + \frac{7}{8} + O\left(\frac{1}{T}\right),$$

where $N(T)$ is the number of zeros $\beta + i\gamma$ of the Riemann zeta function with $0 < \beta < 1$ and $0 < \gamma \leq T$. It is well known that the Riemann hypothesis implies that

$$R(x) = O\left(x^{1/2}\log^2 x\right) \quad \text{and} \quad S(T) = O\left(\frac{\log T}{\log\log T}\right).$$

Furthermore, it is known that the above bound for $R(x)$ is equivalent to the truth of the Riemann hypothesis. We will assume the Riemann hypothesis throughout this paper. Cramér [1] showed, assuming the Riemann hypothesis, that

$$\int_1^X (R(x))^2\frac{dx}{x^2} \sim C\log X, \qquad C = \sum_\gamma{}' \frac{m^2(\gamma)}{|\frac{1}{2} + i\gamma|^2},$$

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where $m(\gamma)$ is the multiplicity of the zero $\frac{1}{2} + i\gamma$, and the dash indicates that the sum is over distinct values of γ. A. Selberg [7] showed that

$$\int_1^T (S(t))^2\,dt \sim \pi^{-2} T \log\log T.$$

Furthermore, he considered the following integrals:

$$J(T,\beta) = \int_1^{T^\beta} \left(R\left(x + \frac{x}{T}\right) - R(x) \right)^2 \frac{dx}{x^2}$$

and

$$I(T,\alpha) = \int_1^T \left(S\left(t + \frac{\alpha}{L}\right) - S(t) \right)^2 dt,$$

where $L = (2\pi)^{-1}\log T$. We remark that the average spacing between consecutive zeros $\frac{1}{2} + i\gamma$ with $\gamma \leqslant T$ is asymptotic to L^{-1}.

Selberg [8] showed that the Riemann hypothesis implies that $J(T,\beta)$ is $\ll T^{-1}\log^2 T$, for each positive β. In fact, it is known that

$$J(T,\beta)\begin{cases} \sim \frac{1}{2}\beta^2 T^{-1}\log^2 T, & \text{for } 0 < \beta \leqslant 1, \\ \ll \beta T^{-1}\log^2 T, & \text{for } \beta > 1. \end{cases} \tag{1}$$

The above asymptotic formula can be evaluated easily by a simple method (see [4]), while the bound is due to Montgomery [5].

The following result for $I(T,\alpha)$ was first obtained by Selberg and later by A. Fujii [2]:

$$I(T,\alpha)\begin{cases} \sim \pi^{-2} T \log\alpha, & \text{for } \alpha \to \infty,\ \alpha \leqslant L; \\ \ll T\log(2+\alpha), & \text{for } \alpha \ll 1. \end{cases}$$

We will show in §3 that for $\alpha \to 0$,

$$I(T,\alpha) \ll T\alpha. \tag{2}$$

Before stating our main result, we will introduce some notation and definitions. Let

$$N^*(T) = \sum_{0<\gamma\leqslant T}{}' \, m^2(\gamma)$$

and denote by $N(T,\alpha)$ the number of pairs of zeros $\frac{1}{2} + i\gamma$, $\frac{1}{2} + i\gamma'$ with $0 < \gamma \leqslant T$ and $0 < (\gamma' - \gamma)L \leqslant \alpha$. We remark that in this notation

$$N(T) = \sum_{0<\gamma\leqslant T}{}' \, m(\gamma).$$

From this definition and the Riemann-von Mangoldt formula it follows easily that

$$\overline{\lim}\, N^*(T)/TL \geqslant 1.$$

Furthermore, Montgomery [6] has shown that

$$N^*(T) \leqslant \left(\tfrac{4}{3} + o(1)\right) TL. \tag{3}$$

As for $N(T,\alpha)$ we have the following results from [4].

