TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 275, Number 1, January 1983

# ON THE ABSENCE OF POSITIVE EIGENVALUES OF SCHRÖDINGER OPERATORS WITH LONG RANGE POTENTIALS

BY

HUBERT KALF AND V. KRISHNA KUMAR

ABSTRACT. In this paper we consider the problem of obtaining upper bounds for the positive bound states associated with the Schrödinger operators with long range potentials. We have extended the size of the class of long range potentials for which one can establish the nonexistence of positive eigenvalues, improving upon the recent results of G. B. Khosrovshahi, H. A. Levine and L. E. Payne (Trans. Amer. Math. Soc. **253** (1979), 211–228).

**1. Introduction.** Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be an exterior domain, that is, an open connected set with

$$\Omega \supset D_R := \{x \mid x \in \mathbf{R}^n, |x| > R\} \tag{1.1}$$

for some $R > 0$. Let $q_1, q_2, q_3$: $\Omega \to \mathbf{R}$ be functions which, in addition to certain regularity requirements, satisfy the following assumptions. $q_3$ is spherically symmetric,

$$q_3(x) = \tilde{q}_3(|x|) \qquad (x \in D_R),$$

and there exist numbers $K \geq 0$, $L \geq 0$, $M \in [0, 1/4)$ and for every $\varepsilon > 0$ a number $\rho > R$ such that

$$\begin{gathered} |x||q_1(x)| \leq K + \varepsilon, \qquad |q_2(x)| \leq \varepsilon, \\ x(\nabla q_2)(x) \leq L + \varepsilon \qquad (x \in D_\rho) \end{gathered}$$

and

$$\left| \int_s^t \sigma \tilde{q}_3(\sigma)\, d\sigma \right| \leq M \qquad (t \geq s \geq \rho).$$

Then it was shown by Khosrovshahi, Levine and Payne [7][1] that the Schrödinger equation

$$-\Delta u + gu = \lambda u \quad \text{with } q := q_1 + q_2 + q_3 \tag{1.2}$$

Received by the editors September 8, 1981 and, in revised form, November 30, 1981.
1980 *Mathematics Subject Classification*. Primary 35J10, 47A40.
*Key words and phrases*. Schrödinger operator, long range potential, eigenvalue.

[1] They require $n \geq 3$, but their arguments need only minor alterations in the case $n = 2$. Inequality (1.3) appears on p. 214 of [7] with $(1 - 2M)^2$ under the square root, but it follows from the last line on p. 215 and the first on p. 216 that this is a printing error. Professor Levine kindly informed us of a criticism of Professor Mišnaevskiĭ that inequality (C3) on p. 227 is, as it stands, not correct unless $q_3$ is spherically symmetric. If $q_3$ does not have this property, there is an additional term loc. cit., l. 6f., from an integration by parts involving the Laplace-Beltrami operator. We shall further comment upon this at the end of the appendix to the present paper. We must confess that we committed the same error in our first version of this paper.


0002-9947/82/0000-0336/$04.25

has no nontrivial solution in $L^2(\Omega)$ if

$$\lambda > \max\left\{\left[\frac{K+\sqrt{K^2+2L(1-2M)}}{2(2-2M)}\right]^2, \frac{2K^2+L(1-4M)}{2(1-4M)^2}\right\}. \tag{1.3}$$

This is the first and up to now, as far as we know, the only multidimensional result on the absence of eigenvalues involving an integral condition on the potential. In the case $M = 0$, the entry on the left in the curly brackets of (1.3) is greater than or equal to the one on the right and the result reduces to the fundamental results of Kato [**6**] and Agmon [**1**] or more precisely, to a certain generalization of these results. We refer to [**3**, **8**] for detailed accounts on the absence of positive eigenvalues of Schrödinger operators.

In this paper we shall relax the condition on $M$ to $M \in [0, 1/2)$ and decrease the bound (1.3) to

$$\left[\frac{K+\sqrt{K^2+2L(1-2M)}}{2(1-2M)}\right]^2. \tag{1.4}$$

It appears that this improvement, however small it is, cannot be achieved on the basis of the convexity argument used in [**7**, Lemma 1]. Instead we have to draw on a technically much more elaborate argument due to Eidus [**4**] and Roze [**9**]. However, it must be admitted that the general strategy of the proof is the one used in [**5** and **7**]. That is why the presentation of our proof, which is given in §§3 and 4, is at times rather condensed. (Incidentally, we note that copying the proof of [**5**] would only lead to the improvement of $M \in [0, 1/3)$ on account of the restrictions ensuing from relationships (4.11)–(4.13) of this paper.) It goes almost without saying that, mutatis mutandis, the proof also holds good in one dimension, in which case it reduces to very simple arguments.

It is tempting to conjecture that the bound (1.4) is the best possible, but we have been unable to prove this.

An interesting testing example, for instance in one dimension, is

$$q(x) := \frac{A \sin Bx}{x} \qquad (x \geqslant 1) \tag{1.5}$$

where $A$, $B \neq 0$. Choosing $q_2 = q_3 = 0$ or $q_1 = q_3 = 0$ in the result above would lead to the exclusion of nontrivial solutions of (1.2) in $L^2((1, \infty))$ for $\lambda > A^2$ and $\lambda > \frac{1}{2}|AB|$, respectively (see [**7**]). In contrast to this, the choice $q_1 = q_2 = 0$ gives a condition on the ratio between amplitude and frequency, and our result shows that there are no nontrivial $L^2$-solutions for $\lambda > 0$ if $|A/B| < \frac{1}{4}$. Unfortunately we are rather off the mark in this particular example, because an asymptotic analysis due to Atkinson [**2**] of the solutions of (1.2) with the potential (1.5) shows that this equation has no nontrivial solution unless $|B| = 2$. If this condition is satisfied, then there exists a nontrivial solution of integrable square for $\lambda = 1$ if and only if $|A| > 2$. (We refer to [**3**] for more details.) So, at any rate, this example tells us that 2 is an upper bound for $M$.

