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NONLINEAR MAPPINGS THAT ARE GLOBALLY EQUIVALENT TO A PROJECTION

BY

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ABSTRACT. The Rank theorem gives conditions for a nonlinear Fredholm map of positive index to be locally equivalent to a projection. In this paper we wish to find conditions which guarantee that such a map is globally equivalent to a projection. The problem is approached through the method of line lifting. This requires the existence of a locally Lipschitz right inverse, $F^{\downarrow}(x)$, to the derivative map $F'(x)$ and a global solution to the differential equation $P'(t) = F^{\downarrow}(P(t))(y - y_0)$. Both these problems are solved and the generalized Hadamard-Levy criterion

$$\int_0^\infty \inf_{|x|<s} \left(1/|F^{\downarrow}(x)|\right) ds = \infty$$

is shown to be sufficient for F to be globally equivalent to a projection map (Theorem 3.2). The relation to fiber bundle mappings is explored in §4.

1. Introduction. In this paper we investigate the structure of "underdetermined" nonlinear systems and mappings. By an underdetermined system or mapping we mean, in the finite-dimensional case, a system represented by a mapping $F\colon \mathbf{R}^n \to \mathbf{R}^p$ where $n > p$. The analog in infinite dimensions is a Fredholm map $F\colon X \to Y$ having positive Fredholm index [**13**]. Here X and Y are Banach spaces.

For finite linear systems, with rank equal to p, represented by a linear map $L\colon \mathbf{R}^n \to \mathbf{R}^p$ ($n > p$), an elementary result based on Gaussian elimination states that the system L is equivalent to a projection map $P\colon \mathbf{R}^n \to \mathbf{R}^p$. In other words there is a change of basis Φ so that $L \circ \Phi^{-1} = P$. The object of this paper is to find conditions on a nonlinear system which insure that it is equivalent to a projection map. The key to the analysis is the use of a technique we call the method of line lifting. This method, successfully used in the investigation of the global inversion problem [**10**, **11**], is applied to the problem of solving an equation $F(x) = y_1$. Instead of solving it directly, we pick a point x_0 and, denoting $y_0 = F(x_0)$, we solve for a path $P(t)$ so that $P(0) = x_0$ and $F(P(t)) = L(t) = (1 - t)y_0 + ty_1$. Then $P(1)$ is the solution to $F(x) = y_1$. At first sight it seems that we are faced with a more difficult problem. However this tactic proves to be quite successful. (In fact, under the name continuation method, it has been used as the basis for various global Newton's method numerical schemes [**16**].)

In §2 we describe, in Definition 2.4, the method of line lifting and its use in solving the problem of global equivalence to a projection. We introduce a condition

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(L) which insures the success of the method. The main result is Theorem 2.9: a Fredholm map F with positive index having a derivative map $F'(x)$ which is a surjective linear map, locally Lipschitz in x, is globally equivalent to a projection iff condition (L) is satisfied.

§3 is concerned with criteria guaranteeing that a map F satisfy condition (L). The conditions of properness and closedness, much used in the global inversion problem, are shown not to be useful in the present case. This can be seen from the fact that a linear projection is neither a proper nor a closed map. We use instead a generalization of the Hadamard-Levy (H-L) criterion, Theorem 3.2. This criterion was used in the global inversion problem **[10]** and applied to differential equations **[11]**.

Finally in §4 we show, in Theorem 4.3, how the method of line lifting relates to the concept of a fiber bundle map. Covering spaces have been used in the global inversion problem but cannot be applied to the present case. The introduction of fiber bundles allows one to employ the formidable topological tools and methods of fiber bundle theory. As an application, one can describe the structure of solutions for the mappings considered in this paper.

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2. Global equivalence to a projection and the method of line lifting. Throughout we shall assume that our map F is a C' map between Banach spaces X and Y. We further suppose that F is a Fredholm map with positive Fredholm index **[13, 15]**. For example, maps between $\mathbf{R}^n$ and $\mathbf{R}^p$ are Fredholm maps with index $n - p$. Our assumption then requires n greater than p. The simplest Fredholm map with a positive index is a linear projection $P\colon X \to Y$, where Y is a closed subspace of X having a finite-dimensional complement.

