DIMENSION OF STRATIFIABLE SPACES

BY

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Abstract. We define a subclass, denoted by \( EM_3 \), of the class of stratifiable spaces, and obtain several dimension theoretical results for \( EM_3 \) including the coincidence theorem for \( \dim \) and \( \text{Ind} \). The class \( EM_3 \) is countably productive, hereditary, preserved under closed maps and, furthermore, the largest subclass of stratifiable spaces in which a harmonious dimension theory can be established.

1. Introduction. Beyond metric spaces, the following line of generalized metric spaces has been established by many authors [S, C, B, H, Ok]:

\[
\text{metric} \rightarrow \text{Lašnev}^1 \rightarrow M_1 \rightarrow \text{stratifiable} \rightarrow \text{paracompact } \sigma. 
\]

After Katětov and Morita's work for metric spaces, the first attack to this line in dimension theory was done by Leibo [L.] who proved the equality \( \dim X = \text{Ind} X \) for any Lašnev space \( X \). Nagami extended this result by defining \( L \)-spaces [N.] and free \( L \)-spaces [N.]. Free \( L \)-spaces form a countably productive and hereditary class containing every Lašnev space and included in the class of \( M_1 \)-spaces. It is now desired to develop a satisfactory dimension theory of a still larger class of generalized metric spaces, say, \( M_1 \)-spaces or stratifiable spaces.

In this direction we define a subclass of stratifiable spaces in terms of a special kind of \( \sigma \)-closure-preserving collection.

Definition 1.1. Let \( X \) be a space. A collection \( \mathcal{S} \) of subsets of \( X \) is called an encircling net (or, for short, \( E \)-net) if for any point \( x \) and any open neighborhood \( U \) of \( x \), there exists a subcollection \( \mathcal{T} \) of \( \mathcal{S} \) such that \( x \in X - \mathcal{T}^* \subseteq U \) and \( \mathcal{T}^* \) is a closed set of \( X \) (where \( \mathcal{T}^* \) denotes the union of the members of \( \mathcal{T} \)).

By \( EM_3 \) we denote the class of stratifiable spaces with \( \sigma \)-closure-preserving \( E \)-nets, and by \( M_3 \) the class of stratifiable spaces.

The class \( EM_3 \) is countably productive, hereditary and preserved under closed maps as well as perfect maps (Corollary 3.9).

Our first main result is a characterization of members of \( EM_3 \) as those spaces which are the perfect (closed) images of zero-dimensional stratifiable spaces (Theorem 3.8). This means that \( EM_3 \) is just the maximal perfect subclass of \( M_3 \) in the sense of Nagami [N.].
The second main results appear in Theorems 4.2 and 4.3 and consist of the following theorems for EM3:
(a) the equidimensional $G_s$-envelope theorem,
(b) the dimension raising theorem,\(^2\)
(c) the decomposition theorem,
(d) the coincidence theorem for dim and Ind.

These theorems for EM3 extend the corresponding theorems for free L-spaces \([N_4]\) as well as those for Lašnev spaces \([L_{11}, L_{2}, O_{11}]\).

It is an open problem whether the inclusion $EM_3 \subset M_3$ is proper. But the characterization above implies that $EM_3$ is the largest\(^3\) subclass of $M_3$ in which the dimension raising theorem holds. We also see in Corollary 4.5 that $EM_3$ is the largest\(^3\) subclass of $M_3$ in which the decomposition theorem and the equidimensional $G_s$-envelope theorem simultaneously hold.

Our arguments are based on Gruenhage and Junnila's result that a stratifiable space is an $M_2$-space \([G, J]\). Indeed, though we use the word "stratifiable" in view of its significance, what we need is only the existence of a $\sigma$-closure-preserving quasi-base.

**Conventions.** Throughout this paper a space is a *Hausdorff topological space*, and a map means an onto continuous one. Let $X, Y$ be spaces and let $f : X \to Y$ be a map. For a collection $\mathcal{F}$ of subsets of $X$, the symbol $\mathcal{F}^*$ denotes the union of all members of $\mathcal{F}$, and $f(\mathcal{F})$ means the collection of subsets of $Y$ of the form $\{f(F) : F \in \mathcal{F}\}$. For a subset $Z$ of $X$ we denote by $\overline{Z}$ (or $\text{Cl} Z$) the closure of $Z$, by $\text{Int} Z$ the interior of $Z$, and by $\text{Bd} Z$ the boundary of $Z$.

2. Encircling nets and large encircling nets. Encircling nets are naturally strengthened as follows:

**Definition 2.1.** Let $X$ be a space. A collection $\mathcal{E}$ of subsets of $X$ is called a *large encircling net* (or, simply, an *LE-net*) if for any disjoint closed sets $C$ and $K$ of $X$, there exists a subcollection $\mathcal{E}_0 \subset \mathcal{E}$ such that $C \in \mathcal{E}_0 \subset X - K$ and $\mathcal{E}_0^*$ is a closed set of $X$.

**Remarks.** Since an LE-net is a net in the usual sense, it follows from Siwiec-Nagata [SN] that a space with a $\sigma$-closure-preserving LE-net is a $\sigma$-space. But a space with a $\sigma$-closure-preserving $E$-net is not necessarily a $\sigma$-space as will be seen in Example 2.8. On the other hand it is trivial that a regular $\sigma$-space $X$ with $\text{ind} X \leq 0$ admits a $\sigma$-closure-preserving $E$-net, and that a normal $\sigma$-space $X$ with $\text{dim} X \leq 0$ admits a $\sigma$-closure-preserving LE-net.