**2. Description of the theorem and of the idea of its proof.** We consider the general Schrödinger differential expression

$$T := D^2 + q \quad \text{with } D := i^{-1}\nabla - b$$

on $D_R$, the exterior of some ball of radius $R > 0$, arranging for simplicity our assumptions on $q$: $D_R \to \mathbf{R}$ and $b$: $D_R \to \mathbf{R}^n$ such that any weak solution of the Schrödinger equation

$$Tu = \lambda u \tag{2.1}$$

for some $\lambda \in \mathbf{R}$ is, in fact, equivalent to a classical solution. By "curl $b$" we mean the matrix $(\partial_j b_k - \partial_k b_j)$.

THEOREM. *Assume that $q_1$, $q_2$, $q_3$: $D_R \to \mathbf{R}$ are locally Hölder continuous and $b^{(1)}$, $b^{(2)}$: $D_R \to \mathbf{R}^n$ are differentiable with locally Hölder continuous derivatives. Let $q := q_1 + q_2 + q_3$ with*

$$q_3(x) = \tilde{q}_3(|x|) \qquad (x \in D_R)$$

*for some $\tilde{q}_3$: $(R, \infty) \to \mathbf{R}$ and $b := b^{(1)} + b^{(2)}$. Suppose there exist numbers $\gamma \in (0, 2)$, $K \geqslant 0$, $L \geqslant 0$ and $M \in [0, \gamma/4)$ with the following properties: For every $\varepsilon > 0$ there is a $\rho > R$ such that*

$$|x||q_1(x)| + |(\operatorname{curl} b^{(1)})(x) \cdot x| \leqslant K + \varepsilon, \tag{2.2}$$

$$|q_2(x)| \leqslant \varepsilon, \qquad x(\nabla q_2)(x) + (2-\gamma)^{-1}|(\operatorname{curl} b^{(2)})(x) \cdot x|^2 \leqslant L + \varepsilon$$

*for all $x \in D_\rho$ and*

$$\left|\int_s^t \sigma \tilde{q}_3(\sigma)\, d\sigma\right| \leqslant M \qquad (t \geqslant s \geqslant \rho).$$

*Let $u$ be a weak solution in $L^2(D_R)$ of (2.1) for some*

$$\lambda > \Lambda := \left[\frac{K + \sqrt{K^2 + L(\gamma - 4M)}}{\gamma - 4M}\right]^2. \tag{2.3}$$

*Then there exists some $\tilde{R} > R$ such that $u$ is equivalent to the null function on $D_{\tilde{R}}$. If $b^{(2)} = 0$, then $\gamma = 2$ is admissible in which case $\Lambda$ is equal to the bound given in (1.4).*

REMARK. As in [5] one can weaken the local regularity assumptions on the potentials considerably and can incorporate, in particular, multiparticle interactions. In this case (2.2) has to be replaced with a condition on

$$x(\nabla q_2)(x) + \text{const } q_2(x) + (2-\gamma)^{-1}|(\operatorname{curl} b^{(2)})(x) \cdot x|^2.$$

This new condition is to be arranged such that the quantity $I_1(v_m)$ occurring in Step 1 of the proof below becomes again nonnegative. When the local regularity assumptions on the coefficients of (2.1) are such that this equation does not admit any solutions with compact support, which is indeed the case in the theorem above, it follows that any solution in $L^2(D_R)$ is in fact equivalent to the null function on $D_R$. We refer to the end of the appendix for the possibility of relaxing the assumption that $q_3$ is spherically symmetric.

Since the proof of the theorem is technically quite involved, we should like to say something on the general idea that lies behind it. Using the classical picture, one has to show that a particle with energy $\lambda > \Lambda$ will eventually move to infinity. In other words using the language of ordinary differential equations, one has to show that the trivial solution of the Schrödinger equation is unstable, which is accomplished by means of a Lyapunov-type function (this is explained with more detail in **[3]**). As in **[5]**, a candidate for such a function is found by integrating an identity leading to the virial theorem.

To be more specific, the procedure is as follows: We consider the function

$$v_m(x) := e^{h_m(|x|)}u(x) \qquad (x \in D_R)$$

where $h_m$ is an appropriately chosen function depending on a parameter $m > 0$ and $u$ is a solution of (2.1). $v_m$ is then a solution of an equation of the type (3.2). Extending in §3 an idea from **[5]**, we associate with $v_m$ a function which plays the central role in the proof in §4 and which may be called a Lyapunov-type function. This is the function $G(t, v_m)$ $(t > R)$ in equation (3.4) below. Now we assume that the assertion of the theorem is not true. Then there exist arbitrarily large values of $t$ for which

$$\int_{|x|=t} |u(x)|^2 > 0.$$

It is readily seen that for such $t$ there exists an $m > 0$ such that $G(t, v_m) > 0$. The assumptions of the theorem ascertain that

$$tG(t, v_m) \geqslant sG(s, v_m) \qquad (t \geqslant s \geqslant R_1, m \geqslant m_0) \tag{2.4}$$

for some $m_0 > 0$ and $R_1 > R$. This is proved in Step 1. Thus there exists an $m_1 > 0$ which makes $G(\cdot, v_{m_1})$ eventually positive. Next we show in Steps 2 and 3 by a bootstrap argument due to Eidus **[4]** and Roze **[9]** that $v_m \in L^2(D_R)$ $(m > 0)$. This implies $G(\cdot, v_m) \in L^1((R_1, \infty))$ and therefore $\liminf_{t\to\infty} t\,|\,G(t, v_m)\,| = 0$ for every $m > 0$. Hence we can conclude from (2.4) that $G(\cdot, v_{m_1})$ is eventually nonpositive which contradiction proves the theorem.

**3. Some preliminary lemmas.** In addition to (1.1) we employ the following notation:

$$S_t := \{x \mid x \in \mathbf{R}^n, |x| = t\} \qquad (t > 0),$$
$$B_{st} := \{x \mid x \in \mathbf{R}^n, s < |x| < t\} \qquad (t > s > 0),$$
$$\nu := x/|x| \qquad (x \in \mathbf{R}^n \setminus \{0\}).$$

$r$ is, as usual, the function $x \mapsto |x|$ $(x \in \mathbf{R}^n \setminus \{0\})$; the radial derivative of a function $f$ of several variables is written as $f'$. If $g$ is a function of one variable, we shall also use "$g$" for the function $x \mapsto g(|x|)$ to simplify the notation; the meaning will always be clear from the context.