$$N(T,\alpha)\begin{cases} \sim TL\alpha & \text{for } \alpha \to \infty,\ \alpha \leqslant TL, \\ \ll TL\alpha & \text{for } \alpha > 1, \\ \ll TL & \text{for } \alpha \leqslant 1. \end{cases} \tag{4}$$

Moreover, the above asymptotic formula cannot hold for $\alpha \cong 1$.[1] In this range, Gallagher [3] obtained the following quantitative result for $\alpha = 1$ assuming the Riemann hypothesis:

$$\overline{\lim}\, N(T, 1)/TL \leq .99$$

and

$$\underline{\lim}\, N(T, 1)/TL \geq .14.$$

We remark that H. L. Montgomery [6] has conjectured that

$$N(T, \alpha) \sim TL\int_0^\alpha \left\{1 - \left(\frac{\sin \pi a}{\pi a}\right)^2\right\} da \tag{4'}$$

as $T \to \infty$, uniformly in each interval $0 < \alpha_0 \leq \alpha \leq \alpha_1 < \infty$. We have shown in [4, pp. 206–208] that (4′) implies that for $\beta \cong 1$ and $\alpha \cong 1$,

$$J(T, \beta) \sim T^{-1}\log^2 T \min\{\beta - \tfrac{1}{2}, \tfrac{1}{2}\beta^2\}$$

and

$$I(T, \alpha) \sim T\int_{-1}^1 \left(\frac{\sin \pi b}{\pi b}\right)^2 |b|\, db.$$

For $\beta \to \infty$ and $\alpha \to 0$ we obtained

(A) $J(T, \beta) \sim \beta T^{-1}\log^2 T$ and

(B) $I(T, \alpha) \sim \alpha T$.

In this note we will show that hypotheses (A) and (B) are equivalent. Moreover, they are equivalent to the *essential simplicity hypothesis*:

(C_1) $N^*(T) \sim TL$,

(C_2) $N(T, \alpha) = o(TL)$ ($\alpha \to 0$ as $T \to \infty$).

THEOREM 1. *Assume the Riemann hypothesis. The three hypotheses* (A), (B) *and* (C_1, C_2) *are equivalent.*

The meaning of essential simplicity hypothesis is that the zeros $\rho = \frac{1}{2} + i\gamma$ with $\gamma \leq T$ are almost all simple zeros; moreover, "nearby" zero pairs are few.

2. Lemmas. Let

$$A(T) = \sum_{\gamma>0}{}' a(\gamma)m(\gamma) \quad \text{and} \quad A^*(T) = \sum_{\gamma>0}{}' a(\gamma)m^2(\gamma)$$

where

$$a(t) = a(t, T) = \left(\frac{\sin t/2T}{t/2T}\right)^2.$$

The sums, here and later, are over distinct values of γ. We remark that from now on the symbols T_0 and T_1 will mean $T_0 = T_0(T)$ such that $T_0/T \to 0$ and $T_1 = T_1(T)$ such that $T_1/T \to \infty$ as $T \to \infty$.

[1] By $f \cong g$ we mean $f \ll g$ and $g \ll f$.

LEMMA 1. *We have*

$$A(T) \sim \tfrac{1}{2} T \log T. \tag{5}$$

Moreover, (C_1) *is equivalent to*

$$A^*(T) \sim \tfrac{1}{2} T \log T. \tag{6}$$

PROOF. First we note that for each T_0

$$\sum_{\gamma \leqslant T_0}{}' a(\gamma) m(\gamma) \leqslant \sum_{\gamma \leqslant T_0}{}' a(\gamma) m^2(\gamma) \ll T_0 L = o(TL).$$

Next, for each T_1 the partial sums of $A(T)$ and $A^*(T)$ of the range $[2^k T_1, 2^{k+1} T_1]$ are

$$\ll \left(\frac{T}{2^k T_1}\right)^2 N(2^{k+1} T_1) \ll 2^{-k+1} T_1^{-1} T^2 L = o(2^{-k+1} TL).$$

