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GENERIC ALGEBRAS

BY

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ABSTRACT. The familiar (merely) generic algebras in a variety $\mathcal{V}$ are those which separate all the different operations of $\mathcal{V}$, or equivalently lie in no proper Birkhoff subcategory. Stronger notions are considered, the strongest being *canonicalness* of a (small) subcategory $\mathcal{Q}$ of $\mathcal{V}$, defined: the structure functor takes inclusion $\mathcal{Q} \subset \mathcal{V}$ to an isomorphism of varietal theories. Intermediate are *dominance* and *exemplariness*: lying in no proper varietal subcategory, respectively full subcategory. It is shown that, modulo measurable cardinals, every finitary variety has a canonical set (subcategory) of one or two algebras, the possible second one being the empty algebra. Without reservation, every variety with rank has a dominant set of one or two algebras (the second as before). Finally, in modules over a ring R, the generic module R is shown to be (a) dominant if exemplary, and (b) dominant if R is countable or right artinian. However, power series rings R and some others are not dominant R-modules.

Introduction. A *variety* (of algebras) will mean, in this paper, a varietal (or tripleable) category over sets. This is substantially just the class of all algebras having certain operations and satisfying certain defining equations—e.g. semigroups, rings, lattices—but note that the operations may be infinitary, and it is not even required that their arities be bounded. However, the main results below (Theorems 2, 3, 4) require a cardinal bound on the arities (a *rank*), and two-thirds of them require all operations to be finitary.

An algebra A is called *generic* in the variety $\mathcal{V}$ in case one can determine $\mathcal{V}$ from A alone, in the sense that the defining equations of $\mathcal{V}$ are those which hold identically in A. It is almost trivial that each variety with rank has a generic algebra. But $\mathcal{V}$ is recoverable from generic A only if we have already decided what the operations are. Observe, for instance, that a free group G on two generators is generic in the variety $\mathcal{V}$ of semigroups; though G belongs to the smaller variety $\mathcal{W}$ of groups, $\mathcal{W}$ is not a "variety of semigroups" (not defined, among semigroups, by equations).

We shall consider three strengthenings of "generic", increasing toward the beginning of the alphabet: *exemplary*, *dominant*, *canonical*. The strongest is the simplest: an algebra A is *canonical* in a variety $\mathcal{V}$ if A is generic in $\mathcal{V}$ and every function of n variables, $h: A^n \to A$ (for any cardinal n), which commutes with all endomorphisms of A is an operation (possibly composite) of $\mathcal{V}$.

If these cardinals n are restricted, the existence of relatively canonical algebras is easy (and it has the following history: Whenever operations $h(x_1,\dots,x_n)$ were

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identified as words and, however large n may be, as elements of a free algebra on $\aleph_0$ generators, in effect there it was. For groups Schreier's work made it plain in the 1920's. Of course, the process of passing from an algebra, or precisely from a convenient countable set of them, to the canonically associated variety was not explicitly considered until it became part of a larger system in Lawvere's work [**11**].) We are concerned with unrestricted n, and thus, in a sense, all our results involve (hypothetical) infinitary algebras. The existence theorem for canonical algebras says, with two reservations, that a variety with only finitary operations has a canonical algebra. Reservation 1 is universe-theoretic; in fact, the theorem is totally false if there is a proper class of measurable cardinals. (Set theorists regard this as a desirable state—perhaps because it makes general topology, functional analysis, and general algebra so cardinally obscure that only a professional set theorist can understand them.) "Totally false" means that no variety with rank and containing an algebra with more than one element could have a canonical algebra or (via extension of the definition) a canonical set of algebras. Reservation 2 concerns one algebra, the empty algebra, and does not affect varieties such as groups which have no empty algebra. At worse we add it, to get a 2-element canonical set.

Now as the abstract has already warned you, to understand this paper thoroughly requires a knowledge of basic functorial semantics. The source paper is [**12**]; a more readable reference is [**10**], which is also a pioneering study of canonicalness beginning with A (the problem: find $\mathcal{V}$). Some such knowledge will henceforth be assumed—but not in every line.