Let $R > 0$ and $\lambda \in \mathbf{R}$. We shall assume throughout this section that $q_1$, $q_2$, $q_3$: $D_R \to \mathbf{R}$ are locally Hölder continuous and that $b$: $D_R \to \mathbf{R}^n$ is differentiable with locally Hölder continuous derivatives, although we could, as we remarked before, manage with much weaker hypotheses. We put $q := q_1 + q_2 + q_3$ and

$$Q(t) := \int_s^t \sigma \tilde{q}_3(\sigma)\, d\sigma \qquad (t \geqslant s > R).$$

Let $u$ be a weak solution of (2.1) on $D_R$. By a standard regularity result, $u$ is equivalent to a function in $C^2(D_R)$ and this latter function is again denoted by "$u$". Let $h \in C^3((R, \infty))$ be real-valued and

$$v(x) := e^{h(|x|)}u(x) \qquad (x \in D_R). \tag{3.1}$$

Then $v$ satisfies the differential equation

$$D^2 v + 2ij\nu Dv + (q - k - \lambda)v = 0 \tag{3.2}$$

where

$$j = h' \quad \text{and} \quad k(t) := j^2(t) - \frac{n-1}{t} j(t) - j'(t) \qquad (t > R.) \tag{3.3}$$

Now we can formulate an identity associated with (3.2) which plays a central role in our proof.

LEMMA 1. *Let $f \in C^1((R, \infty))$ and $g \in C^2((R, \infty))$ be real-valued functions and*

$$\begin{aligned} G(t, v) := \int_{S_t} \{ & f[2|\nu Dv|^2 - e(v)] \\ & + [f(q_1 + q_3 + k) - \tfrac{1}{2}t^{-1}g' + t^{-2}Q(fQ - g)]|v|^2 \\ & - \tfrac{1}{2}t^{-1}(2fQ - g)(|v|^2)'\} \end{aligned} \qquad (t > R) \tag{3.4}$$

*where*

$$e(v) := |Dv|^2 + (q - \lambda)|v|^2.$$

*Then*

$$tG(t, v) - sG(s, v) = \int_{B_{st}} [I_1(v) + I_2(v)] \tag{3.5}$$

*where*

(3.6)

$$\begin{aligned} I_1(v) := {} & [(2 - n - 2Q)f - rf' + g]|Dv|^2 + 2rf'|\nu Dv|^2 \\ & - 2f \operatorname{curl} b \cdot x \operatorname{Re}(\bar{v}Dv) - 2[rfq_1 + r^{-1}Q(fQ - rf' - g)]\operatorname{Im}(\bar{v}\nu Dv) \\ & + \{[(n + 2Q)f + rf' - g](\lambda - q_2) - rfq_2' - (2fQ - g)q_1 \\ & + r^{1-n}(-\tfrac{1}{2}(r^{n-1}g')' + Q[Q(r^{n-2}f)' - (r^{n-2}g)'])\}|v|^2, \end{aligned}$$

$$\begin{aligned} I_2(v) := {} & 2j[2rf|\nu Dv|^2 + (2fQ - g)\operatorname{Im}(\bar{v}\nu Dv)] \\ & + \{[(n + 2Q)f + rf' - g]k + rfk'\}|v|^2. \end{aligned} \tag{3.7}$$

*Moreover,*

$$(3.8)\qquad \int_{B_{st}} g\left[e(v) - k|v|^2\right] = \frac{1}{2}\int_{S_\rho} g\left(|v|^2\right)'\Big|_s^t + \int_{B_{st}} (2gj + g')\operatorname{Im}(\bar{v}\nu Dv)$$

*for* $t \geqslant s > R$.

Equations (3.5)–(3.7) are a variant of Lemma 4 in [5] which follow from the identity given there by repeated integration by parts on all terms involving $q_3$. To make the present paper self-contained, a detailed proof is given in the appendix.

We also need the following Dirichlet type result.

LEMMA 2. *Suppose that the assumptions of the theorem are satisfied and that the function $v$ in* (3.1) *lies in $L^2(D_R)$. Furthermore assume that the functions $j$ and $k$ in* (3.3) *are bounded. Then $Dv \in L^2(D_s)$ for $s > R$.*

We omit the simple proof which is rather an immediate consequence of relationship (3.8) with $g = 1$.

**4. Proof of the theorem.** Let $u$ be a solution in $L^2(D_R)$ of (2.1) for some $\lambda > \Lambda$, where $\Lambda$ is given by (2.3), and put

$$(4.1)\qquad v_m := u\exp(mr^l) \qquad (l \in (0,1),\ m \geqslant 0).$$

This function is a solution of (3.2) with

$$(4.2)\qquad j = lmr^{l-1} \quad\text{and}\quad k = lmr^{-2(1-l)}\left[lm - (n-2+l)r^{-l}\right].$$

We use the abbreviation

$$g_0 := n - \gamma + \kappa,$$

where $\kappa > 0$ is a number to be specified later, and consider the function (3.4) with $v = v_m, f = 1$ and $g = g_0$.

*Step* 1. *Proof of the existence of numbers $m_0 > 0$ and $R_1 > R$ such that* (2.4) *holds.* With the above-mentioned choice of functions the integrals defined in (3.5)–(3.7) read

$$\begin{aligned}(4.3)\qquad I_1(v_m) := {}& (2 - \gamma + \kappa - 2Q)|Dv_m|^2 - 2\operatorname{curl} b \cdot x\operatorname{Re}(\bar{v}_m Dv_m)\\ & -2\left[rq_1 + r^{-1}Q(Q - g_0)\right]\operatorname{Im}(\bar{v}_m\nu Dv_m)\\ & +\left[(\gamma - \kappa + 2Q)(\lambda - q_2) - rq_2' - (2Q - g_0)q_1\right.\\ & \left. + (n-2)Qr^{-2}(Q - g_0)\right]|v_m|^2,\end{aligned}$$

$$\begin{aligned}(4.4)\qquad I_2(v_m) := {}& 2j\left[2r|\nu Dv_m|^2 + (2Q - g_0)\operatorname{Im}(\bar{v}_m\nu Dv_m)\right]\\ & +\left[(\gamma - \kappa + 2Q)k + rk'\right]|v_m|^2.\end{aligned}$$

First we show that there exist numbers $c_0 > 0$ and $R_0 > R$ such that

$$(4.5)\qquad I_1(v_m) \geqslant c_0\left[|v_m|^2 + |Dw_m|^2\right] \qquad (m \geqslant 0)$$

on $D_{R_0}$.