**Proposition 2.2.** A metric space admits a $\sigma$-locally finite LE-net.

**Proof.** Let $M$ be a metric spaces and \{\(\mathcal{S}_i\); \(i = 1, 2, \ldots\)\} a sequence of locally finite closed covers of $M$ such that, for each $i$, the diameter of each member of $\mathcal{S}_i$ is smaller than $1/i$. Let $C, K$ be disjoint closed sets of $M$ and put

$$\mathcal{F}_i = \{ E \in \mathcal{S}_i ; E \cap C \neq \emptyset \text{ and } E \cap K = \emptyset \}.$$  

\(^2\) The dimension raising theorem for a topological class $\mathcal{C}$ is: If $X \in \mathcal{C}$ and $\text{dim} X \leq n$, then $X$ is the image of a space $X_0 \in \mathcal{C}$ with $\text{dim} X_0 \leq 0$ under a perfect map of order not greater than $n + 1$.

\(^3\) When using this word we take no account of infinite-dimensional spaces in the sense of dim.
It is then clear that \( \bigcup_{i=1}^{\infty} \mathcal{S}_i^* \) is a closed set of \( X \) including \( C \) but not meeting \( K \). Hence \( \bigcup_{i=1}^{\infty} \mathcal{S}_i \) is a \( \sigma \)-locally finite \( LE \)-net on \( M \), which completes the proof.

**Proposition 2.3.** The property of having a \( \sigma \)-closure-preserving \( LE \)-net is preserved under closed maps.

We thus have

**Proposition 2.4.** A Lašnev space admits a \( \sigma \)-closure-preserving \( LE \)-net, and hence it is a member of \( EM_3 \).

**Lemma 2.5.** If \( \mathcal{S} \) is an \( E \)-net (resp. \( LE \)-net) on a space, then \( \{ E : E \in \mathcal{S} \} \) is an \( E \)-net (resp. \( LE \)-net) on the space.

**Proposition 2.6.** The property of having a \( \sigma \)-closure-preserving \( E \)-net is countably productive, hereditary and preserved under perfect maps.

**Proof.** Let \( X_i, i = 1, 2, \ldots, \) be spaces with \( \sigma \)-closure-preserving \( E \)-nets \( \mathcal{S}_i \). It is then clear that

\[
\left\{ E_j \times \prod_{i=1, i \neq j}^{\infty} X_i : E_j \in \mathcal{S}_j, j = 1, 2, \ldots \right\}
\]

is a \( \sigma \)-closure-preserving \( E \)-net on \( \prod_{i=1}^{\infty} X_i \).

By the preceding lemma it is obvious that the property is hereditary.

Let \( X \) be a space with a \( \sigma \)-closure-preserving \( E \)-net \( \mathcal{S} \) and let \( f : X \to Y \) be a perfect map onto a space \( Y \). By Lemma 2.5 we may assume that every finite intersection of members of \( \mathcal{S} \) is again a member of \( \mathcal{S} \). To show that \( f(\mathcal{S}) \) is an \( E \)-net on \( Y \) let \( y \in Y \) and let \( U \) be an open neighborhood of \( y \). There exist subcollections \( \mathcal{S}_i, 1 \leq i \leq k \), of \( \mathcal{S} \) such that \( f^{-1}(y) \subset X - \bigcap_{i=1}^{k} \mathcal{S}_i^* \subset f^{-1}(U) \) and \( \mathcal{S}_i^* \) is a closed set of \( X \). It then follows from assumption that \( f(\bigcap_{i=1}^{k} \mathcal{S}_i^*) \) is a closed set of \( Y \) written as a union of members of \( f(\mathcal{S}) \) such that \( y \in Y - f(\bigcap_{i=1}^{k} \mathcal{S}_i^*) \subset U \). This completes the proof.

**Proposition 2.7.** Let \( X \) be a space (resp. a semistratifiable space). Then the following statements are equivalent:

1. \( X \) admits a \( \sigma \)-closure-preserving \( LE \)-net (resp. \( E \)-net).
2. \( X \) admits a \( \sigma \)-locally finite \( LE \)-net (resp. \( E \)-net).
3. \( X \) admits a \( \sigma \)-discrete \( LE \)-net (resp. \( E \)-net).

**Proof.** It follows from Lemma 2.5 and a remark above that a space with a \( \sigma \)-closure-preserving \( LE \)-net admits a \( \sigma \)-closure-preserving net of closed sets, and therefore it is semistratifiable. Hence the proposition is immediate from Lemma 2.5 and the following fact, which is essentially due to Siwiec and Nagata [SN]: Let \( X \) be a semistratifiable space and \( \mathcal{S} \) a \( \sigma \)-closure-preserving collection of closed sets of \( X \). Then there exists a \( \sigma \)-discrete collection \( \mathcal{S} \) of closed sets of \( X \) such that each member of \( \mathcal{S} \) is a union of members of \( \mathcal{S} \).

As for famous pathological spaces, we have the following results which imply particularly that the existence of \( \sigma \)-closure-preserving \( E \)-nets does not mean, in general, that of \( \sigma \)-closure-preserving \( LE \)-nets (but, for stratifiable spaces, the former means the latter as will be seen in Theorem 3.8).
Examples 2.8. (1) The Michael line $I(M)$ has a $\sigma$-discrete $E$-net, but does not have a $\sigma$-closure-preserving $LE$-net.

(2) The same is true for the Sorgenfrey line $R(S)$.

(3) $[0, \omega_1]$ does not admit a $\sigma$-closure-preserving $E$-net.

(4) The quotient space $I(M)/Q$ obtained by identifying the rational points in $I(M)$ does not admit a $\sigma$-closure-preserving $E$-net. In particular the property of having a $\sigma$-closure-preserving $E$-net is not preserved under closed maps.

Proof. (1) and (2) (simultaneously). Let $\mathcal{F}$ be a $\sigma$-discrete net of closed sets in the unit interval $I$ (resp. the real line $R$) with the usual topology. It is easy to see that $\mathcal{F}$ is a $\sigma$-discrete $E$-net on $I(M)$ (resp. $R(S)$). But $I(M)$ (resp. $R(S)$) does not admit a $\sigma$-closure-preserving $LE$-net because it is not a $\sigma$-space.