A subcategory $\mathcal{A}$ of a variety $\mathcal{V}$ is called *exemplary* in $\mathcal{V}$ if it lies in no proper full subvariety (subobject in the category of varieties); *dominant* if it lies in no proper subvariety. We will be interested only in full subcategories $\mathcal{A}$ (though we may have $\mathcal{A} \subset \mathcal{B} \subset \mathcal{V}$ where $\mathcal{B}$ is not full); the set of objects of full $\mathcal{A} \subset \mathcal{V}$ may be called *exemplary*, *dominant*, or *canonical* in $\mathcal{V}$ if $\mathcal{A}$ is so. The definition of a canonical subcategory unfortunately takes us into "tractability", treated sketchily by Linton [**12**] and usually avoided since. Cheer up, we can avoid it too. Let us call a category over sets, i.e. a category $\mathcal{C}$ together with a functor $U: \mathcal{C} \to \mathcal{S}$, a *datum*. Tractability is a certain property of data, defined in [**12**], which all our data have; it suffices to observe that $\mathcal{C} \to \mathcal{S}$ is tractable if (1) $\mathcal{C}$ is small, or (2) $\mathcal{C}$ is a variety and the functor $\mathcal{C} \to \mathcal{S}$ is the forgetful functor. Now recall further [**12**] that there are two categories (bigger than the universe: one uses a set theory with universes) which we may call Theo and Tract, Theo consisting of varietal theories and interpretations between them and Tract of tractable data $\mathcal{C} \to \mathcal{S}$ and their morphisms (functors $\mathcal{C} \to \mathcal{C}'$ making a commutative triangle over $\mathcal{S}$). There are adjoint functors Sem: Theo $\to$ Tract, Str: Tract $\to$ Theo (Sem $\mathbf{T}$ is the category of $\mathbf{T}$-algebras; Str $\mathcal{C}$ generalizes, from a category $\mathcal{A}$ on one object with underlying set A, the varietal theory of all n-ary operations $A^n \to A$ commuting with morphisms). Then we may define a tractable subdatum $\mathcal{C} \to \mathcal{S}$ of a variety $\mathcal{V} \to \mathcal{S}$ to be *canonical* in $\mathcal{V}$ if Str takes the insertion $i: \mathcal{C} \to \mathcal{V}$ to an isomorphism of theories.

EXAMPLES. Groups form a full proper subvariety of semigroups, or of monoids; so no group or category of groups is exemplary in monoids or in semigroups. Monoids

form a subvariety of semigroups. but not a full one. Indeed one can show that a free monoid on two generators is an exemplary semigroup.

Rings do not form a subvariety of groups, since two different rings may have (the same set of elements and) exactly the same underlying additive group. The crucial difference is that while the entire structure of a monoid (or group) is determined by its underlying semigroup, the structure of a ring is not determined by its additive group. That is the difference in semantic terms; in suitable infinitary languages, there is a syntactic equivalent. If $\mathcal{W}$ is a subvariety of a variety $\mathcal{V}$ with rank, then each operation of $\mathcal{W}$ is definable in terms of $\mathcal{V}$-operations. (That is "definable", not "expressible". For instance, the 0-ary operation e of groups is defined $y = e$ iff $yy = y$. The equation $yy = y$ has no such effect in general semigroups, but the definition need only work in groups.) This particular theorem will not be needed below. It is given, in different language, by Hodges and Shelah [**3**, Corollary 10]. We shall want a syntactic description of full subvarieties $\mathcal{W} \subset \mathcal{V}$. The classical finitary result, due to H. J. Keisler and more general, is in our terms that $\mathcal{W}$ is a full subvariety if and only if each operation of $\mathcal{W}$ can be defined from $\mathcal{V}$-operations by means of conjunction, disjunction and existential quantifiers. One calls such definitions *positive existential.* The infinitary generalization (without assuming a rank) and some sharper versions are in [**8**]. In fact we shall want a further sharpening, Theorem 1 below, which makes essential use of the infinitary language to eliminate all quantifiers. (Warm thanks to Stephen Schanuel for insisting on wondering if all those existences could be accidental.)

(STILL EXAMPLES). In the variety of distributive lattices with 0 and 1, the lattice $A = \{0, 1\}$ is generic; but it is not exemplary since it lies in the full subvariety of Boolean algebras. In semilattices with 0 and 1, A is exemplary (proof near the end of §1) but not dominant. For, lattices form a subvariety of semilattices, since the semilattice structure determines the order which determines the whole structure of a lattice. Turn now and consider A as a rather trivial algebra, a double-pointed set (the theory has two constants, 0 and 1, nothing more). We can show (§1) that it is a dominant one; but it is not canonical since A has no endomorphisms except the identity, but not all functions $A^n \to A$ are constant.

Returning to the results of this paper: every variety with rank has a dominant pair of algebras, of which one is empty (omit it if nonexistent).

The last topic here investigated is dominance and exemplariness for modules over a ring R. First, they are the same problem: every exemplary subcategory of a category of modules is dominant. Then we ask when a free module on one generator, i.e. the ring R itself, is dominant. The basic counterexample: a ring $A[[t]]$ of all formal power series in a central indeterminate t, over any nonsingleton ring A, is a nondominant $A[[t]]$-module, say not *self-dominant.* On the other hand, any non-self-dominant ring R must resemble a power series ring in having a descending sequence of right ideals I_n with zero intersection and a sequence of nonzero elements c_n of I_n such that all infinite series $\sum c_i x_i$ ($x_i \in R$) have sums s in the sense that for each n, $s - c_1x_1 - \cdots - c_nx_n \in I_{n+1}$. In particular, countable rings are self-dominant.