In fact, for every $\varepsilon > 0$ there is a number $\rho > R$ with

$$2\left[rq_1 \operatorname{Im}(\bar{v}_m \nu D v_m) + \operatorname{curl} b^{(1)} \cdot x \operatorname{Re}(\bar{v}_m D v_m)\right] \leqslant (K+\varepsilon)\left(\frac{1}{\theta}|Dv_m|^2 + \theta |v_m|^2\right) \qquad (\theta > 0)$$

and

$$2\left|\operatorname{curl} b^{(2)} \cdot x \operatorname{Re}(\bar{v}_m D v_m)\right| \leqslant (2-\gamma)|Dv_m|^2 + (2-\gamma)^{-1}(\operatorname{curl} b^{(2)} \cdot x)^2 |v_m|^2$$

on $D_\rho$. Hence

$$\begin{aligned} I_1(v_m) \geqslant & \left[\varepsilon + \kappa - 2M - \varepsilon - \frac{K+\varepsilon}{\theta} + o(1)\right] |Dv_m|^2 \\ & + \Big\{(\gamma - \kappa - 2M)\lambda - (K+\varepsilon)\theta \\ & \quad - \left[rq_2' + (2-\gamma)^{-1}(\operatorname{curl} b^{(2)} \cdot x)^2 + (\gamma - \kappa + 2Q)q_2\right] + o(1)\Big\} |v_m|^2 \end{aligned}$$

on $D_\rho$. We put $\theta := (K+\varepsilon)/(\kappa - 2M - \varepsilon)$ and give $\kappa$ the value where the function

$$f(x) := \frac{1}{\gamma - 2M - x}\left(L + \frac{K^2}{x - 2M}\right) \qquad (2M < x < \gamma - 2M)$$

takes its minimum, this minimum being the number $\Lambda$ in (2.3).

Since

$$(K+\varepsilon)^2/(\kappa - 2M - \varepsilon) = K^2/(\kappa - 2M) + O(\varepsilon),$$

$(4.5)$[2] follows provided that $\varepsilon$ is chosen sufficiently small.

For a suitable choice of $l \in (0,1)$ we next prove the existence of numbers $m_0 > 0$ and $R_1 > R_0$ with

$$I_2(v_m) \geqslant 0 \qquad (m \geqslant m_0) \tag{4.6}$$

on $D_{R_1}$. Indeed,

$$\begin{aligned} I_2(v_m) \geqslant & j\left[4r - (2M + g_0)r^\delta\right] |\nu D v_m|^2 \\ & + \left[(\gamma - \kappa + 2Q)k + rk' - j(2M + g_0)r^{-\delta}\right] |v_m|^2 \end{aligned}$$

for every $\delta > 0$ and

$$ck + rk' = lmr^{-2(1-l)}\left\{lm[c - 2(1-l)] - (n-2+l)(c+l-2)r^{-l}\right\}$$

for any $c > 0$. Since we have chosen $\kappa$ such that $\gamma - \kappa - 2M > 0$, we can find numbers $l \in (0,1)$ and $\delta$ with

$$\gamma - \kappa - 2M > 2(1-l) \quad \text{and} \quad 1 - l < \delta < 1.$$

This proves (4.6) and, in conjunction with (4.5), (2.4).

---

[2] In one dimension it is obvious that $G(\cdot, v_0)$ is eventually positive unless $v_0 = u = 0$. (2.4) with $m = 0$ is therefore incompatible with $G(\cdot, u) \in L^1((R_1, \infty))$, so that for $n = 1$ the proof of the theorem is now already completed.

*Step* 2. *Proof that* $r^{m/2}u \in L^2(D_R)$ *for all* $m \geqslant 0$. Let $a \geqslant 0$ and $w_a := r^a u$. We now consider the function (3.4) with $v = w_a$, but keep to our previous choice $f = 1$, $g = g_0$. Since $w_a$ satisfies (3.2) with

(4.7) $$j = ar^{-1} \quad \text{and} \quad k = a(a+2-n)r^{-2},$$

these functions now replace the functions $j$ and $k$ from (4.2) in (4.3)–(4.4). With the numbers $c_0 > 0$ and $R_0 > R$ from the previous step we therefore have

(4.8) $$I_1(w_a) \geqslant c_0(|Dw_a|^2 + |w_a|^2) \qquad (a \geqslant 0)$$

on $D_{R_0}$.

For $a = 0$, $w_a = u$ and from Lemma 2 we have $Du \in L^2(D_{R_0})$. Then (3.5) takes on the form

(4.9) $$tG(t,u) - sG(s,u) = \int_{B_{st}} I_1(u) \qquad (t \geqslant s > R).$$

In view of

$$\liminf_{t\to\infty} t\,|G(t,u)| = 0$$

(4.8)–(4.9) now imply

(4.10) $$c_0\int_{D_s}(|Du|^2 + |u|^2) \leqslant -sG(s,u)$$
$$\leqslant s\int_{S_s} e(u) - \int_{S_s} q_3|u|^2 + c_1\int_{S_s}(|Du|^2 + |u|^2)$$

for all $s \geqslant R_0$.