(3) For any $\sigma$-closure-preserving collection $\mathcal{F}$ of $[0, \omega_1]$, $\mathcal{F}$ fails to be an $E$-net at $\omega_1$; indeed, $\text{Cl}(\{\bar{F}: F \in \mathcal{F}, \omega_1 \not\in \bar{F}\}) \cap \{\omega_1\} = \emptyset$.

(4) If $I(M)/Q$ had a $\sigma$-closure-preserving $E$-net, then every point in $I(M)/Q$, in particular the quotient image of $Q$, would be a $G_\delta$-set of $I(M)/Q$; but this is impossible because $Q$ is not a $G_\delta$-set of $I(M)$.


Lemma 3.1 [O2, Lemma 3.1]. Let $X$ be a submetrizable space (that is, $X$ admits a weaker metric topology), and let $\mathcal{U}$ be a $\sigma$-discrete collection of cozero sets of $X$. Then there exists a metric space $M$ and a one-to-one map $f: X \to M$ such that $f(U)$ is an open set of $M$ for every $U \in \mathcal{U}$.

The following lemma plays a fundamental role in this paper.

Lemma 3.2. Let $X$ be a paracompact $\sigma$-space and let $\mathcal{F} = \bigcup_{i=1}^\infty \mathcal{F}_i$ be a collection of closed sets of $X$ such that $\mathcal{F}_i$ is closure-preserving for each $i$. Then there exist a metric space $M$ and a one-to-one map $f: X \to M$ such that $f(F)$ is a closed set of $M$ for every $F \in \mathcal{F}$ and such that $f(\mathcal{F}_i)$ is closure-preserving in $M$ for every $i$.

Proof. Let $\mathcal{B} = \bigcup_{i=1}^\infty \mathcal{B}_i$ be a net of $X$ consisting of closed sets such that $\mathcal{B}_i$ is discrete for each $i$. For each $i$ let $\mathcal{V}_i = \{V_i(F): B \in \mathcal{B}_i\}$ be a discrete collection of open sets of $X$ such that $B \subset V_i(F)$ for each $B \in \mathcal{B}_i$. For $i, j = 1, 2, \ldots, B \in \mathcal{B}_i$, put

$$W_i(B) = V_i(B) \cap \left( X - \{F: F \in \mathcal{B}_j, F \cap B = \emptyset\}^* \right).$$

Then $W_i(B)$ is an open set of $X$, and $\{W_i(B): B \in \mathcal{B}_i\}$ is discrete in $X$. Hence Lemma 3.1 applies to give a metric space $M$ and a one-to-one map $f: X \to M$ such that $f(W_i(B))$ is an open set of $M$ for every $B \in \mathcal{B}_i$, $i, j = 1, 2, \ldots$. It is then obvious that for each $i$, $f(\mathcal{F}_i)$ is a closure-preserving collection of closed sets of $M$. This completes the proof.

Definition 3.3. Let $X \in EM_3$ and let $\{\mathcal{F}, \mathcal{V}, \mathcal{S}, \mathcal{S}\}$ be a quartet of collections of subsets of $X$. The quartet is called an $E$-quartet if we can write $\mathcal{F} = \bigcup_{i=1}^\infty \mathcal{F}_i$, $\mathcal{V} = \bigcup_{i=1}^\infty \mathcal{V}_i$, $\mathcal{S} = \bigcup_{i=1}^\infty \mathcal{S}_i$, $\mathcal{S} = \bigcup_{i=1}^\infty \mathcal{S}_i$, and if the following four conditions are satisfied:

(1) $\mathcal{F}$ is a net on $X$ consisting of closed sets.
(2) For each i, \( V_i \) is a discrete collection of open sets of X written as \( V_i = \{ V_i(F) : F \in T_i \} \) in such a manner that \( F \subseteq V_i(F) \) for each \( F \in T_i \).

(3) \( S \) is an E-net on X consisting of closed sets and \( S_i \) is closure-preserving for each i.

(4) \( S \) is a quasi-base\(^4\) for X consisting of closed sets and \( S_i \) is closure-preserving for each i.

By Heath [H], Gruenhage [G] and Junnila [J], each member of \( EM_3 \) admits an E-quartet.

**Definition 3.4.** Let X be a member of \( EM_3 \) with an E-quartet \( \{ T, V, \mathcal{S}, S \} \). A map \( f: X \to Y \) onto a normal space Y is called an E-map with respect to the E-quartet if the following five conditions are satisfied:

1. \( f \) is one-to-one.
2. \( f(F) \) is a closed set for every \( F \in T \).
3. \( f(V) \) is an open set for every \( V \in V \).
4. \( f(E) \) is a closed set for every \( E \in \mathcal{S} \), and \( f(S_i) \) is closure-preserving in Y for every i.
5. \( f(S) \) is a closed set for every \( S \in S \), and \( f(S_i) \) is closure-preserving in Y for every i.

Noting that \( \{ X - V : V \in V \} \) is a \( \sigma \)-closure-preserving collection of closed sets of X, we have the following result by virtue of Lemma 3.2.

**Proposition 3.5.** Let X be a member of \( EM_3 \). Then for any E-quartet of X there exist a metric space M and an E-map \( f: X \to M \) with respect to the E-quartet.

The following lemma is well known (see, for example, [E, 2.3.16]).

**Lemma 3.6.** Let X be a space and let \( C, K \) be disjoint closed sets of X. Let \( \mathcal{U} \) be a countable open cover of X such that for each \( U \in \mathcal{U} \), either \( \overline{U} \cap C = \emptyset \) or \( \overline{U} \cap K = \emptyset \). Then \( C \) and \( K \) are separated by a closed set \( S \) such that \( S \subseteq \{ \text{Bd } U : U \in \mathcal{U} \}^* \).

Now we have the following result frequently used later.

**Proposition 3.7.** Let X be a member of \( EM_3 \) with an E-quartet \( \{ T, V, \mathcal{S}, S \} \). Let \( f: X \to Y \) be an E-map with respect to the E-quartet onto a normal space Y. Then \( \text{Ind } X \leq \text{Ind } Y \).