DEFINITION 2.1. F is (globally) equivalent to a projection map P if there is a homeomorphism Φ and a space Z where $\Phi\colon X \to Z$, $P\colon Z \to Y$ and $F \cdot \Phi^{-1} = P$.

A local result in this direction is

PROPOSITION 2.2. *If $F'(x)$ is a surjective linear map then, on some open set O about x, F is locally equivalent to a projection map $P\colon Y \oplus \operatorname{Ker} F'(x) \to Y$.*

PROOF. The Fredholm assumption **[17]** allows us to write $X = K \oplus \operatorname{Ker} F'(x)$ (here K is a closed subspace and $K \cap \operatorname{Ker} F'(x) = \{0\}$). Let G be the projection map of X onto $\operatorname{Ker} F'(x)$. We define $\Phi = F \oplus G$. Since $U'(x) = F'(x) \oplus G$ is an invertible linear map of X onto $Y \oplus \operatorname{Ker} F'(x)$, we can use the inverse function theorem **[4]** to find an open set O about x so that Φ is a homeomorphism on O. Clearly $F \cdot \Phi^{-1} = P$ (on O).

Some consequences of $F'(x)$ being a surjective linear map are: (i) F is an open map; (ii) the fibers $F^{-1}(y)$ are manifolds with dimension that of $\operatorname{Ker} F'(x) = \operatorname{index} F$.

A point x for which $F'(x)$ is surjective is called a regular point, otherwise it is a singular point.

We wish to find conditions on F which lead to global equivalence to a projection map. The notion of line lifting, used in the study of the global inversion problem **[10b]**, will prove to be equally useful in our situation.

DEFINITION 2.3. F lifts lines if for any line $L(t) = (1 - t)y_0 + ty_1$ $(0 \leq t \leq 1)$ lying in the range of F and each point x in $F^{-1}(y_0)$ there is a path $P_x(t)$ so that $F(P_x(t)) = L(t)$ and $P_x(0) = x$.

We will soon see that line lifting is the key to global equivalence to a projection. First we must be able to solve the line lifting problem. We do this by recasting it into a differential equation.

Applying the chain rule to $F(P(t)) = (1 - t)y_0 + ty_1$we have $F'(P(t))P'(t) = y_1 - y_0$. If $F'(x)$ has a right inverse $F^{\downarrow}(x)$ (i.e. $F'(x)F^{\downarrow}(x) = I$), then we have the differential equation

$$(*) \qquad P'(t) = F^{\downarrow}(P(t))(y_1 - y_0); \qquad 0 \leq t \leq 1,$$
$$P(0) = x.$$

DEFINITION 2.4. The technique of solving the lifting problem by using differential equation $(*)$ is called (in this paper) the method of line lifting.

Not only do we require the existence of $F^{\downarrow}(x)$, we also require that it be locally Lipschitz in x in order to guarantee the unique solvability of system $(*)$ **[4]**. If X and Y are Hilbert spaces, we can actually construct the required right inverse. With $F'(x)^*$ denoting the Hilbert space adjoint, we have

LEMMA 2.5. *If X and Y are Hilbert spaces and $F'(x)$ is a surjective linear map, locally Lipschitz in x, there is a locally Lipschitz right inverse* $F^{\downarrow}(x) = F'(x)^*[F'(x)F'(x)^*]^{-1}$.

PROOF. The proof of the lemma is straightforward, it only being necessary to check that the indicated inverse exists. To this end, write $X = K_x \oplus K_x^{\perp}$ where K_x is the kernel of the linear map $F'(x)$. Now $F'(x)$ is a one-one map on $K_x^{\perp}$ and, by hypotheses, maps this space onto Y. From this it follows that the Hilbert space adjoint $F'(x)^*$ is a one-one map on Y whose range is $K_x^{\perp}$. Thus the map $F'(x)F'(x)^*$ is a one-one linear map of Y onto Y and so is invertible. (The local Lipschitz property follows from the differentiability of the maps $A \to A^*$ and $A \to A^{-1}$ **[4]**.)