I am indebted to John Dauns and Stephen Schanuel for discussions of some of this material.

1. Generalities. A datum is precisely a functor $U\colon \mathcal{C} \to \mathcal{S}$ from its category $\mathcal{C}$ to the category $\mathcal{S}$ of sets, but it will usually be called simply $\mathcal{C}$. The varietal theory Str $\mathcal{C}$ is determined by specifying its n-ary operations for each cardinal n, and the rules for composing operations; and they are as follows. The n-ary operations are the natural transformations $U^n \to U$. Of course the formalities **[12]** involve indices which we can be casual about. To compose n-ary w with n operations v_α in (some of) m variables, write each v_α as an m-ary operation $U^m \to U$; then the ordered n-tuple $\{v_\alpha\}$ gives a natural transformation $U^m \to U^n$, which one follows with $w\colon U^n \to U$.

We must refer, in treating one of the examples, to Linton's *equational* theories which are bigger than varietal theories **[12]** (for instance, the theory of complete Boolean algebras). Their virtue (for our purpose) is that they have finite colimits. We get back to varietal theories quickly by means of the observation that an equational theory which has a generic algebra A is varietal. (That is, it has only a set of n-ary operations, no larger than the set of functions $A^n \to A$, for each n.)

For a general small subcategory $\mathcal{Q}$ of a varietal category $\mathcal{S}^{\mathbf{T}}$, the insertion i: $\mathcal{Q} \to \mathcal{S}^{\mathbf{T}}$ induces a morphism of theories Str i: $\mathbf{T} \to$ Str $\mathcal{Q}$ with which there are associated three kinds of "image". $\mathcal{Q}$ is generic if and only if interpreting the formal operations of $\mathbf{T}$ as operations on the underlying sets of objects of $\mathcal{Q}$ is one-to-one; this means precisely that Str i is monic. (And it depends only on the objects of $\mathcal{Q}$, not on the morphisms.) Equivalently, of course: Str i factors across no proper strict quotient of $\mathbf{T}$, but has image $\mathbf{T}$.

It is convenient to take "dominant" before "exemplary". $\mathcal{Q} \to \mathcal{S}^{\mathbf{T}}$ is dominant when it does not factor through a tripleable proper subcategory. Those subcategories are given **[3]** by morphisms $\mathbf{T} \to \mathbf{T}'$ such that every operation $\alpha(x_1, x_2, \ldots)$ of $\mathbf{T}'$ is formally definable in terms of operations of $\mathbf{T}$; that is, it is a theorem of $\mathbf{T}'$ that $y = \alpha(x_1, x_2, \ldots)$ if and only if Φ, where Φ is a properly formed statement in the language of the image of $\mathbf{T} \to \mathbf{T}'$ (set-theoretic image). These are the epimorphisms of varietal theories. The universal such factorization is given by the stable dominion $\mathbf{T}^*$ of Str i: $\mathbf{T} \to$ Str $\mathcal{Q}$, the largest subtheory of Str $\mathcal{Q}$ into which $\mathbf{T}$ goes epically. (For proof of existence, the reader has the choice of avoiding the Hodges-Shelah construction **[3]** and using the usual transfinite downward construction, or excavation, of stable dominions **[6]**; or of using **[3]** and the trivial lemma: $\mathbf{T} \to \mathbf{T}^*$ is epic if $\mathbf{T}^*$ is generated by subtheories into which $\mathbf{T}$ goes epically.)

Applying this, i: $\mathcal{Q} \to \mathbf{S}^{\mathbf{T}}$ is dominant if and only if the stable dominion of Str i is $\mathbf{T}$, which is to say that Str i is an extremal monomorphism.

For full tripleable subcategories, everything is the same except that the definitions of operations must be positive existential, i.e. constructed without use of negation or universal quantifiers **[8]**. These epics $\mathbf{T} \to \mathbf{T}'$ may reasonably be called *positive epics*, and $\mathbf{T}'$ a *positive quotient* of $\mathbf{T}$. Then i: $\mathcal{Q} \to \mathcal{S}^{\mathbf{T}}$ is exemplary if and only if Str i factors across no positive proper quotient of $\mathbf{T}$ ($\mathbf{T}$ is, as it were, the positive image).

THEOREM 1. *A morphism of varietal theories F: $\mathbf{T} \to \mathbf{T}'$ is positive epic if and only if every operation of* $\mathbf{T}'$ *belongs to a set S of operations $w_j(x_1, x_2, \ldots)$ of* $\mathbf{T}'$ *such that the*

conjunction of "$y_j = w_j(x_1, x_2, \ldots)$" *for all* $w_j \in S$ *is equivalent, as a theorem of* $\mathbf{T}'$, *to a conjunction of* $F(\mathbf{T})$-*equations* "$\varphi_k((x_i),(y_j)) = \psi_k((x_i),(y_j))$".