**Proof.** The proof is by induction on \( \text{Ind } Y \). If \( Y = \emptyset \) then the proposition is trivial. Suppose that the proposition is valid when \( \text{Ind } Y \leq n - 1 \) and consider the case of \( \text{Ind } Y = n \). To show \( \text{Ind } X \leq n \), let \( C, K \) be disjoint closed sets of X. For the time being, fix a point \( x \) in \( X - C \) arbitrarily. We show that there exists an open neighborhood \( W \) of \( x \) such that \( \overline{W} \cap C = \emptyset \) and \( \text{Ind } \text{Bd } W \leq n - 1 \). Let \( \mathcal{S}(x) \) be a subcollection of \( \mathcal{S} \) such that \( x \in X - \mathcal{S}(x)^* \subseteq X - C \) and \( \mathcal{S}(x)^* \) is a closed set. Write \( \mathcal{S}(x) = \bigcup_{i=1}^{\infty} \mathcal{S}_i(x) \) where \( \mathcal{S}_i(x) \subseteq \mathcal{S}_i \). Put \( \mathcal{S}_i(x) = \{ S \in \mathcal{S}_i : S \cap \mathcal{S}(x)^* = \emptyset \} \)

\(^4\) A collection \( \mathcal{S} \) of subsets of a space X is called a quasi-base for X if for any point x and any open neighborhood \( U \) of x there exists a member \( S \) of \( \mathcal{S} \) such that \( x \in \text{Int } S \subseteq S \subseteq U \).
and \( S(x) = \bigcup_{i=1}^{\infty} S_i(x). \) Fix \( i_0 \) so that \( x \in \operatorname{Int} S_{i_0}(x)^* \). By (3t) and (4t) there exist open sets \( O_j, j = 1, 2, \ldots, \) of \( Y \) such that

\[
f\left( \bigcup_{j=1}^{\infty} S_j(x)^* \right) \cup f(S_{i_0}(x)^*) \subset O_j \subset \overline{O_j} \subset Y - f(S_j(x)^*)
\]

and

\[
\operatorname{Ind} \operatorname{Bd} O_j \leq n - 1.
\]

Define \( W = \bigcap_{j=1}^{\infty} f^{-1}(O_j) \). Then

\[
x \in W \subset \overline{W} \subset \bigcap_{j=1}^{\infty} f^{-1}(\overline{O_j}) \subset X - S(x)^* \subset X - C.
\]

To show that \( W \) is open, let \( x' \in W \). Since \( x' \in X - S(x)^* \) and \( S(x)^* \) is a closed set, it follows from (4q) that \( x' \in \operatorname{Int} S_{m}(x)^* \) for some \( m \). Then

\[
x' \in \bigcap_{j=1}^{m-1} f^{-1}(O_j) \cap \operatorname{Int} S_{m}(x)^* \subset W,
\]

which implies that \( W \) is open. To show \( \operatorname{Ind} \operatorname{Bd} W \leq n - 1 \), note that, for any subset \( Z \) of \( X, f|Z: Z \to f(Z) \) is again an \( E \)-map with respect to the \( E \)-quartet \( \{ S|Z, S|Z, S|Z, S|Z \} \) on \( Z \). Hence we may apply induction hypothesis to obtain \( \operatorname{Ind} f^{-1}(\operatorname{Bd} O_j) \leq n - 1, j = 1, 2, \ldots, \) which yields

\[
\operatorname{Ind} \operatorname{Bd} W \leq \operatorname{Ind} \left( \bigcup_{j=1}^{\infty} \operatorname{Bd} f^{-1}(O_j) \right)
\]

\[
= \max \left\{ \operatorname{Ind} \operatorname{Bd} f^{-1}(O_j) : j = 1, 2, \ldots \right\}
\]

\[
\leq \max \left\{ \operatorname{Ind} f^{-1}(\operatorname{Bd} O_j) : j = 1, 2, \ldots \right\} \leq n - 1.
\]

Hence \( W \) is a required open neighborhood of \( x \); we have thus finished "local" separation.

Now put

\[
\mathcal{F}_i(C) = \{ F \in \mathcal{F} : F \subset W \text{ for some open set } W \text{ with } \overline{W} \cap C = \emptyset \text{ and } \operatorname{Ind} \operatorname{Bd} W \leq n - 1 \}.
\]

Then by (1q) and by the "local" separation above, we have \( \bigcup_{i=1}^{\infty} \mathcal{F}_i(C)^* = X - C \). For each \( F \in \mathcal{F}_i(C) \), fix such a \( W \) and denote it by \( W_i(C, F) \). On the other hand, by (1t) and (2t), there exist open sets \( H_i(F), F \in \mathcal{F}_i \), of \( Y \) such that \( f(F) \subset H_i(F) \subset \overline{f(V_i(F))} \) and \( \operatorname{Ind} \operatorname{Bd} H_i(F) \leq n - 1 \) (where the set \( V_i(F) \) is as in Definition 3.3(2q)). By induction hypothesis again,

\[
\operatorname{Ind} \operatorname{Bd} f^{-1}(H_i(F)) \leq \operatorname{Ind} f^{-1}(\operatorname{Bd} H_i(F)) \leq n - 1.
\]

Put for each \( F \in \mathcal{F}_i(C) \),

\[
D_i(C, F) = W_i(C, F) \cap f^{-1}(H_i(F)).
\]

Then

\[
\operatorname{Ind} \operatorname{Bd} D_i(C, F) \leq \max \{ \operatorname{Ind} \operatorname{Bd} W_i(C, F), \operatorname{Ind} f^{-1}(H_i(F)) \} \leq n - 1.
\]
Put $D_i(C) = \{D_i(C, F) : F \in \mathcal{F}(C)\}$. Since $D_i(C, F) \subset V_i(F)$, (2$_q$) implies that $\{D_i(C, F) : F \in \mathcal{F}(C)\}$ is discrete. Thus $\text{Ind} \text{Bd} D_i(C) \leq n - 1$, $i = 1, 2, \ldots$. By the same discreteness and by the fact $D_i(C, F) \subset W_i(C, F) \subset \text{Cl} W_i(C, F) \subset X - C$, we have $C \cap \text{Cl} D_i(C) = \emptyset$ for every $i = 1, 2, \ldots$. We also obtain $\bigcup_{i=1}^{\infty} D_i(C) = X - C$ because $\bigcup_{i=1}^{\infty} \mathcal{F}(C) = X - C$.