The Banach space case is trickier for we cannot write down $F^{\downarrow}(x)$ as we did for a Hilbert space.

LEMMA 2.6. *Let X and Y be Banach spaces and $F'(x)$ a surjective linear map locally Lipschitz in x. Then a locally Lipschitz right inverse $F^{\downarrow}(x)$ exists.*

PROOF. Let a be a point in X. The Fredholm structure enables us to write $X = K_a \oplus \operatorname{Ker} F'(a)$. Let G be the projection onto $\operatorname{Ker} F'(a)$. The map $h_a(x) = F'(x) \oplus G$ is an invertible linear map of X onto $Y \oplus \operatorname{Ker} F'(a)$ for x in a sufficiently small open set O_a about a. This map $h_a(x)$ is also locally Lipschitz in x as is $h_a^{-1}(x)$. Let $H_a(x) = h_a^{-1}(x)$ restricted to $Y \times \{0\}$. Now $H_a(x)$ is a locally Lipschitz right inverse of $F'(x)$ for x in O_a. For each a in X we have such a construction. Let $\{U_a, \lambda_a\}$ be a locally Lipschitz partition of unity subordinate to the covering O_a **[9b]**. Setting $H_a(x) = 0$ if x is not in U_a, we define $F^{\downarrow}(x) = \Sigma\, \lambda_a(x)H_a(x)$.

Now that we have conditions which insure the unique solvability of $(*)$, we require a condition which insures that the local solution can be extended to all t between 0 and 1.

DEFINITION 2.7. Let y_1 and y_0 be any points in Y and let $L(t) = (1-t)y_0 + ty_1$, for $0 \leqslant t \leqslant 1$, be the line segment joining them. Suppose we can find a path $P(t)$, and some number b so that $F(P(t)) = L(t)$ for $0 \leqslant t < b$. We say F satisfies condition (L) if there exists a sequence $\{t_n\}$, converging to b, so that $\lim P(t_n)$ exists.

With condition (L) we can solve the differential equation $(*)$.

PROPOSITION 2.8. *Let F be a Fredholm map with a locally Lipschitz derivative $F'(x)$. If $F'(x)$ is a surjective linear map and F satisfies condition (L), then differential equation $(*)$ has a unique solution defined on the interval* $[0, 1]$.

The proof only requires an application of condition (L) and then the unique solvability insured by the locally Lipschitz right-hand side to guarantee that the solution exists on the full interval $[0, 1]$.

We are now in the position to state and prove the main theorem of this section.

THEOREM 2.9. *Let F: $X \to Y$ be a Fredholm map having positive index. Suppose $F'(x)$ is a surjective linear map locally Lipschitz in x. Then F is globally equivalent to a projection iff F satisfies condition (L).*

PROOF. If $F = P \cdot \Phi$, then any line can be lifted and so condition (L) must be satisfied.

On the other hand, we first observe that condition (L) implies that F maps onto Y. This is accomplished by applying Proposition 2.8 to lift the line $(1-t)y_0 + ty$, where y_0 is in the range of F and y is any point in Y.

Now let y_0 be any point in Y. Then for every x in X, Proposition 2.8 guarantees the existence of a unique path $P_x(t)$ which lifts the line $L(t) = (1-t)F(x) + ty_0$ with $P_x(0) = x$.

Define the map $E(x) = P_x(1)$. The unique solvability of $(*)$ is insured by the local Lipschitz condition. This implies that the map $E(x)$ is one-one on each fiber $F^{-1}(y)$. In addition, the continuous dependence of $(*)$ on its initial conditions **[4]** implies that $E(x)$ is a continuous map of X onto $F^{-1}(y_0)$.

Let $\Phi(x) = (F(x), E(x))$ and let P be the projection of $Y \times F^{-1}(y_0)$ onto Y. Clearly $F = P \cdot \Phi$. To complete the proof requires showing that Φ is a homeomorphism of X onto $Y \times F^{-1}(y_0)$.