PROOF. "If" is trivial. For the converse, any $\mathbf{T}'$-operation $w((x_i)_{i\in I})$ belongs to a free $\mathbf{T}'$-algebra S_0 on generators x_i. If w is one of the variables x_i, "$y = x_i$" is the required single equation. Otherwise let S be S_0 minus the generators and let Σ be the set of all $F(\mathbf{T})$-equations in variables x_i ($i \in I$) and $y_j \in S$ that are true in S_0. For any values x_i^*, y_j^* for these variables in a $\mathbf{T}'$-algebra A, they simultaneously satisfy Σ if and only if the function $*$: $S_0 \to A$ which they define underlies a morphism from Sem $F(S_0)$ to Sem $F(A)$. Sem F being full, $*$ underlies a morphism of $\mathbf{T}'$-algebras just in this case; and that is in turn equivalent to $y_j^* = w_j(x_1^*, x_2^*, \ldots)$ for all j.

Note, A and B must be $\mathbf{T}'$-algebras; nothing was said about expressing the laws of $\mathbf{T}'$ in terms of $F(\mathbf{T})$. Of course it can be done if complexity of sentences is unlimited, but I do not know how reasonably.

Returning to "generic", one has Birkhoff's semantic construction of an image: $\mathfrak{A} \subset \mathbb{S}^{\mathbf{T}}$ is generic if and only if every object of $\mathbb{S}^{\mathbf{T}}$ is a strict quotient of a subobject of a product of objects of $\mathfrak{A}$.

A sufficient condition for dominance of $\mathfrak{A} \subset \mathbb{S}^{\mathbf{T}}$ (for use in Theorem 3) is that the smallest subcategory $\mathcal{C}$ containing $\mathfrak{A}$ and closed under limits, the *limit closure* of $\mathfrak{A}$, contains all free algebras and their morphisms; for a tripleable subcategory is closed under limits and also under U-split coequalizers (PTT, see e.g. [**13**, p. 147]), which yields from free algebras, all algebras. I do not know if this sufficient condition is necessary. Of course, if the full limit closure contains the free algebras, $\mathfrak{A}$ is exemplary.

It can do no harm to spell out: $\mathcal{C} \subset \mathfrak{D}$ is closed under limits. This means that every limit L in $\mathfrak{D}$ of a diagram in $\mathcal{C}$ is in $\mathcal{C}$ and is a limit in $\mathcal{C}$ of that diagram. This guarantees many morphisms. For instance, every automorphism f of an object X of $\mathcal{C}$; for the singleton diagram X has f: $X \to X$ as one of its limits.

Finally, i: $\mathfrak{A} \to \mathbb{S}^{\mathbf{T}}$ canonical means that Str i: $\mathbf{T} \to$ Str $\mathfrak{A}$ is invertible. The theory $\mathbf{T}$ is nearly the same thing as the full subcategory $\mathfrak{T}$ of $\mathbb{S}^{\mathbf{T}}$ on the free algebras. (Slightly differing conventions are in use [**12**, **14**]; $\mathbf{T}$ is $\mathfrak{T}$ with a functor $\mathbb{S} \to \mathfrak{T}$ giving free generators.) The functor carrying Str i is the restriction to $\mathfrak{T}$ of the right subregular representation of $\mathbb{S}^{\mathbf{T}}$ over $\mathfrak{A}$, and thus Str i is invertible if and only if $\mathfrak{A}$ is right adequate for free algebras [**4**].

The implications canonical $\to$ dominant $\to$ exemplary $\to$ generic do not run backwards, in general. Undoubtedly there exist broad conditions for inverting some of them. But not finiteness conditions; the examples in the Introduction show that, when we complete the justification of what was said there. (1) The two-element semilattice A is exemplary. For every semilattice is isomorphic with a semilattice of sets, i.e. embeddable in a power A^m; and every subsemilattice of A^m is the equalizer of two morphisms into an A^n. Thus the full limit closure is everything, A is exemplary. (2) A is a dominant double-pointed set. First, every epimorphism $\mathbf{D} \to \mathbf{T}$ from the theory $\mathbf{D}$ of double-pointed sets is surjective; for, suppose the contrary. Then some free $\mathbf{T}$-algebra B has an element formed from the generators by an operation not in $\mathbf{D}$, so a larger one C has two such elements. The set C admits a

different (but isomorphic) **T**-structure with those two elements exchanged, which does not change the **D**-structure. This gives a model M, on the set C, of an equational theory $\mathbf{T} +_{\mathbf{D}} \mathbf{T}$. But M is generic in a varietal specialization **V**. We have two different morphisms $\mathbf{T} \to \mathbf{V}$ agreeing on **D**, a contradiction. Now these epics are determined; $\mathbb{S}^{\mathbf{D}}$ has only two proper subvarieties, and neither contains A.