Here $c_1$ is an appropriate constant and we have used the fact that $sq_1$ is bounded for large $s$. Now multiplying by $s^m$ on either side of (4.10) and integrating over $[R_0, t]$ we obtain on an application of (3.8) with $g = r^{m+1}$ and $j = 0$ ($k = 0$)

(4.11) $$c_0\int_{R_0}^{t} s^m\left[\int_{D_s}(|Du|^2 + |u|^2)\right] ds$$
$$\leqslant \frac{1}{2}\rho^{m+1}\int_{S_\rho}(|u|^2)'\Big|_{R_0}^{t} - \rho^m\int_{S_\rho} Q|u|^2\Big|_{R_0}^{t} + c_2\int_{B_{R_0 t}} r^m(|Du|^2 + |u|^2),$$

where $c_2$ is an appropriate positive constant. We proceed by induction. We assume that $m \in \mathbf{Z}$, $m \geqslant 0$, is a number for which

$$r^{m/2}u \in L^2(D_{R_0}) \quad \text{and} \quad r^{m/2}Du \in L^2(D_{R_0}).$$

Then $D(r^{m/2}u) \in L^2(D_{R_0})$ and hence

$$\int_{D_{R_0}}\left[r^{m-1}|u|^2 + r^m|(|u|^2)'|\right] < \infty;$$

here we have used the fact that $r^m \operatorname{Im}(\bar{u}\nu Du) = \operatorname{Im}[r^{m/2}u\nu D(r^{m/2}u)]$. Hence there exists a sequence $\{t_j\}$ with $t_j \to \infty$ as $j \to \infty$ and

$$t_j\int_{S_{t_j}}\left(t_j^{m-1}|u|^2 + t_j^m|(|u|^2)'|\right) \to 0 \quad \text{as } j \to \infty.$$

Letting $t \to \infty$ through $\{t_j\}$ in (4.11) in conjunction with the above result we obtain

$$(4.12)\quad c_0\int_{R_0}^{\infty} s^m\left[\int_{D_s}\left(|Du|^2+|u|^2\right)\right] ds$$
$$\leqslant -\frac{1}{2}R_0^{m+1}\int_{S_{R_0}}\left(|u|^2\right)' + R_0^m\int_{S_{R_0}} Q|u|^2 + c_2\int_{D_{R_0}} r^m\left(|Du|^2+|u|^2\right).$$

However,

$$\int_{R_0}^{\infty} s^m\left[\int_{D_s}\left(|Du|^2+|u|^2\right)\right] ds = \frac{1}{m+1}\int_{D_{R_0}}\left(r^{m+1}-R_0^{m+1}\right)\left(|Du|^2+|u|^2\right).$$

Now combining this with (4.12) and noting that $u,\ Du \in L^2(D_{R_0})$, by induction, we conclude

$$r^{m/2}u \in L^2\left(D_{R_0}\right) \quad\text{and}\quad r^{m/2}Du \in L^2\left(D_{R_0}\right) \quad \text{for } m \in \mathbf{Z},\ m \geqslant 0.$$

*Step* 3. *Proof that* $v_m$ *defined by* (4.1) *belongs to* $L^2(D_R)$. This is done as follows. Defining

$$\phi(\tau) := \tau\int_{S_\tau}|u|^2 \qquad (\tau \geqslant R)$$

we claim that for any $l \in (0,1)$ and $p > 0$ there exists an $R_2 \geqslant R_0$ such that

$$(4.13)\qquad \phi'(\tau) + lp\tau^{l-1}\phi(\tau) \leqslant 0 \qquad (\tau \geqslant R_2)$$

which in turn would imply that

$$\phi(\tau) \leqslant C\exp(-p\tau^l) \qquad (\tau \geqslant R_2)$$

for some constant $C$. Consequently when $l \in (0,1)$, choosing $p > 2m$, we obtain

$$\int_{D_{R_2}}|v_m|^2 = \int_{R_2}^{\infty}\tau^{-1}\exp\left[(2m-p)\tau^l\right]\exp(p\tau^l)\phi(\tau)\,d\tau < \infty.$$

So, it remains to establish the inequality (4.13). Let $a \geqslant n-2$ and $\eta > 0$. With the function $w_a$ from Step 2 we define

$$\tilde{G}(t,w_a) := G(t,w_a) + \eta t^{-1}\int_{S_t}|w_a|^2,$$

so that

$$(4.14)\qquad t\tilde{G}(t,w_a) - s\tilde{G}(s,w_a) = \int_{B_{st}}\left[I_1(w_a) + \tilde{I}_2(w_a)\right] \qquad (t \geqslant s > R)$$

where

$$\tilde{I}_2(w_a) := I_2(w_a) + \eta\left[-2\,\mathrm{Im}(w_a\nu Dw_a) + (n-1)r^{-1}|w_a|^2\right].$$

We recall that $I_1(w_a)$ and $I_2(w_a)$ are given by (4.3)–(4.4) with "$w_a$" replacing "$v_m$", while $j$ and $k$ are as in (4.7). For $I_1(w_a)$ we have already found the estimate (4.8). In order to estimate $\tilde{I}_2(w_a)$ from below, we choose a number $\theta > 0$ such that $\eta/\theta < c_0/2$

and observe $(\gamma - \kappa - 2M)k \geqslant 0$. Hence

$$\tilde{I}_2(w_a) \geqslant \{a[4 - (2M + g_0)r^{-1}] - \eta\theta\} \, |\nu D w_a|^2 - \{ar^{-1}[2(a + 2 - n)r^{-1} + 2M + g_0] + \eta/\theta\} \, |w_a|^2. \tag{4.15}$$

Now, fix numbers $l \in (0, 1)$ and $p > 0$ and consider the function

$$a(\tau) := \tfrac{1}{2}(lp\tau^l + n + 2M) \qquad (\tau > R_0).$$

The reason for this particular choice will become apparent from equation (4.19) below. $a$ is monotonically increasing. Also there exists a number $\rho_1 \geqslant R_0$ such that $a(\tau) > n - 2$ together with

$$a(\tau)[4 - (2M + g_0)\tau^{-1}] - \eta\theta > 0 \tag{4.16}$$

and

$$a(\tau)\tau^{-1}\{2[a(\tau) + 2 - n]\tau^{-1} + g_0\} < \eta/\theta \tag{4.17}$$

for $\tau \geqslant \rho_1$. Thus

$$t\tilde{G}(t, w_{a(\tau)}) - s\tilde{G}(s, w_{a(\tau)}) \geqslant 0 \qquad (t \geqslant s \geqslant \tau \geqslant \rho_1) \tag{4.18}$$

as a consequence of (4.14)–(4.17) and (4.8).