Φ is one-one by the construction of E. We now construct Φ^{-1}. Let (y, a) be any point in $Y \times F^{-1}(y_0)$. Finding Φ^{-1} requires finding an $\bar{x}$ so that $F(\bar{x}) = y$ and $P_{\bar{x}}(1) = a$ (where $F(P_{\bar{x}}(t)) = (1-t)y_0 + ty$).

To do this, let $L_y(t)$ be the line $(1-t)y_0 + ty$. Let us denote $Y_a(t)$ as the lift of this line found by solving $(*)$ with $Y_a(0) = a$. Let $\bar{x} = Y_a(1)$. Our claim is that $Y_a(1)$ is $\Phi^{-1}(y, a)$. By the construction of Φ this requires showing that $E(\bar{x}) = P_{\bar{x}}(1) = a$. $P_{\bar{x}}(t)$ is the path, defined in the construction of Φ, which lifts the line $L_y(1-t)$ above, with $P_{\bar{x}}(0) = \bar{x}$. Now $P_{\bar{x}}(1-t)$ and $Y_a(t)$ are paths, found by solving $(*)$, which lift the same line and have the point $\bar{x}$ in common. By unique solvability, they must be identical, i.e. $P_{\bar{x}}(1-t) = Y_a(t)$. In particular $P_{\bar{x}}(1) = Y_a(0) = a$, as was to be shown. Setting $\Phi^{-1}(y, a) = Y_a(1)$, the continuity of Φ^{-1} follows once more from the corresponding continuity theorems for differential equations **[4]**.

3. The generalized Hadamard-Levy theorem. We have seen in Theorem 2.9 how condition (L) is used in the method of line lifting. We consider several hypotheses on the map F which insure the satisfaction of condition (L). These are properness, closedness and what has been called the Hadamard-Levy criterion [**6**, **7**, **10b**].

A map is proper if $F^{-1}(C)$ is a compact set whenever C is a compact set. In [**10b**] we have shown that properness does insure condition (L). Proper maps have had wide consideration in the global inversion problem [**1**, **3**, **8**, **9a**, **12**]. However the class of proper maps excludes our prototype map, the linear projection. In fact in [**2**] it is shown that a proper Fredholm map of positive index must have a singularity, i.e., there is a point x for which $F'(x)$ fails to be a surjective map. Thus Theorem 2.9 cannot be applied.

The condition of closedness, i.e., the image of a closed set is closed, was first used by Browder [**3**] in connection with the global inversion problem, This also insures condition (L) [**10b**]. Again linear projection maps fail to be closed. We shall show that this condition, if satisfied by a nonlinear map, yields the existence of a singularity.

PROPOSITION 3.1. *Let F be a closed Fredholm map of positive index having a locally Lipschitz derivative. Then there is a point x so that $F'(x)$ is not surjective.*

PROOF. Suppose there are no singularities. Then we may apply Theorem 2.9 to conclude that F is equivalent to a projection map P on $Y \times F^{-1}(y_0)$ to Y (where y_0 can be chosen to be any point in Y). Since F is a closed map, then P must also be a closed map. We shall show that F is a proper map by showing that the inverse image of a point is compact. By Proposition 2.2 $F^{-1}(y_0)$ is a finite-dimensional manifold. We need only show that it is bounded. If not let $\{x_n\}$ be an unbounded sequence. Let $\{y_n\}$ be any sequence in Y converging to y_0. Then $C = \{(y_n, x_n)\}$ is a closed set in $Y \times F^{-1}(y_0)$, and so $P(C) = \{y_n\}$ should be a closed set in Y, which it is not. Thus $F^{-1}(y_0)$ is bounded and closed, thus compact. So F is proper [**15**] and must have a singular point [**2**].

We introduce the generalized Hadamard-Levy criterion in the next theorem.

THEOREM 3.2. *Suppose F: $X \to Y$ is a Fredholm map with positive index whose derivative is locally Lipschitz. Suppose further that $F'(x)$ is a surjective linear map whose right inverse $F^{\downarrow}(x)$ satisfies*

$$(**) \qquad \int_0^\infty \inf_{|x|<s} \left(\frac{1}{|F^{\downarrow}(x)|} \right) ds = \infty .$$

Then F is globally equivalent to a projection.