As a matter of fact, we have not touched the bottom of the cardinals, and there is an unresolved question in that vicinity. At the bottom, if all the operations of a theory are 0-ary, its generic algebras are dominant, by the argument on **D** above. I do not know whether, when all operations are at most unary, exemplary algebras are dominant. There is some difficulty in finding examples because every epimorphism of monoids is positive. (This follows from the description of the epimorphisms **[6]**; the extension to categories, where every bimorphism is positive epic, is explicit in **[7]**.) However, as we shall see in a moment, a monoid theory (theory of M-sets) can have a proper quotient with operations of higher arity, and such a quotient need not be positive. Our example factors through a positive quotient with nonunary operations; and indeed, the question whether exemplary families of M-sets are dominant comes down to: Does every proper quotient of a monoid theory factor through a proper positive quotient?

The example is the theory **T** of sets A with a bijection $m\colon A \times A \to A$, a unary operation d, and a constant o such that for all x, $dm(o, x) = dm(x, o) = o$. Its operations are generated by m, d, o, and the coordinates p_1, p_2 of m^{-1}. Evidently they are definable over the monoid **M** generated by p_1, p_2 and d. But since two sets of power $\aleph_0$ are equivalent, there is a bijection $Z \times Z \to Z$ whose restriction to pairs in the set N of positive numbers is a bijection upon N. Defining $dx = \min((p_1 x)^2, (p_2 x)^2)$, N becomes a **T**-algebra with $o = 1$, Z a **T**-algebra with $o = 0$ and $N \subset Z$ an **M**-homomorphism which is not **T**-homomorphic. So $\mathbf{M} \subset \mathbf{T}$ is a nonpositive quotient.

2. Sufficient conditions. Observe that for every category $\mathcal{Q} \to \mathbb{S}$ over finite sets, there are infinitary operations on the sets of $\mathcal{Q}$ which are preserved by all the mappings in $\mathcal{Q}$. If $\mathcal{F}$ is any ultrafilter on an index set N, an N-tuple $\{x_\nu\}$ in a finite set "converges" to a definite value x, the unique value which is taken on a set of indices belonging to $\mathcal{F}$. Naming the functor $U\colon \mathcal{Q} \to \mathbb{S}$, one checks trivially that $\mathcal{F}$-convergence gives a natural transformation $U^n \to U$. Since there are ultrafilters on arbitrarily large sets which are not supported by smaller subsets, this shows that a variety having a canonical set of finite algebras cannot have a rank.

The argument applies as well if the sets X, underlying objects of $\mathcal{Q}$, are not finite but the ultrafilter $\mathcal{F}$ on N is such that every function $N \to X$ is constant on some member of $\mathcal{F}$: $\mathcal{F}$ is *X-multiplicative*. A *measurable cardinal* is a cardinal n such that a set N of that size has a nonprincipal ultrafilter that is X-multiplicative for all smaller sets X. Evidently:

If every cardinal is less than some measurable cardinal then a nontrivial variety having a canonical set of algebras cannot have a rank.

On the contrary hypothesis, however, there are some affirmative results. We should note in passing that there are affirmative results in any case; the compact

spaces, and the complete atomic Boolean algebras, form perfectly good rankless varieties having canonical algebras. More: every algebra A in $\mathbb{S}^{\mathbf{T}}$ is canonical in a suitable variety $\mathbb{S}^{\operatorname{Str} A}$. Perhaps the varieties having canonical algebras are not worse behaved than the varieties having rank. But rankless varieties are less convenient, and we drop them.

THEOREM 2. *If* $\mathbf{T}$ *is an algebraic (finitary varietal) theory and* m *is a cardinal not less than any measurable cardinal nor than the number of* $\aleph_0$*-ary operations of* $\mathbf{T}$, *then in* $\mathbb{S}^{\mathbf{T}}$ *a free algebra on* m *generators with (if possible) an empty algebra forms a canonical set.*

PROOF. First, existence of an empty algebra E is equivalent to nonexistence of 0-ary operations in $\mathbf{T}$; and including E in a subcategory $\mathcal{Q}$ guarantees that $\operatorname{Str}\mathcal{Q}$ has no 0-ary operations. For $0 < n < p$, the n-ary operations of any varietal theory are identifiable among the p-ary operations since $[n]$ is a definable retract of $[p]$. If there are 0-ary operations, $[0]$ is also a definable retract of $[p]$. Hence it suffices to show that $i\colon \mathcal{Q} \to \mathbb{S}^{\mathbf{T}}$ induces a morphism $\operatorname{Str} i$ bijective on p-ary operations for $p > m$.