Summing over k, we get

$$\sum_{\gamma > T_1}{}' a(\gamma) m(\gamma) \leqslant \sum_{\gamma > T_1}{}' a(\gamma) m^2(\gamma) = o(TL).$$

From the Riemann-von Mangoldt formula $N(T) \sim TL$, we get

$$A(T) \sim \sum_{T_0 < \gamma \leqslant T_1}{}' a(\gamma) m(\gamma) = \frac{1}{2\pi} \int_{T_0}^{T_1} a(t) \log \frac{t}{2\pi}\, dt + \int_{T_0}^{T_1} a(t)\, d(o(T \log T)).$$

The first integral is asymptotic to

$$TL \int_{T_0/T}^{T_1/T} \left(\frac{\sin \frac{1}{2}\theta}{\frac{1}{2}\theta} \right)^2 d\theta \sim TL \int_0^{\infty} \left(\frac{\sin \frac{1}{2}\theta}{\frac{1}{2}\theta} \right)^2 d\theta = \tfrac{1}{2} T \log T,$$

provided

$$\log T_0 \sim \log T_1 \sim \log T.$$

The remainder integral is

$$o(T \log T) + o(T \log T) \int_{T_0}^{T_1} | a'(t) |\, dt = o(T \log T),$$

since $a'(t) \ll T^{-1}(1 + t/T)^{-2}$ and

$$\int_{T_0/T}^{T_1/T} (1 + t/T)^{-2}\, d(t/T) \leqslant \int_0^{\infty} (1 + \theta)^{-2}\, d\theta = 1.$$

This proves (5).

An entirely similar method will also show that (C_1) implies (6). To show the converse, we first remark that from (5) and (6) we get

$$A^*(T) - A(T) = o(TL).$$

Since $a(\gamma) \gg 1$ for $\gamma \leqslant T$ and the sum is nonnegative, it follows that

$$N^*(T) - N(T) = \sum_{0 < \gamma \leqslant T} (m^2(\gamma) - m(\gamma)) = o(TL),$$

from which we obtain (C_1). This completes the proof of Lemma 1.

For each β, define

$$A(T,\beta) = \sum_{\substack{\gamma>0 \\ (\gamma'-\gamma)L>0}} a(\gamma)S((\gamma'-\gamma)L\beta)$$

where

$$S(b) = \left(\frac{\sin \pi b}{\pi b}\right)^2.$$

We remark that from now on the symbol β will mean $\beta = \beta(T)$ such that $\beta \to \infty$ as $T \to \infty$.

LEMMA 2. *The hypothesis* (C_2) *is equivalent to*

(7) $$A(T,\beta) = o(TL) \qquad (\beta \to \infty \text{ as } T \to \infty).$$

PROOF. First we will show that

(8) $$\sum_{\substack{\gamma>T_1 \\ (\gamma'-\gamma)L>0}} a(\gamma)S((\gamma'-\gamma)L\beta) = o(TL)$$

provided $T_1/T \to \infty$ as $T \to \infty$.

For each fixed $\alpha > 1$, the sum over the range $2^kT_1 < \gamma \leqslant 2^{k+1}T_1$ and $0 < (\gamma'-\gamma)L \leqslant \alpha$ is

$$\ll \left(\frac{T}{2^kT_1}\right)^2 \cdot 2^{k+1}T_1\alpha L = o(2^{-k+1}TL);$$

here we have used the bound for $N(T,\alpha)$ given by (4). Similarly, over the range $2^kT_1 < \gamma \leqslant 2^{k+1}T_1$ and $2^l\alpha < (\gamma'-\gamma)L \leqslant 2^{l+1}\alpha$, the sum is

$$\ll \left(\frac{T}{2^kT_1}\right)^2 \left(\frac{1}{2^l\alpha}\right)^2 \cdot 2^{k+l+2}T_1\alpha L = o(2^{-k-l+2}TL).$$

Summing on $k \geqslant 0$ and $l \geqslant 0$, we get (8). Now we have

(9) $$A(T,\beta) = A_1(T,\beta) + o(TL),$$

where

$$A_1(T,\beta) = \sum_{\substack{0<\gamma\leqslant T_1 \\ (\gamma'-\gamma)L>0}} a(\gamma)S((\gamma'-\gamma)L\beta).$$

To prove Lemma 2, it suffices to show that (C_2) is equivalent to

(10) $$A_1(T,\beta) = o(TL)$$

provided $T_1/T \to \infty$ sufficiently slowly.