Quite similarly we can obtain open subsets $D_i(K)$, $i = 1, 2, \ldots$, such that $\text{Ind} \text{Bd} D_i(K) \leq n - 1$, $K \cap \text{Cl} D_i(K) = \emptyset$ and $\bigcup_{i=1}^{\infty} D_i(K) = X - K$. Hence, applying Lemma 3.6, we have a closed set $B$ separating $C$ and $K$ such that

$$B \subset \left( \bigcup_{i=1}^{\infty} \text{Bd} D_i(C) \right) \cup \left( \bigcup_{i=1}^{\infty} \text{Bd} D_i(K) \right) .$$

By the countable sum theorem for Ind, we have $\text{Ind} B \leq n - 1$. Thus Ind $X \leq n$, which completes the proof of Proposition 3.7.

We can now prove a characterization theorem for $EM_3$.

**THEOREM 3.8.** The following statements about a space $X$ are equivalent:

1. $X$ is a stratifiable space with a $\sigma$-closure-preserving $E$-net.
2. $X$ is the perfect image of a stratifiable space $X_0$ with $\dim X_0 \leq 0$.
3. $X$ is the closed image of a stratifiable space $X_0$ with $\text{ind} X_0 \leq 0$.
4. $X$ is a stratifiable space with a $\sigma$-closure-preserving $LE$-net.

**PROOF.** The implications (2) $\rightarrow$ (3) and (4) $\rightarrow$ (1) are obvious. To show (1) $\rightarrow$ (2) let $X$ be a member of $EM_3$ with an $E$-quartet $\{\mathcal{S}, \mathcal{V}, \mathcal{S}_p, \mathcal{S}_f\}$. By Proposition 3.5 there exists an $E$-map $f : X \rightarrow M$ onto a metric space $M$ with respect to $\{\mathcal{S}, \mathcal{V}, \mathcal{S}_p, \mathcal{S}_f\}$. By Morita [M], $M$ is the image of a metric space $P$ with $\dim P \leq 0$ under a perfect map $g$. Now let $T$ be the fiber product of $P$ and $X$ with respect to $g$ and $f$, that is,

$$T = \{(p, x) \in P \times X : g(p) = f(x)\}$$

with the topology induced from $P \times X$. Let $t_p, t_X$ be the restrictions to $T$ of the projections from $P \times X$ onto $P$ and $X$, respectively. We thus have the following commutative diagram:

$$\begin{array}{ccc}
X & \xrightarrow{t_X} & T \\
\downarrow f & & \downarrow t_p \\
M & \xleftarrow{g} & P
\end{array}$$

It is a well-known property of fiber products that the perfectness of $g$ implies the perfectness of $t_X$ (see [Pe, Lemma 7.5.13]). $T$ is stratifiable by [C, Theorems 2.3, 2.4]. Hence what should be proved is the zero-dimensionality of $T$. By Proposition 2.2, $P$ admits an $E$-quartet $\{\mathcal{S}_p, \mathcal{V}_p, \mathcal{S}_p, \mathcal{S}_f\}$. Now define

$$\mathcal{S}_T = \{t_p^{-1}(F_p) \cap t_X^{-1}(F) : F_p \in \mathcal{S}_p, F \in \mathcal{S}\},$$

$$\mathcal{V}_T = \{t_p^{-1}(V_p) \cap t_X^{-1}(V) : V_p \in \mathcal{V}_p, V \in \mathcal{V}\},$$

$$\mathcal{S}_T = \{t_p^{-1}(S_p) \cap t_X^{-1}(S) : S_p \in \mathcal{S}_p, S \in \mathcal{S}\},$$

and

$$\mathcal{S}_T = \{t_p^{-1}(E_p) : E_p \in \mathcal{S}_p\} \cup \{t_X^{-1}(E) : E \in \mathcal{S}\}.$$
Then it is easy to see that the quartet \( \{ T, T', S, T' \} \) is an \( E \)-quartet of \( T \). Furthermore, the map \( t \) is an \( E \)-map with respect to \( \{ T, T', S, T' \} \) because, in general, \( t_\chi(t_\chi^{-1}(P') \cap \chi^{-1}(X')) = P' \cap g^{-1} \circ f(X') \) for any \( P' \subset P \) and \( X' \subset X \), and because \( f \) is an \( E \)-map with respect to \( \{ T, T', S, T' \} \). Hence, applying Proposition 3.7, we have \( \text{Ind } T \leq 0 \). Thus the implication (1) \( \rightarrow \) (2) has been proved.

To show (3) \( \rightarrow \) (4) let \( X_0 \) be a stratifiable space with \( \text{ind } X_0 \leq 0 \) and let \( f: X_0 \rightarrow X \) be a closed map. Note that every net on \( X_0 \) is an \( E \)-net; hence \( X_0 \) is a member of \( \text{EM}_3 \) by Heath [H]. It now follows from the implication (1) \( \rightarrow \) (2) that \( X_0 \) is the image of a stratifiable space \( X_1 \) with \( \dim X_1 \leq 0 \) under a perfect map \( h \). Since every net on \( X_1 \) is an \( LE \)-net, it follows from Heath [H] again that \( X_1 \) admits a \( \sigma \)-closure-preserving \( LE \)-net. Hence, applying Proposition 2.3 to the closed map \( f \circ h \), we see that \( X \) admits a \( \sigma \)-closure-preserving \( LE \)-net. On the other hand \( X \) is stratifiable by Borges [B, Theorem 3.1]. This completes the proof of Theorem 3.8.