Now, by Step 2, $G(\cdot, w_{a(\tau)})$ is integrable at $\infty$ and hence (using $w$ instead of $w_{a(\tau)}$)

$$\lim_{t \to \infty} \inf t \, |\tilde{G}(t, w)| = 0.$$

Then by letting $t \to \infty$ in (4.18) we have

$$0 \leqslant -s\tilde{G}(s, w) = s\int_{S_s} [e(w) - 2\,|\nu D w|^2 - (q_1 + q_3 + k)\,|w|^2] - \int_{S_s} [\eta + s^{-1}Q(Q - g_0) + (2Q - g_0)\mathrm{Im}(\bar{w}\nu D w)].$$

Since

$$|(2Q - g_0)\mathrm{Im}(\bar{w}\nu D w)| \leqslant s|\nu D w|^2 + (1/4s)(2M + g_0)^2|w|^2,$$

we infer

$$0 \leqslant s\int_{S_s} [e(w) - k\,|w|^2 - |\nu D w|^2] - s\int_{S_s} q_3\,|w|^2 - \int_{S_s} \{\eta - s\,|q_1| - s^{-1}[M(M + g_0) + \tfrac{1}{4}(2M + g_0)^2]\}\,|w|^2.$$

Now we multiply both sides of the above inequality by $s^{-2a(\tau)}$, integrate over $[\tau, t]$ and use (3.8) with $g = r^{1-2a(\tau)}$. Writing "$a$" for "$a(\tau)$" we thus obtain

$$0 \leqslant \frac{1}{2}\rho^{1-2a}\int_{S_\rho} (|w|^2)'\Big|_\tau^t + \int_{B_{\tau t}} r^{-2a}[\mathrm{Im}(\bar{w}\nu D w) - r\,|Dw|^2 - rq_3\,|w|^2] - \int_{B_{\tau t}} r^{-2a}[\eta - r\,|q_1| + o(1)]\,|w|^2.$$

Integration by parts of the term involving $q_3$ yields

$$0 \leqslant \frac{1}{2}\rho^{1-2a}\int_{S_\rho}(|w|^2)'\Big|_\tau^t - \rho^{-2a}\int_{S_\rho} Q|w|^2\Big|_\tau^t$$

$$-\int_{B_{\tau t}} r^{1-2a}\big[|Dw|^2 + (2Q-1)r^{-1}\operatorname{Im}(\bar{w}\nu Dw)\big]$$

$$-\int_{B_{\tau t}} r^{-2a}\big[\eta - r|q_1| + o(1) - (n-1-2a)r^{-1}Q\big]|w|^2.$$

Now we choose $\eta > K$. Then there exists an $R_2 \geqslant \rho_1$ such that the last two terms of the above inequality are negative for $t \geqslant \tau \geqslant R_2$. Returning to the notation $a(\tau)$ and $w_{a(\tau)}$, we find

$$\tfrac{1}{2}\tau^{1-2a(\tau)}\int_{S_\tau}(|w_{a(\tau)}|^2)' = \tfrac{1}{2}\{\phi'(\tau) + [2a(\tau) - n]\tau^{-1}\phi(\tau)\}.$$

Hence

$$0 \leqslant \tfrac{1}{2}t^{1-2a(\tau)}\int_{S_t}(|w_{a(\tau)}|^2)' + Mt^{-2a(\tau)}\int_{S_t}|w_{a(\tau)}|^2 \tag{4.19}$$
$$-\tfrac{1}{2}\{\phi'(\tau) + [2a(\tau) - n - 2M]\tau^{-1}\phi(\tau)\}.$$

Since $w_{a(\tau)} \in L^2(D_R)$, it is possible to find a sequence $\{t_j\}$ such that the first term in (4.19) is nonpositive along $t_j$ and the second term goes to zero in the limit along $t_j$ as $j \to \infty$. This proves (4.14) completing Step 3.

*Step* 4. *Derivation of a contradiction*. In view of what has been said at the end of §3, it remains to show that for every $t \geqslant R_1$ with

$$\int_{S_t}|u|^2 > 0 \tag{4.20}$$

there exists a number $m > 0$ with $G(t, v_m) > 0$. However, this is clear since $G(t, v_m)$ is $\exp(2mt^l)$ times a quadratic in $m$ with the leading coefficient a positive multiple of (4.20).

**5. Appendix.** We use the notation and the assumptions from §3 except that we do *not* require $q_3$ to be spherically symmetric.

We first claim that

(5.1)

$$\begin{aligned}
0 &= \operatorname{Re}\big[(D^2 + 2ij\nu D + q - k - \lambda)v\cdot(-2ifx\,\overline{Dv} + g\bar{v})\big]\\
&= -\nabla\{2f\operatorname{Re}[(Dv)x\,\overline{Dv}] - \tfrac{1}{2}(\nabla g)|v|^2 + \tfrac{1}{2}g\nabla|v|^2\}\\
&\quad + r^{1-n}\{r^n f[|Dv|^2 + (q_2 - k - \lambda)|v|^2]\}'\\
&\quad + [(2-n)f - rf' + g]|Dv|^2 + 2r(2fj + f')|\nu Dv|^2\\
&\quad - 2\{f\operatorname{curl} b\cdot x\operatorname{Re}(\bar{v}Dv) + [rf(q - q_2) + gj]\operatorname{Im}(\bar{v}\nu Dv)\}\\
&\quad + [(nf + rf' - g)(\lambda - q_2 + k) - rf(q_2 - k)' + g(q - q_2) - \tfrac{1}{2}\Delta g]|v|^2.
\end{aligned}$$

Apart from a slight change in notation this is basically the identity (3.3) in **[5]** which was stated there without proof.