We remark that for $0 < \gamma \leqslant T$ and $0 < (\gamma'-\gamma)L \leqslant \alpha$ each term in the sum of $A_1(T,\beta)$ over this range is $\gg 1$ provided $\alpha \leqslant (2\beta)^{-1}$. Since $A_1(T,\beta)$ is a sum of nonnegative terms we have

$$N(T,\alpha) \ll A_1(T,\beta).$$

Hence (C_2) follows from (10).

To show the converse, we split $A_1(T, \beta)$ into two parts, Σ' and Σ'', according as $0 < (\gamma' - \gamma)L \leqslant \alpha$ or $(\gamma' - \gamma)L > \alpha$ where $\alpha < 1$ will be determined later. We observe that if $T_1/T \to \infty$ sufficiently slowly then we may replace T by T_1 in (C_2). Since

$$\sum{}' \ll N(T_1, \alpha),$$

we get $\Sigma' = o(TL)$ from (C_2). To deal with Σ'', we note that the partial sum over the range $2^l\alpha < (\gamma' - \gamma)L \leqslant 2^{l+1}\alpha$ is

$$\ll (2^l\alpha\beta)^{-2} N(T_1, 2^{l+1}\alpha) \ll \begin{cases} 2^{-l+1}\alpha^{-1}\beta^{-2}T_1 L & \text{for } 2^{l+1}\alpha \geqslant 1, \\ 2^{-2l}(\alpha\beta)^{-2}T_1 L & \text{otherwise.} \end{cases}$$

Suppose that $(\alpha\beta)^{-1} \to 0$ and $T_1/T \to \infty$ sufficiently slowly as $T \to \infty$; then on summing over $l \geqslant 0$, we get

$$\sum{}'' = o(TL). \tag{11}$$

This proves Lemma 2.

LEMMA 3. *We have*

$$\int_0^\beta J(T, b)\, db = \beta^2 T^{-2} \log T\{A^*(T) + 2A(T, \beta) + o(TL)\} \tag{12}$$

where $\beta = \beta(T)$ such that $\beta \to \infty$ as $T \to \infty$.

PROOF. We begin by quoting the following result from §3 of [4]: for each fixed $\beta > 0$,

$$J(T, \beta) = T^{-2} \log T\left\{\int_0^\beta \left(2 \operatorname{Re} \sum c(\gamma) e(\gamma L b)\right)^2 db + o(TL)\right\} \tag{13}$$

where

$$c(\gamma) = \frac{1 - e^{i\gamma/T}}{i\gamma/T},$$

and the remainder term is independent of β. We remark that (13) also holds provided $\beta \to \infty$ sufficiently slowly.

We will show that after smoothing the integral in (13), its partial sum over $\gamma > T_1$ is negligible; therefore the remaining part of that double integral corresponds to terms on the right-hand side of (12). That is

$$\int_0^\beta \int_0^b \left(2 \operatorname{Re} \sum_{\gamma > T_1} c(\gamma) e(\gamma L w)\right)^2 dw\, db = o(\beta T L) \qquad (\beta \to \infty \text{ as } T \to \infty) \tag{14}$$

and

(15)

$$\int_0^\beta \int_0^b \left(2 \operatorname{Re} \sum_{0 < \gamma \leqslant T_1} c(\gamma) e(\gamma L w)\right)^2 dw\, db = \beta^2\{A^*(T) + 2A(T, \beta) + o(TL)\}.$$