**Corollary 3.9.** The class \( \text{EM}_3 \) is countably productive, hereditary and preserved under closed maps.

**Proof.** This is immediate from Theorem 3.8, Proposition 2.6 and the analogous result for \( M_3 \) due to Ceder [C] and Borges [B].

A topological class \( \mathcal{C} \) is called **perfect** (Nagami [N], also see [N2]) if it is countably productive, hereditary, preserved under perfect maps, included in the class of normal spaces, and every member of \( \mathcal{C} \) is the perfect image of a zero-dimensional (in the sense of \( \dim \)) member of \( \mathcal{C} \). Theorem 3.8 and Corollary 3.9 say

**Corollary 3.10.** The class \( \text{EM}_3 \) is the maximal perfect subclass of \( M_3 \).

Recently Itô [I] has presented a free \( L \)-space, a certain closed image of which is not a free \( L \)-space. But we have

**Corollary 3.11.** Every closed image of a free \( L \)-space is a member of \( \text{EM}_3 \).

**Proof.** By Nagami [N4, Theorem 2.10] and Theorem 3.8, every free \( L \)-space is a member of \( \text{EM}_3 \) (it is also easy to directly prove that every free \( L \)-space admits a \( \sigma \)-closure-preserving \( E \)-net). Hence this corollary is immediate from Corollary 3.9.

### 4. Dimension for \( \text{EM}_3 \)

We begin with the equidimensional \( G_\sigma \)-envelope theorem. To show this, the following lemma is useful.

**Lemma 4.1 (Oka [O4, Lemma 3.3]).** Let \( X \) be a hereditarily normal space and let \( f: X \rightarrow L \) be a map onto a metric space \( L \). Then for any subset \( Y \subset X \), there exist a \( G_\sigma \)-set \( Z \) of \( X \), a metric space \( M \) and maps \( g: Z \rightarrow M \), \( h: M \rightarrow f(Z) \) such that

- (i) \( Y \subset Z \),
- (ii) \( \dim g(Y) \leq \dim Y \) and
- (iii) \( f \circ Z = h \circ g \).

**Theorem 4.2.** Let \( X \in \text{EM}_3 \) and let \( Y \) be a subset of \( X \) with \( \dim Y \leq n \). Then there exists a \( G_\sigma \)-set \( G \) of \( X \) such that \( Y \subset G \) and \( \dim G \leq n \).

**Proof.** Let \( f: X \rightarrow L \) be an \( E \)-map onto a metric space \( L \) with respect to an \( E \)-quartet, say \( \{ T, T', S, T' \} \), on \( X \). By the above lemma there exist a \( G_\sigma \)-set \( Z \) of \( X \), a
metric space $M$ and maps $g: Z \to M$, $h: M \to f(Z)$ satisfying (i), (ii), (iii) above. Since $\dim g(Y) \leq n$ and $M$ is metrizable, we can find a $G_\delta$-set $H$ of $M$ such that $g(Y) \subseteq H$ and $\dim H \leq n$ (see, for example, [E, 4.1.19]). Define $G = g^{-1}(H)$. Then $G$ is a $G_\delta$-set of $Z$, and hence of $X$. To show $\dim G \leq n$, note that $g | G$ is an $E$-map with respect to $\{\mathcal{G}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$ because $f | G$ is so and because $f | G = h \circ g | G$ by (iii). Hence by Proposition 3.7 we have $\text{Ind} G \leq \text{Ind} H$. Consequently
\[
\dim G \leq \text{Ind} G \leq \text{Ind} H = \dim H \leq n,
\]
as required. This completes the proof.

The following theorem occupies the central position in dimension theory of $EM_3$. The key argument of the proof has already appeared in the proof of Theorem 3.8.

**Theorem 4.3.** The following statements about a space $X$ are equivalent:

1. $X \in EM_3$ and $\dim X \leq n$.
2. $X$ is the image of a stratifiable space $X_0$ with $\dim X_0 \leq 0$ under a perfect map of order not greater than $n + 1$.
3. $X$ is a stratifiable space which is the union of $G_\delta$-sets $X_i$, $1 \leq i \leq n + 1$, with $\dim X_i \leq 0$.
4. $X \in EM_3$ and $\text{Ind} X \leq n$.

**Proof.** (1) $\rightarrow$ (2). Let $X$ be a member of $EM_3$ such that $\dim X \leq n$. Let $\{\mathcal{G}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$ be an $E$-quartet of $X$. By Proposition 3.5 there exist a metric space $L$ and an $E$-map $f: X \to L$ with respect to the $E$-quartet. By Pasynkov's factorization theorem [P, Theorem 29], there exist a metric space $M$ and maps $g: X \to M$, $h: M \to L$ such that $\dim M \leq n$ and $f = h \circ g$. It then follows from Morita [M] that $M$ is the image of a metric space $P$ with $\dim P \leq 0$ under a perfect map $r$ such that $\text{ord} r \leq n + 1$. Let $T$ be the fiber product of $P$ and $X$ with respect to $r$ and $g$, and let $t_P$, $t_X$ be the restrictions to $T$ of the projections from $P \times X$ onto $P$ and $X$, respectively. We thus obtain the following commutative diagram:

\[
\begin{array}{cccc}
X & \xleftarrow{t_X} & T \\
\downarrow f & & \downarrow t_P \\
L & \xleftarrow{h} & M & \xleftarrow{r} & P
\end{array}
\]

It is obvious that $t_X$ is a perfect map of order not greater than $n + 1$ and that $T$ is a stratifiable space. Note that $g$ is an $E$-map with respect to $\{\mathcal{G}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$ because $f$ is so and $f = h \circ g$. Now, as in the proof of Theorem 3.8, $t_P$ is also an $E$-map with respect to a certain $E$-quartet of $T$, and hence $\dim T \leq 0$ by Proposition 3.7.