PROOF OF (5.1). We treat the terms involving the function $g$ first. Comparing

$$D^2 v = -\Delta v + (b^2 + i\nabla b)v + 2ib\nabla v \tag{5.2}$$

and

$$|Dv|^2 = |\nabla v|^2 + b^2|v|^2 - 2b\,\mathrm{Im}(\bar{v}\nabla v), \tag{5.3}$$

we find

$$\mathrm{Re}(\bar{v}D^2 v) = |Dv|^2 - \left[\mathrm{Re}(\bar{v}\Delta v) + |\nabla v|^2\right] = |Dv|^2 - \tfrac{1}{2}\Delta|v|^2$$

or

$$g\,\mathrm{Re}(\bar{v}D^2 v) = \tfrac{1}{2}\nabla\left[(\nabla g)|v|^2 - g\nabla|v|^2\right] + g|Dv|^2 - \tfrac{1}{2}(\Delta g)|v|^2.$$

Thus

$$\begin{aligned} 0 &= g\,\mathrm{Re}\left[(D^2 + 2ij\nu D + q - k - \lambda)v \cdot \bar{v}\right] \\ &= \tfrac{1}{2}\nabla\left[(\nabla g)|v|^2 - g\nabla|v|^2\right] \\ &\quad + g\left[e(v) - k|v|^2\right] - \tfrac{1}{2}(\Delta g)|v|^2 - 2gj\,\mathrm{Im}(\bar{v}\nu Dv). \end{aligned} \tag{5.4}$$

When we rewrite

$$\nabla\left[(\nabla g)|v|^2 - g\nabla|v|^2\right] - (\Delta g)|v|^2$$

as

$$-\nabla\left(g\nabla|v|^2\right) + (\nabla g)\nabla|v|^2$$

and note that

$$\left(|v|^2\right)' = -2\,\mathrm{Im}(\bar{v}\nu Dv), \tag{5.5}$$

integration of (5.4) over $B_{st}$ yields relationship (3.8) in Lemma 1.

Using (5.5) again, we next observe that

$$\begin{aligned} 2\,\mathrm{Re}&\left[(q - k - \lambda)v\left(-ifx\overline{Dv}\right)\right] \\ &= 2rf(q - q_2)\mathrm{Im}\left(v\nu\overline{Dv}\right) + rf(q_2 - k - \lambda)\left(|v|^2\right)' \\ &= -2rf(q - q_2)\mathrm{Im}(\bar{v}\nu Dv) + r^{1-n}\left[r^n f(q_2 - k - \lambda)|v|^2\right]' \\ &\quad + \left[(nf + rf')(\lambda - q_2 + k) - rf(q_2 - k)'\right]|v|^2. \end{aligned} \tag{5.6}$$

Finally we assert that

$$\begin{aligned} 2\,\mathrm{Re}\left[(D^2 v)\left(-ifx\overline{Dv}\right)\right] &= -\nabla\left\{2f\,\mathrm{Re}\left[(Dv)\left(x\overline{Dv}\right)\right]\right\} + r^{1-n}\left(r^n f|Dv|^2\right)' \\ &\quad + \left[(2 - n)f - rf'\right]|Dv|^2 + 2rf'|\nu Dv|^2 \\ &\quad - 2f\,\mathrm{curl}\, b \cdot x\,\mathrm{Re}(\bar{v}Dv). \end{aligned} \tag{5.7}$$

Once we have verified this identity, relationship (5.1) follows upon adding (5.4) and (5.6)–(5.7).

For the sake of brevity we write "c.c." to indicate that the complex conjugate of the terms preceding this sign has to be added. We write the right-hand side of (5.7) as follows:

$$-rf\big[(\nabla Dv)\nu\overline{Dv} + (Dv)\nabla(\nu\overline{Dv}) + \text{c.c.}\big] + rf\nu\nabla\big[(Dv)\overline{Dv}\big]$$
$$+2f\big[|Dv|^2 - |\nu Dv|^2 - \operatorname{curl} b\cdot x\operatorname{Re}(\bar{v}Dv)\big]$$
$$= rf\big\{-\big[\tfrac{1}{i}\Delta v - (\nabla b)v - b\nabla v\big]\nu\overline{Dv}$$
$$-(Dv)\big[(\overline{Dv}\cdot\nabla)\nu + (\nu\nabla)\overline{Dv}$$
$$+(\operatorname{curl}\nu)\overline{Dv} + (\operatorname{curl}\overline{Dv})\nu\big] + (Dv)\big[(\nu\nabla)\overline{Dv}\big] + \text{c.c.}\big\}$$
$$+2f\big[|Dv|^2 - |\nu Dv|^2 - \operatorname{curl} b\cdot x\operatorname{Re}(\bar{v}Dv)\big]$$
$$= rf\big\{\tfrac{1}{i}\big[-\Delta v + (b^2 + i\nabla b)v + 2ib\nabla v\big]\nu\overline{Dv} - (b\nabla v - ib^2v)\nu\overline{Dv}$$
$$-(Dv)\big[r^{-1}\overline{Dv} - r^{-1}\nu(\nu\overline{Dv}) - (\operatorname{curl} b\cdot\nu)\bar{v} - (\nu b)\nabla\bar{v} + b\bar{v}'\big] + \text{c.c.}\big\}$$
$$+2f\big[|Dv|^2 - |\nu Dv|^2 - \operatorname{curl} b\cdot x\operatorname{Re}(\bar{v}Dv)\big]$$
$$= -2rf\operatorname{Re}\big[i(D^2v)\nu\overline{Dv}\big] + 2f\operatorname{curl} b\cdot x\operatorname{Re}(\bar{v}Dv) - 2f\operatorname{curl} b\cdot x\operatorname{Re}(\bar{v}Dv).$$

The other terms sum up to zero:

$$-(b\nabla v - ib^2v)\nu\overline{Dv} + (Dv)\big[(\nu b)\nabla\bar{v} - b\bar{v}'\big] + \text{c.c.}$$
$$= -i(bDv)\big(-\tfrac{1}{i}\bar{v}'\big) + (b\nabla v)\nu b\bar{v} + ib^2(\nu b)|v|^2 + \tfrac{1}{i}\nu b|\nabla v|^2$$
$$-(b\nabla\bar{v})\nu bv - (bDv)\bar{v}' + \text{c.c.} = 0. \quad \square$$

Proof of Lemma 1. There are two terms in (5.1) which involve the potential $q_3$, namely,

$$gq_3|v|^2 \quad\text{and}\quad -2rfq_3\operatorname{Im}(\bar{v}\nu Dv).$$

Let $t \geqslant s > R$. Writing

$$\hat{Q}(x) := \int_s^{|x|}\sigma q_3\Big(\sigma\frac{x}{|x|}\Big)\,d\sigma \qquad (x\in D_R),$$

integration by parts gives

$$\int_{B_{st}} gq_3|v|^2 = t^{-1}\int_{S_t} g\hat{Q}|v|^2 - \int_{B_{st}}\hat{Q}\big[r^{1-n}(r^{n-2}g)'|v|^2 - 2r^{-1}g\operatorname{Im}(\bar{v}\nu Dv)\big]. \tag{5.8}$$