PROOF. We apply Theorem 2.9 by showing that F satisfies condition (L). So suppose a path $P(t)$ is defined for $0 \leqslant t < b$. For ease of writing let $B(x) = 1/|F^{\downarrow}(x)|$ and $h(s) = \inf_{|x|<s} B(x)$. Choose $\gamma < b$, and for each n, choose a partition $0 \leqslant t \leqslant \cdots \leqslant t_{n+1} = \gamma$ which refines the partition previously chosen at step $(n-1)$. Let $\bar{t}_i$ denote a number, lying in the interval $[t_i, t_{i+1}]$, that satisfies $\sup_{[t_i, t_{i+1}]} |P'(t)| = |P'(\bar{t}_i)|$.

Let $L(t) = (1 - t)y_0 + y_1$ be the line joining y_0 to y_1. Then using the fact that $|P(t)|$ has bounded variation on $[0, \gamma]$, we have (recall that $P(t)$ satisfies (*) of §2):

$$b|y_1 - y_0| \geqslant \int_0^\gamma \frac{1}{|F^\downarrow(P(t))|}|F^\downarrow(P(t)(y_1 - y_0))|\,dt = \int_0^\gamma B(P(t))\,|P'(t)|\,dt$$

$$= \lim_{n\to\infty} \sum_0^n B\big(P(\bar{t}_i)\big)\,|P'(\bar{t}_i)|\,(t_{i+1} - t_i)$$

$$\geqslant \lim_{n\to\infty} \sum_0^n B\big(P(\bar{t}_i)\big)(|P(t_{i+1})| - |P(t_i)|)$$

$$= \int_0^\gamma B(P(t))\,d|P(t)| \geqslant \int_0^\gamma \inf_{|x|<|P(t)|} B(x)\,d|P(t)|$$

$$= \int_0^\gamma h(|P(t)|)\,d|P(t)| = \int_{|P(0)|}^{|P(\gamma)|} h(s)\,ds.$$

From (**) and the above estimate we conclude that $\{P(t)\}$ is bounded on $[0, b)$. Also from (**), we have that $\sup\{s \mid h(s) > 0\} = \infty$, and so $B(x)$ is bounded from below on any bounded set (since $h(s)$ is nonincreasing). In particular, there is a $\lambda > 0$ so that $B(P(t)) > \lambda$ for t in the interval $[0, b)$.

We are now ready to show that condition (L) is satisfied. In fact we shall show that for every ε, there is a δ so that any numbers r and s satisfying $|r - b| < \delta$ and $|s - b| < \delta$ also satisfy $|P(r) - P(s)| < \varepsilon$. First of all, the existence of $\int_0^b |P'(t)|\,dt$ follows from the following series of inequalities: For any $\gamma < b$,

$$\int_0^\gamma |P'(t)|\,dt \leqslant \frac{1}{\lambda}\int_0^\gamma B(P(t))\,|P'(t)|\,dt \leqslant \frac{b}{\lambda}|y_1 - y_0|\,.$$

Finally we establish condition (L):

$$|P(r) - P(s)| = \left|\int_s^r P'(t)\,dt\right| \leqslant \int_s^r |P'(t)|\,dt.$$

That this last integral can be made less than ε if r and s are sufficiently close to b follows from the existence of the integral $\int_0^b |P'(t)|\,dt$.