(2) $\rightarrow$ (3). Let $t: X_0 \to X$ be a perfect map from a stratifiable space $X_0$ with $\dim X_0 \leq 0$ onto a space $X$ such that $\text{ord} t \leq n + 1$. Put $Y_i = \{x \in X: |t^{-1}(x)| = i\}$, $1 \leq i \leq n + 1$. It then follows from Nagami [N2, Lemma 4] that $\dim Y_i \leq 0$ for each $i = 1, 2, \ldots, n + 1$. Since $X$ is a member of $EM_3$ by Theorem 3.8, we may apply Theorem 4.2 to obtain $G_\delta$-sets $X_i$, $1 \leq i \leq n + 1$, such that $\dim X_i \leq 0$ and $Y_i \subseteq X_i$.

The implication (4) $\rightarrow$ (1) is trivial.
Finally the implication (3) \(\rightarrow\) (4) is assured by the following theorem (but the fact \(\text{Ind } X \leq n\) only is direct from (3) as a consequence general for hereditarily normal spaces).

**Theorem 4.4.** Let \(X\) be a normal \(\sigma\)-space expressed as the finite union of \(G_\delta\)-sets \(X_i\), \(1 \leq i \leq k\), such that \(\dim X_i \leq 0\). Then \(X\) admits a \(\sigma\)-closure-preserving \(LE\)-net.

**Proof.** The proof is by induction on \(k\). When \(k = 1\), the theorem is trivial. Now suppose that the theorem is valid when \(k = m - 1\), and consider the case \(k = m\). Put \(Y_m = X - X_m\). Then by induction hypothesis and Lemma 2.5, the normal \(\sigma\)-space \(Y_m\) admits a \(\sigma\)-closure-preserving \(LE\)-net, say \(\mathcal{D}\), consisting of closed sets of \(Y_m\). Write \(Y_m = \bigcup_{i=1}^{\infty} C_i\) with closed sets \(C_i\) such that \(C_i \subset C_{i+1}\), and put \(\mathcal{D}_i = \mathcal{D} | C_i\). Let \(\mathcal{T}\) be a \(\sigma\)-locally finite net of \(X\). Now consider the \(\sigma\)-closure-preserving collection \(\bigcup_{i=1}^{\infty} \mathcal{D}_i \cup \mathcal{T}\) of \(X\). To show that the collection is an \(LE\)-net on \(X\), let \(C, K\) be disjoint closed sets of \(X\). Since \(X\) is hereditarily normal and \(\text{Ind } X_m \leq 0\), there exists a closed set \(S\) separating \(C\) and \(K\) such that \(S \cap X_m = \emptyset\). Represent \(X\) as the disjoint union \(V \cup S \cup W\), where \(V\) and \(W\) are open sets of \(X\) including \(C\) and \(K\) respectively. Write \(V = \bigcup_{i=1}^{\infty} V_i\) with open sets \(V_i\) such that \(V_i \subset V_{i+1}\) for every \(i\). For each \(i\) take a subcollection \(\mathcal{S}_i\) of \(\mathcal{S}\) such that

\[
(W \cup S) \cap C_i \subset \mathcal{S}_i^* \subset C_i - (V_i \cup C)
\]

and \(\mathcal{S}_i^*\) is a closed set of \(C_i\). Now put

\[
B = W \cup \left( \bigcup_{i=1}^{\infty} \mathcal{S}_i^* \right).
\]

It is easy to see that \(B\) is a closed set of \(X\) including \(K\) and not meeting \(C\). Since \(W\) is the union of some members of \(\mathcal{T}\), \(B\) is the union of some members of \(\bigcup_{i=1}^{\infty} \mathcal{S}_i \cup \mathcal{T}\). Thus \(\bigcup_{i=1}^{\infty} \mathcal{S}_i \cup \mathcal{T}\) is a \(\sigma\)-closure-preserving \(LE\)-net on \(X\). This completes the proof of Theorem 4.4 and, therefore, of Theorem 4.3.

**Remark.** Slightly modifying the above proof, we can weaken the condition “\(X_i\) is \(G_\delta\)” in Theorem 4.4 to “\(X_i\) is either \(G_\delta\) or \(F_\sigma\).”

As a trivial version of Theorem 4.4, we have the following result which tells us that the dimension theory does not work well in the remainder \(M_3 - EM_3\).

**Corollary 4.5.** Let \(X\) be a normal \(\sigma\)-space not admitting a \(\sigma\)-closure-preserving \(LE\)-net. Then either

1. \(X\) cannot be decomposed into finitely many zero-dimensional (in the sense of \(\dim\)) subsets, or
2. there exists a zero-dimensional (in the sense of \(\dim\)) subset of \(X\) not admitting an equidimensional \(G_\delta\)-envelope.

As an immediate consequence of Theorem 4.3, we have

**Corollary 4.6.** Let \(X\) be a stratifiable space with \(\text{ind } X \leq 0\). Then \(\dim X = \text{Ind } X\).

**Remark.** This result, however, is generalized to paracompact \(\sigma\)-spaces in my recent paper [Oka].
We conclude this section with the following result, an immediate consequence of Corollary 3.11 and Theorem 4.3.

**Corollary 4.7.** Let $X$ be the closed image of a free $L$-space. Then $\dim X = \text{Ind } X$.

### 5. Other spaces admitting $\sigma$-closure-preserving $E$-nets.

Let $C$ be a topological property. A space is called *peripherally* $C$ if every point in the space admits an open neighborhood base, the boundary of each member of which is $C$.

**Theorem 5.1.** (1) A peripherally $\sigma$-discrete, paracompact $\sigma$-space admits a $\sigma$-closure-preserving $E$-net.

(2) A peripherally $\sigma$-compact, stratifiable space admits a $\sigma$-closure-preserving $E$-net.