Similarly,

$$J := -2\int_{B_{st}} rfq_3\operatorname{Im}(\bar{v}\nu Dv) \tag{5.9}$$
$$= \int_{S_t} f\hat{Q}(|v|^2)' - \int_{B_{st}}\hat{Q}\big[r^{1-n}(r^{n-1}f)'(|v|^2)' + f(|v|^2)''\big].$$

We write $\nabla_S$ and $\Delta_S$ for the gradient and the Laplacian on the unit sphere in $\mathbf{R}^n$, respectively. Then

$$\Delta v = v'' + (n-1)r^{-1}v' + r^{-2}\Delta_S v.$$

Owing to (3.2) and (5.2)–(5.3) we thus obtain

$$\begin{aligned}(|v|^2)'' &= 2\left[|v'|^2 + \mathrm{Re}(\bar{v}v'')\right]\\ &= 2\Big[|v'|^2 - r^{-2}\mathrm{Re}(\bar{v}\Delta_S v) - \tfrac{1}{2}(n-1)r^{-1}(|v|^2)'\\ &\qquad - \mathrm{Re}(\bar{v}D^2 v) + b^2|v|^2 - 2b\,\mathrm{Im}(\bar{v}\nabla v)\Big]\\ &= 2\Big[|v'|^2 - |\nabla v|^2 - r^{-2}\mathrm{Re}(\bar{v}\Delta_S v) - \tfrac{1}{2}(n-1)r^{-1}(|v|^2)'\\ &\qquad + |Dv|^2 + (q-k-\lambda)|v|^2 - 2j\,\mathrm{Im}(\bar{v}\nu Dv)\Big].\end{aligned}$$

We insert this into (5.9):

(5.10)

$$\begin{aligned}J &= \int_{S_t} f\hat{Q}(|v|^2)'\\ &\quad + \int_{B_{st}} 2\hat{Q}\{(2fj+f')\mathrm{Im}(\bar{v}\nu Dv) - f[|Dv|^2 + (q_1+q_2-k-\lambda)|v|^2]\}\\ &\quad - \int_{B_{st}} 2f\hat{Q}q_3|v|^2 + \tilde{J}.\end{aligned}$$

Here

(5.11)

$$-\int_{B_{st}} 2f\hat{Q}q_3|v|^2 = -t^{-1}\int_{S_t} f\hat{Q}^2|v|^2 + \int_{B_{st}} \hat{Q}^2\left[r^{1-n}(r^{n-2}f)'|v|^2 - r^{-1}f\,\mathrm{Im}(\bar{v}\nu Dv)\right]$$

and

$$\begin{aligned}\tilde{J} &:= \int_{B_{st}} 2f\hat{Q}\left[|\nabla v|^2 - |v'|^2 + r^{-2}\mathrm{Re}(\bar{v}\Delta_S v)\right]\\ &= \int_{B_{st}} 2r^{-2}f\{\hat{Q}|\nabla_S v|^2 - \mathrm{Re}[(\nabla_S v)\nabla_S(\hat{Q}\bar{v})]\}\\ &= -\int_{B_{st}} 2r^{-2}f(\nabla_S\hat{Q})\mathrm{Re}(\bar{v}\nabla_S v).\end{aligned}$$

If $q_3$ is spherically symmetric, then $\tilde{J} = 0$, and Lemma 1 follows by integrating (5.1) over $B_{st}$ and using (5.8)–(5.11). □

Now

$$|\tilde{J}| \leq \int_{B_{st}} r^{-1}|f\nabla_S\hat{Q}||v||\nabla v|.$$

Since an arbitrarily small factor in front of

$$-\int_{B_{st}} (|v|^2 + |Dv|^2)$$

would not affect our reasoning (nor that in [7]), we could replace the requirement that $q_3$ be rotationally symmetric by the assumption that $b$ is bounded at infinity and that there is for every $\varepsilon > 0$ a $\rho > R$ such that

$$r^{-1}|\nabla_S\hat{Q}| < \varepsilon$$

on $D_\rho$.

ACKNOWLEDGEMENTS. V. Krishna Kumar is indebted to the DAAD (Federal Republic of Germany) for financial support and to the University of Calicut for granting study leave to visit the Technische Hochschule Darmstadt during 1980–1981.

## REFERENCES

1. S. Agmon, *Lower bounds for solutions of Schrödinger equations*, J. Analyse Math. **23** (1970), 1–25.

2. F. V. Atkinson, *The asymptotic solution of second-order differential equations*, Ann. Mat. Pura Appl. **37** (1954), 347–378.

3. M. S. P. Eastham and H. Kalf, *Schrödinger-type operators with continuous spectra*, Research Notes in Mathematics 65, Pitman, Boston, London, Melbourne, 1982.

4. D. M. Eidus, *The principle of limit amplitude*, Russian Math. Surveys **24** (1969), no. 3, 97–167.

5. H. Kalf, *Non-existence of eigenvalues of Schrödinger operators*, Proc. Roy. Soc. Edinburgh Sect. A **79** (1977), 157–172.

6. T. Kato, *Growth properties of solutions of the reduced wave equation with variable coefficients*, Comm. Pure Appl. Math. **12** (1959), 403–425.

7. G. B. Khosrovshahi, H. A. Levine and L. E. Payne, *On the positive spectrum of Schrödinger operators with long range potentials*, Trans. Amer. Math. Soc. **253** (1979), 211–228.

8. M. Reed and B. Simon, *Analysis of operators*, Methods of Modern Mathematical Physics. IV, Academic Press, New York, 1978.

9. S. N. Roze, *On the spectrum of an elliptic operator of second order*, Math. USSR Sb. **9** (1969), 183–197.

FACHBEREICH MATHEMATIK, TH DARMSTADT, SCHLOSSGARTENSTRASSE 7, D-6100 DARMSTADT, FEDERAL REPUBLIC OF GERMANY

*Current address* (V. Krishna Kumar): Department of Mathematics, University of Calicut, Pin: 673 635, India