We may now ignore E on account of the uniqueness of functions with empty domains. Let F be a free algebra on a set $M = \{x_\mu\}$ of m generators, and $S \supset M$ its underlying set. Note that S has only m elements (since it is a union of free algebras of at most m elements generated by the m finite subsets of M). Since m is infinite, $\operatorname{Str} i$ is injective. It remains to show that every function $h\colon S^p \to S$ commuting with all endomorphisms is an operation of $\mathbf{T}$.

We shall study mainly the restriction of h to M^p. Indeed, once we have a finite set of indices α_i and an operation w of $\mathbf{T}$ such that $h((u_\alpha)) = w(u_{\alpha_1}, u_{\alpha_2}, \ldots)$ for each $(u_\alpha) \in M^p$, the same holds for every (u_α) in S^p. For there are only m different possible values for u_α, so there exist (v_α) in M^p and an endomorphism e of F satisfying $e(v_\alpha) = u_\alpha$ for all α; hence $h((u_\alpha)) = e(w(v_{\alpha_1}, \ldots)) = w(u_{\alpha_1}, \ldots)$.

Recall that the set of generators on which an element t of the free finitary algebra F depends is uniquely determined unless t is pseudoconstant, i.e. generated by each of two disjoint subsets I, J of M. (If t is generated by each of K, $L \subset M$ then it is generated by any nonempty set $H \supset K \cap L$, for one can map K into H by a function fixing all points of L; thus F has an endomorphism which leaves t fixed but also takes it into the subalgebra generated by H.) We are interested in the numbers of generators, say $n(u)$, on which values $h(u)$ of h depend—and of course in which generators. Call $n(u)$ zero if $h(u)$ is pseudoconstant.

For u in M^p, $h(u)$ depends at most on the coordinates of u, since endomorphisms fixing u fix $h(u)$. Moreover, since h commutes with the automorphisms induced by permutations of M, n is constant on orbits under that group. In more detail, let P be a p-element index set, coding the power set M^p as M^P; to each partition $\mathcal{U}$ of P into at most m subsets U_α there is associated a finite subset $S(\mathcal{U})$ such that for any $u \in M^P$ whose fibers are the elements of $\mathcal{U}$, $h(u)$ depends on just those coordinates u_π for which π belongs to an element of $S(\mathcal{U})$. And we can construct u^* from u by identifying all the unused u_α; the fibers of u^* are the $n(u)$ elements of $S(\mathcal{U})$ and the complement of their union, and $h(u^*) = h(u)$.

If a partition $\mathcal{V}$ refines $\mathcal{U}$, then each element of $S(\mathcal{U})$ contains an element of $S(\mathcal{V})$, for there are associated elements v, u of M^P and an endomorphism e with $e(v_\pi) = u_\pi$ for all π, so $h(u) = e(h(v))$.

Call an element U of a finite partition $\mathcal{U}$ of P *unbounded* in $\mathcal{U}$ if there are partitions (and therefore there are finite partitions) $\mathcal{V}$ containing $\mathcal{U} - \{U\}$ for which $S(\mathcal{V})$ is arbitrarily large; otherwise U is *bounded* in $\mathcal{U}$. If U is bounded in $\mathcal{U}$, $S(\mathcal{V})$ having the maximum size k on partitions $\mathcal{V} \supset \mathcal{U} - \{U\}$, consider some $\mathcal{V}$ attaining the maximum value and let $V_1, \dots, V_r$ be the elements of $S(\mathcal{V})$ which are subsets of U. If we further refine $\mathcal{V}$ by carving some V_i into m or fewer parts V_{ij}, making a new partition $\mathcal{W}$, $S(\mathcal{W})$ has at least k elements since $\mathcal{W} < \mathcal{V}$, but at most k since $\mathcal{W} \supset \mathcal{U} - \{U\}$; looking more closely, $S(\mathcal{W})$ consists of $S(\mathcal{V})$ with just one V_{ij} replacing V_i. The subsets V_{ij} of V_i which are in $S(\mathcal{W}_j)$, $\mathcal{W}_j = (\mathcal{V} - \{V_i\}) \cup \{V_{ij}, V_i - V_{ij}\}$, form an m-multiplicative ultrafilter. It must be principal. We conclude that $V_1, \dots, V_r$ contain points $p_1, \dots, p_r$ such that the refinement $\mathcal{W}$ of $\mathcal{V}$ which replaces each V_i with $\{p_i\}$ and its relative complement has $S(\mathcal{W})$ consisting of the singletons $\{p_i\}$ and $k - r$ elements of $\mathcal{U}$. Therefore $S(\mathcal{W}^*)$ is the same, where $\mathcal{W}^* = (\mathcal{U} - \{U\}) \cup \{\{p_1\}, \dots, \{p_r\}, U - \{p_1, \dots, p_r\}\}$.