First we remark that the integral in (14) is

$$\ll \int_{-\beta}^{\beta}\left(1-\frac{|b|}{\beta}\right)\left|\sum_{\gamma>T_1} c(\gamma)e(\gamma Lb)\right|^2 db$$

$$\ll L^{-1}\int_{-\beta L}^{\beta L}\left|\sum_{\gamma>T_1} c(\gamma)e(\gamma y)\right|^2 dy \qquad (y=Lb).$$

Using the same method as that in §3 of [4] for bounding means of exponential sums, the last integral can be shown to be

$$\ll \int_{\frac{1}{2}T_1}^{\infty}\left(\frac{1}{U}\sum_{t}^{t+U}|c(\gamma)|\right)^2 dt \ll \left(\frac{T}{U}\right)^2\int_{\frac{1}{2}T_1}^{\infty}(\Delta_U N)^2\frac{dt}{t^2}$$

where

$$\Delta_U N(t) = N(t+U) - N(t),$$
$$UL=\alpha=(2\beta)^{-1} \quad \text{and} \quad c(t) \ll T/t.$$

We assert that the contribution of the last integral from the interval $(2^jT_1, 2^{j+1}T_1]$ is $\ll (2^jT_1)^{-1}(U(j+L))$. Summing on $j \geqslant -1$, we get a bound

$$\ll (T_1\alpha)^{-1}T^2L = o(\beta TL).$$

The assertion is a consequence of the bounds (3) and (4) for $N^*(T)$ and $N(T,\alpha)$ for $\alpha \to 0$ and the formula that has been proved on p. 209 of [4]:

$$\int_T^{2T}(\Delta_U N)^2\,dt \leqslant UN^*(T) + 2\int_0^{\alpha} N(T,a)\,da + O(L^2). \tag{16}$$

Next, we note that (15) may be written as

$$\beta^2\left\{\sum_{0<\gamma\leqslant T_1}{}' a(\gamma)m^2(\gamma) + 2\sum_{\substack{0<\gamma\leqslant T_1\\(\gamma'-\gamma)L>0}} \overline{c(\gamma)}\,c(\gamma')S((\gamma'-\gamma)L\beta) + \sum_{0<\gamma,\gamma'\leqslant T} c(\gamma)c(\gamma')S((\gamma'+\gamma)L\beta)\right\},$$

since

$$\int_0^{\beta}\int_0^{b} 2\,\mathrm{Re}\,e(aw)\,dw\,db = \left(\frac{\sin\pi a\beta}{\pi a}\right)^2 = \beta^2 S(a\beta),$$

$$(2\,\mathrm{Re}\,S)^2 = 2|S|^2 + 2\,\mathrm{Re}(S^2) \quad \text{and} \quad a(\gamma) = |C(\gamma)|^2.$$

The first sum in the bracket is asymptotic to $A^*(T)$ and therefore to derive (15) it remains to show the estimates

$$\sum_{\substack{0<\gamma\leqslant T_1\\(\gamma'-\gamma)L>0}} \overline{c(\gamma)}\,c(\gamma')S((\gamma-\gamma)L\beta) - A(T,\beta) = o(TL) \tag{17}$$

and

$$\sum_{0<\gamma,\gamma'\leq T_1} c(\gamma)c(\gamma')S((\gamma'+\gamma)L\beta) = o(TL). \tag{18}$$

We remark first that, in (17), $A(T,\beta)$ may be replaced by $A_1(T,\beta)$ because of (9). Also the partial sum in (17) over the range $(\gamma'-\gamma)L > \alpha$ is $o(TL)$ provided that α satisfies the property assumed in (11). This can be shown in a similar way to (11). Therefore the left-hand side of (17) may be replaced by

$$\sum_{\substack{0<\gamma\leq T_1\\ 0<(\gamma'-\gamma)L\leq\alpha}} \overline{c(\gamma)}\,(c(\gamma')-c(\gamma))S((\gamma'-\gamma)L\beta) + o(TL).$$

But the above sum is $\ll T_1\alpha L$, which is $o(TL)$ provided $T_1/T \to \infty$ sufficiently slowly.