**Proof.** We shall prove (1) and (2) simultaneously. Let $S$ be a $\sigma$-locally finite net (resp. a $\sigma$-closure-preserving quasi-base) of $X$ consisting of closed sets. To show that $S$ itself is an $E$-net on $X$ let $x$ be a point of $X$ and $V$ an open neighborhood of $x$. Take an open set $U$ such that $x \in U \subseteq \overline{U} \subseteq V$ and $\text{Bd } U$ is $\sigma$-discrete (resp. $\sigma$-compact). Write $U = \bigcup_{i=1}^{\infty} U_i$ with open sets $U_i$ such that $\overline{U}_i \subseteq U_{i+1}$ for every $i$. Write $\text{Bd } U = \bigcup_{i=1}^{\infty} C_i$ with discrete (resp. compact) closed sets $C_i$, $i = 1, 2, \ldots$. There exists, for each $i$, a discrete (resp. finite) subcollection $S_i$ of $S$ such that $C_i \subseteq S_i \subseteq X - (\overline{U_i} \cup \{x\})$. Then $\bigcup_{i=1}^{\infty} S_i \cup (X - \overline{U})$ is a closed set of $X$ including $X - V$, not meeting $\{x\}$ and expressed as a union of members of $S$. Thus $S$ is an $E$-net of $X$, which completes the proof.

Now we have the following generalization of Corollary 4.6.

**Corollary 5.2.** Let $X$ be a peripherally $\sigma$-compact (or peripherally $\sigma$-discrete) stratifiable space. Then $\dim X = \text{Ind } X$.

We next verify a countable sum theorem for $\sigma$-closure-preserving $LE$-nets.

**Theorem 5.3.** Let $X$ be a normal space expressed as the countable union of closed sets $X_i$, $i = 1, 2, \ldots$, each of which admits a $\sigma$-closure-preserving $LE$-net. Then $X$ has a $\sigma$-closure-preserving $LE$-net.

**Proof.** Note that $X$ is perfectly normal because each $X_i$ is. Let $S_i$ be a $\sigma$-closure-preserving $LE$-net of $X_i$. It is clear that $\bigcup_{i=1}^{\infty} S_i$ is a $\sigma$-closure-preserving in $X$. To show that $\bigcup_{i=1}^{\infty} S_i$ is an $LE$-net, let $C$ and $K$ be disjoint closed sets of $X$. Write $X - C = \bigcup_{i=1}^{\infty} V_i$ with open sets $V_i$ such that $\overline{V}_i \subseteq V_{i+1}$. For each $i$ let $S_i$ be a subcollection of $S_i$ such that $S_i^*$ is a closed set of $X_i$ and $C \cap X_i \subseteq \overline{S_i^*} \subseteq X_i - (K \cup \overline{V}_i)$. It is then obvious that $\bigcup_{i=1}^{\infty} S_i^*$ is a closed set of $X$ and $C \subseteq \bigcup_{i=1}^{\infty} S_i^* \subseteq X - K$. This completes the proof.

The following result is immediate from Theorem 5.3, Proposition 2.2 and Ceder [C, Theorem 8.3].

**Corollary 5.4.** A chunk complex (and hence a CW-complex) is a member of $EM_3$.

We list several unsolved problems below.
Problem 5.5. (1) Does every stratifiable space admit a σ-closure-preserving E-net? By virtue of Theorem 3.8, this is equivalent to:

(2) (Nagami [N₁, Problem 4]) Is every stratifiable space a perfect image of a zero-dimensional (in the sense of dim) stratifiable space?

The author also does not know whether the inclusion \( EM \subset M \) (or \( M \subset EM \)) holds or not.

Problem 5.6. Let \( X \) be a paracompact σ-space admitting a σ-closure-preserving E-net. Then:

(1) Does the equality \( \text{dim } X = \text{Ind } X \) hold?

(2) Is \( X \) a perfect image of a zero-dimensional (in the sense of dim) paracompact σ-space? More weakly:

(3) Does \( X \) admit a σ-closure-preserving LE-net?

In the specific case of \( \text{ind } X \leq 0 \), (1) admits an affirmative answer by the inequality \( \text{Ind } X \leq \text{dim } X + \text{ind } X \) for every nonempty paracompact σ-space \( X \) [Oₙ]; (2) is also affirmative, that is, a paracompact σ-space of \( \text{ind } X \leq 0 \) is the perfect image of a paracompact σ-space of \( \text{dim } X \leq 0 \).

To outline the proof, let \( X \) be a nonempty paracompact σ-space with \( \text{ind } X = 0 \). Let \( \mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}_i \) and \( \mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i \) be as in Definition 3.3. Let \( f: X \to M \) be a one-to-one map onto a metric space \( M \) such that \( f(\mathcal{V}_i^*) \) is open and \( f(\mathcal{G}_i^*) \) is closed for every \( i \). In [Oₙ, Lemma 5] it is proved that, in general, \( \text{Ind } X \leq \text{Ind } M + \text{ind } X \) for any such map \( f: X \to M \). The metric space \( M \) is the image of a metric space \( L \) with \( \text{dim } L = 0 \) under a perfect map \( g \). Let \( T \) be the fiber product of \( L \) and \( X \) with respect to \( g \) and \( f \). Let \( t_L, t_X \) be the restrictions to \( T \) of the projections from \( L \times X \) onto \( L \) and \( X \), respectively. Then, since the map \( t_L \) is of the “same type” as \( f \), we have \( \text{Ind } T \leq \text{Ind } L + \text{ind } T = \text{ind } T \). But, in the present case, \( \text{ind } T \leq \text{ind } (L \times X) = 0 \); hence \( \text{Ind } T = 0 \). It is clear that \( T \) is a paracompact σ-space and \( t_X \) is a perfect map. This completes the proof.

Problem 5.7. Let \( X \) be a stratifiable space expressed as the union of countably many metrizable \((G_δ)\) subsets. Does the equality \( \text{dim } X = \text{Ind } X \) hold? More strongly, does \( X \) admit a σ-closure-preserving E-net? (A space of this type is a natural generalization of a Lašnev space in view of Lašnev’s well-known decomposition theorem [La].)

Bibliography


DIMENSION OF STRATIFIABLE SPACES


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