The proof is now reduced to showing that n is a bounded function, i.e. that P is bounded in $\{P\}$. For that will give us a partition $\mathcal{W}^*$ into r singletons and the rest of P, with $S(\mathcal{W}^*)$ consisting of the singletons, and each refinement $\mathcal{X}$ of $\mathcal{W}^*$ having only r elements in $S(\mathcal{X})$, one in each singleton.

Let us prove $(*)$: If $U \in \mathcal{U}$ is unbounded in the finite partition $\mathcal{U}$, there is a refinement $\mathcal{Y}$ consisting of $\mathcal{U} - \{U\}$ and finitely many subsets of U of which at least one is unbounded in $\mathcal{Y}$ and a different one belongs to $S(\mathcal{Y})$. At least (from the preceding) there is a refinement $\mathcal{V}$ consisting of $\mathcal{U} - \{U\}$ and finitely many more elements V_i of which at least two belong to $S(\mathcal{V})$. Suppose each V_i bounded in $\mathcal{V}$. Let us take first the case of two V_i. Since V_1 is bounded in $\mathcal{V}$ there exists $\mathcal{W}^*$, which is $\mathcal{V}$ with V_1 replaced by some singletons in $S(\mathcal{W}^*)$ and the rest R of V_1, as above. Form $\mathcal{W}^{**}$ from $\mathcal{W}^*$ by uniting R with V_2; via commuting with endomorphisms, $S(\mathcal{W}^{**})$ is $S(\mathcal{W}^*)$ with $R \cup V_2$ replacing V_2. In this case, since U is unbounded in $\mathcal{U}$, $R \cup V_2$ cannot be bounded in $\mathcal{W}^{**}$.

Finally, if we have $(s + 1)$ V_i's (each bounded in $\mathcal{V}$), and the case of only s of them has been settled, observe that we may assume each V_i is in $S(\mathcal{V})$. But again, since V_1 is bounded in $\mathcal{V}$, we get some singletons in V_1 with finite union σ as before; and in the partition $(\mathcal{U} - \{U\}) \cup \{U - \sigma\} \cup \{\{x\}\colon x \in \sigma\}$, $U - \sigma$ cannot be bounded. But $U - \sigma$ cuts up into fewer V_i's, namely $R \cup V_2, V_3, \dots, V_{s+1}$; if this partition is $\mathcal{Z}$, all the sets just listed are in $S(\mathcal{Z})$. So $(*)$ is proved.

If P were unbounded in $\{P\}$, this would give us an infinite sequence of successively finer finite partitions $\mathcal{U}^k$, each containing all but one element of the preceding with the exceptional elements nested and with $S(\mathcal{U}^k)$ increasing in size. Such a sequence has a countable common refinement $\mathcal{V}$. Since $m \geqslant \aleph_0$, $\mathcal{V}$ must have a finite subset $S(\mathcal{V})$; but $S(\mathcal{V})$ also must contain all but one element of each of the increasingly large sets $S(\mathcal{U}^k)$, a contradiction.

Let us notice some related previous results. In vector spaces over a division ring D, an m-dimensional one is canonical if no measurable cardinal exceeds m [**5**]. For abelian groups, if $\aleph_0$ is the only measurable cardinal, m need only be 2. Abelian groups are, of course, Z-modules; and this result of Łos has been extended by Gerstner, Kaup, and Weidner [**2**] to some other rings sufficiently like Z. According to C. Jensen [**9**, p. 87] one has

THEOREM. *If there are no measurable cardinals, then R^2 is a canonical R-module, for every commutative principal ideal ring R having infinitely many maximal ideals.*

The theorem is expressed a little differently, referring to homomorphisms $R^p \to R$ instead of operations $(R^2)^p \to R^2$. It is easy to see the equivalence of the two expressions. The use of R^2 guarantees that an operation is homomorphic, since it commutes with the endomorphism $(x, y) \to (x + y, 0)$.

Notice also a complex of related results [**1**, especially 3.1] of Ehrenfeucht, Fajtlowicz and Mycielski. In [**1**] too, the subject is homomorphisms; but to commute with the set E of endomorphisms is precisely to be homomorphic with respect to an obvious algebra structure. Indeed I was forced to improve Theorem 2 because the referee showed that the weaker original version followed rather easily from [**1**].

Before stating the next result we note that for infinitary theories, while the set of variables on which an operation depends is no longer precisely determined, the number of them is—since cardinals are well ordered. Call this number, the minimum number of variables in terms of which an operation α can be expressed, the *actual arity* of α.