We now show that linear projection maps satisfy the criteria of Theorem 3.2. To see this, write $X = H \oplus W$, where W is a finite-dimensional space. Let P: $H \oplus W \to H$ be a projection. Then $P^\downarrow = I_H$ (the identity on H) and $|P^\downarrow| = 1$. So for a projection map

$$\int_0^\infty \inf_{|x|<s}\left(\frac{1}{|P^\downarrow(x)|}\right) ds = \infty.$$

To gain further insight let us look at maps F: $\mathbf{R}^n \to \mathbf{R}^1$. Then $F'(x)$ is just grad $F(x)$ and the requirement that $F'(x)$ be surjective reduces to grad $F(x) \neq 0$. The right inverse can be constructed from Lemma 2.5, where $F^\downarrow(x) = F'(x)^*[F'(x)F'(x)^*]^{-1}$. Writing grad $F(x) = (\partial F/\partial x_1, \dots, \partial F/\partial x_n)$ it is easy to check that $F^\downarrow(x) = (\text{grad } F(x)^*)/|\text{grad } F(x)|^2$, and so $|F^\downarrow(x)| = 1/|\text{grad } F(x)|$.

COROLLARY 3.3. *Let F: $\mathbf{R}^n \to \mathbf{R}^1$ have a locally Lipschitz derivative* grad $F(x)$. *If* $|\text{grad } F(x)| \geqslant \lambda(|x|)$, *where $\lambda(t)$ is a positive function satisfying $\int_0^\infty \lambda(t)\,dt = \infty$, then F is globally equivalent to a projection.*

The proof follows directly from Theorem 3.2 and the discussion following it.

In **[10b]** and **[11]** several other conditions are given which insure that the generalized Hadamard-Levy criterion (**) is satisfied. These conditions can also be adapted to the present case.

4. The method of line lifting and fiber bundle maps. In order to broaden the scope of the ideas introduced in the previous sections, we must take a closer look at the role played by condition (L). The following example will help to clarify the relations between the line lifting property (Definition 2.3), the method of line lifting (Definition 2.4) and condition (L) (Definition 2.7).

Let us look at $F(x, y) = e^{x+y}$, a map of $\mathbf{R}^2 \to \mathbf{R}^1$. Clearly F is not onto. In fact its range is $\mathbf{R}^+ = (0, \infty)$.

Let $L(t) = (1-t)a_0 + ta_1$ be a line in $\mathbf{R}^+$. Let (x_0, y_0) be any point in $F^{-1}(a_0)$. F has the line lifting property. In fact any path of the form

$$\big(x(t), \ln(|L(t)|) - x(t)\big)$$

for any $x(t)$ (with $x(0) = x_0$) is a lift. However, the unique path, as determined by the method of line lifting (where $F^{\downarrow}(x, y) = \frac{1}{2}e^{-(x+y)}(1, 1)$), is

$$x(t) = \big(\ln(|L(t)|) + c\big)/2, \qquad y(t) = \big(\ln(|L(t)|) - c\big)/2.$$

Here $c = 2x_0 - \ln(a_0)$. Furthermore F is globally equivalent to a projection map P. In fact we can follow the construction of the homeomorphism Φ given in the proof of Theorem 2.9. We find that $\Phi = (F(x, y), E(x, y))$, where

$$E(x, y) = P_{(x,y)}(1) = \left(\frac{\ln(a_0) + (x-y)}{2}, \frac{\ln(a_0) + (y-x)}{2}\right),$$

is a homeomorphism between $\mathbf{R}^2$ and $\mathbf{R}^+ \times F^{-1}(a_0)$. Also condition (L) is not satisfied (try extending the lift of $L(t) = 1 - 2t$ with $b = \frac{1}{2}$).

From this example we can make the following observations. Firstly, the method of line lifting (Definition 2.4) enables us to construct a path in such a manner that the map Φ is a one-one map. Secondly, condition (L), which insured the global solvability of (*), is too strong. That is condition (L) actually guarantees that our mapping F is an onto mapping. This fact was observed in the proof of Theorem 2.9. In what follows we will weaken condition (L) so that we can deal with mappings that are not onto.

We continue with the theoretical investigation of equivalence to a projection by introducing the concept of a fiber bundle map (roughly, a map which is locally equivalent to a projection).

DEFINITION 4.1. A mapping $F: A \to B$ is called a fiber bundle mapping if there is a fiber set M and a covering $\{O_\alpha\}$ of B by open sets so that for each O_α in the covering there is a homeomorphism, Φ_α: $F^{-1}[O_\alpha] \to O_\alpha \times M$, with the property that $F = P \circ \Phi_\alpha$ (where P is the projection of $O_\alpha \times M$ onto O_α).