To show (18), we let $T_0 L = o(T_1 L)^{1/2}$ and again we divide the sum into two parts Σ_1 and Σ_2, according as $0<\gamma$, $\gamma'\leq T_0$ or otherwise. Assuming $T_1/T\to\infty$ sufficiently slowly, then

$$\sum\nolimits_1 \ll N(T_0)^2 = o(TL)$$

and

$$\sum\nolimits_2 \ll (T_0 L\beta)^{-2} N(T_1)^2 = o(TL).$$

This proves (18) and also completes the proof of Lemma 3.

3. Proof of (2) and Theorem 1.

PROOF OF (2). We will first prove the formula

$$UN^*(T) + 2\int_0^\alpha N(T,a)\,da = I(T,\alpha) + o(T\alpha) \tag{19}$$

for $\alpha\to 0$ and $T\to\infty$.

From the Riemann-von Mangoldt formula in §1, we get

$$\int_T^{2T}(\Delta_U N)^2\,dt = \int_T^{2T}(\Delta_U M)^2\,dt + 2\int_T^{2T}(\Delta_U M)(\Delta_U S)\,dt + \int_T^{2T}(\Delta_U S)^2\,dt + O(L). \tag{20}$$

Since

$$\Delta_U M(t) \sim \alpha$$

for $T\leq t\leq 2T$ and $T\to\infty$, we have

$$\int_T^{2T}(\Delta_U M)\,dt \sim \alpha^2 T. \tag{21}$$

To estimate the second integral on the right-hand side of (20) we use the bound $S(T)\ll L$ to get

$$\int_T^{2T}(\Delta_U S)(t)\,dt = \int_{2T}^{2T+U} S(t)\,dt - \int_T^{T+U} S(t)\,dt \ll \alpha.$$

Since $\Delta_U M$ is monotonic and $\ll\alpha$, it follows that

$$\int_T^{2T}(\Delta_U M)(\Delta_U S)\,dt \ll \alpha^2. \tag{22}$$

Combining (20), (21) and (22) we get, for $T \to \infty$ and $\alpha \to 0$,

$$\int_T^{2T} (\Delta_U N)^2 \, dt = I(T, \alpha) + o(T\alpha). \tag{23}$$

From (16) and (23) we get (19), and (2) follows from (19) and the bounds (3) and (4).

PROOF OF THEOREM 1. It is obvious that the equivalence of hypotheses (B) and (C_1, C_2) is an immediate consequence of (19). Now we assume that (A) holds; then from (1) we get for $\beta \to \infty$

$$\int_0^\beta J(T, b) \, db \sim \tfrac{1}{2}\beta^2 T^{-1} \log^2 T \qquad (T \to \infty). \tag{24}$$

Combining this with (12) and using Lemmas 1 and 2, we get both (C_1) and (C_2). Conversely, suppose both (C_1) and (C_2) hold; then (12) simplifies to (24). Since $J(T, \beta)$ is a positive increasing function of β, we have

$$(\beta\varepsilon)^{-1} \int_{\beta(1-\varepsilon)}^{\beta} J(T, b) \, db \lesssim J(T, \beta) \lesssim (\beta\varepsilon)^{-1} \int_{\beta}^{\beta(1+\varepsilon)} J(T, b) \, db$$

for each $\varepsilon > 0$. But the integrals here are asymptotic to $\beta^2 \varepsilon (1 \pm \frac{1}{2}\varepsilon) T^{-1} \log^2 T$. By letting $\varepsilon \to 0$, we get (A).

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DEPARTMENT OF MATHEMATICS, FORDHAM UNIVERSITY, BRONX, NEW YORK 10458

Current address: School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540