For any map $F: A \to B$, the fiber over $y \in B$ is defined to be the set $F^{-1}(y)$. If F is a fiber bundle map, then it is easy to see that the fiber set M is homeomorphic to every fiber $F^{-1}(y)$.

A small modification of condition (L) paves the way towards a connection with fiber bundle maps.

DEFINITION 4.2. F satisfies condition (L') if for any $L(t)$ in the range of F and a path $P(t)$ with $F(P(t)) = L(t)$ for $0 \leq t < b$, there is a sequence $\{t_n\}$ converging to b so that $\lim P(t_n)$ exists and is in the domain of F.

THEOREM 4.3. *Let A be an open set in the Banach space X and $F: A \to Y$ a map with a continuous and locally Lipschitz derivative map $F'(x)$. Suppose further that $F'(x)$ is always a surjective linear map. Then condition (L') is both necessary and sufficient for F to be a fiber bundle map between A and $F(A)$.*

PROOF. First observe that the hypotheses imply that $F(A)$ is an open set in Y.

The necessity follows from the path lifting properties of fiber bundle maps.

The proof of sufficiency is the same as in Theorem 2.9 with the modification that for every y_α in $F(A)$, choose O_α to be the open ball about y_α that is contained in $F(A)$.

We refer the reader to the articles [**2** and **5**] for further applications of the use of fiber bundle theory to mapping problems, and to the article [**16**] for the implications that these ideas have for the numerical solution of nonlinear equations.

BIBLIOGRAPHY

1. S. Banach and S. Mazur, *Über mehrdeutige stetige abbildungen*, Studia Math. **5** (1934), 174–178.
2. M. S. Berger and R. A. Plastock, *On the singularities of nonlinear Fredholm operators of positive index*, Proc. Amer. Math. Soc. **79** (1980), 217–221.
3. F. E. Browder, *Covering spaces, fiber spaces and local homeomorphisms*, Duke Math. J. **21** (1954), 329–336.
4. J. Dieudonné, *Foundations of modern analysis*, Academic Press, New York, 1960.
5. C. Earle and J. Eells, *Foliations and fibrations*, J. Differential Geom. **7** (1967), 61–69.
6. J. Hadamard, *Sur les transformations ponctuelles*, Bull. Soc. Math. France **34** (1904), 71–84.
7. P. Levy, *Sur les fonctions des lignes implicites*, Bull. Soc. Math. France **48** (1920), 13–27.
8. J. Ortega and W. Rheinboldt, *Iterative solutions of nonlinear equations in several variables*, Academic Press, New York, 1970.
9. (a) R. S. Palais, *Natural operations on differential forms*, Trans. Amer. Math. Soc. **92** (1959), 125–141.
   (b) ______, *Lyusternik-Schnirelmann theory on Banach manifolds*, Topology **5** (1966), 115–132.
10. (a) R. A. Plastock, *Nonlinear Fredholm maps of index zero and their singularities*, Proc. Amer. Math. Soc. **68** (1978), 317 – 322.
    (b) ______, *Homeomorphisms between Banach spaces*, Trans. Amer. Math. Soc. **200** (1974), 169–183.
11. M. Radulescu and S. Radulescu, *Global inversion theorems and applications to differential equations*, Nonlinear Anal. **4** (1980), 951–965.
12. W. Rheinboldt, *Local mapping relations and global implicit function theorems*, Trans. Amer. Math. Soc. **138** (1969), 183–198.
13. S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–866.
14. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.
15. Yu. G. Borisovich, U. G. Zuyagin, and Yu. Sapronov, *Non-linear Fredholm maps and the Leray-Schauder theory*, Russian Math. Surveys **32**(4) (1977), 1–54.
16. W. C. Rheinboldt, *Solution fields of nonlinear equations and continuation methods*, SIAM J. Numer. Anal. **17** (1980), 221–237.
17. M. S. Berger, *Nonlinearity and functional analysis*, Academic Press, New York, 1977.

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