THEOREM 3. *If* $\mathbf{T}$ *is a varietal theory having rank and m is an infinite cardinal greater than the actual arity of every operation of* $\mathbf{T}$, *then in* $\mathbb{S}^{\mathbf{T}}$ *a free algebra on m generators with (if possible) an empty algebra forms a dominant set.*

PROOF. As we saw in §1, $\mathcal{Q} \subset \mathbb{S}^{\mathbf{T}}$ is dominant if the full subcategory on the free algebras is in the limit closure of $\mathcal{Q}$. We have the free algebra on 0 generators, in $\mathcal{Q}$ if it is empty and as a retract if it is not empty. Having the free algebra F on m generators, we need to get every larger free algebra G and every morphism h: $G \to F$. Then, since every (free) algebra H that we have (with a possible empty exception) is obtained by iterated construction of limits from F, we shall have every morphism $G \to H$ since we have its coordinates $G \to F$.

Given G free on $p > m$ generators and h: $G \to F$, take a free algebra G' on a disjoint sum $P \cup M$ of a p-element set P and the set M of free generators of F. Construct an inverse mapping system indexed by the m-element or smaller subsets T of P, the Tth object X_T being the subalgebra of G' generated by $T \cup M$. For $U \supset T$, X_U is retracted on its summand X_T by sending the generators in $U - T$ to the summand F by means of h. Evidently G' maps to the system , going to X_T by the retraction which on $P - T$ is given by h.

This exhibits G' as limit of the X_T (all of which are isomorphic with F). For, consider any thread $\{x_T\}$. I claim that some summand X_{T^*} contains all x_T. If this were false then we could construct an expanding sequence of m sets T_λ with x_{T_λ} not

in the join of the previous X_{T_κ}. Let U be the union of all T_λ. $x_U \in X_U$ is generated by l elements of $U \cup M$ for some actual arity $l < m$. But the sequence $\{T_\lambda\}$ has an initial segment (not necessarily proper) whose length is the regular cardinal l^+. Let V be its union. The retraction from X_U to X_V takes x_U to x_V in the subalgebra generated by M and at most l elements of V; hence x_V is in X_{T_λ} for some $\lambda < l^+$, and so is $x_{T_{\lambda+1}}$, a contradiction, proving the claim. Now $\{x_T\}$ is the image of $x_{T^*} \in G'$. For if $T \supset T^*$, $x_T \in X_{T^*}$, and $X_T \to X_{T^*}$ is a retraction; $x_T = x_{T^*}$. Since those T are cofinal, this determines the limit.

Finally, G is embeddable in G' as the limit of the diagram consisting of G' and all its automorphisms fixing P. For each element of G' is generated by P and a set L of less than m elements of M, and when fixed under an automorphism taking L into $M - L$, it is generated by P. (This uses the fact that equalities in G' are laws of $\mathbf{T}$; in effect, if $\alpha(x, y) = \alpha(x, z)$ by law, then in particular it is $\alpha(x, x)$.) Composing with the coordinate map to $X_\varnothing$, we have h.

COROLLARY. *Dominance and exemplariness are not universe-dependent; that is, if a category $\mathfrak{A}$ of models of a varietal theory with rank in a universe $\mathfrak{U}$ is dominant, or exemplary, it remains so in any larger universe.*

For the smallest containing tripleable subcategory (tripleable full subcategory) contains the algebras of Theorem 3 and hence all algebras.

Genericity obviously is not universe-dependent. As we have seen, canonicalness is.

One may naturally ask whether Theorem 3 holds for finite cardinals $m \geqslant 2$. At the first case, $m = 2$, I do not know. Here is a little. First, a two-element set T is dominant as follows. Any set S is embeddable in a power P of T; moreover, it can be embedded so that $P - S$ has more than one point, making S an equalizer of two automorphisms of P. Any map $S \to T$ is a composite $S \to P \times T \to T$ of an embedding as equalizer and a coordinate projection. This argument extends to a free G-set on two generators for any group G. I do not know if the result extends to M-sets where M is a three-element chain.

Turning to modules, we have

LEMMA 1. *Every finite-limit-closed subcategory of an abelian category is full; so every exemplary subcategory of a category of modules is dominant.*

PROOF. If A is in a finite-limit-closed subcategory of abelian $\mathcal{C}$, so is Δ: $A \to A \oplus A$. So is the automorphism ε of $A \oplus A$ taking (x, y) to $(x, y - x)$. And so is the second coordinate of $\varepsilon\Delta$, which is 0: $A \to A$. Hence the equalizer $0 \to A$ of 0 and 1: $A \to A$. Since the object 0 is a limit of the empty diagram, we also have morphisms $B \to 0$ and 0: $B \to A$. Finally, if we have the objects A and B, and h: $A \to B$ is a morphism, we get h by injecting A into $A \oplus B$, applying the automorphism $(x, y) \to (x, y + h(x))$, and projecting.

This is a cheering lemma, but it does not really affect the work to follow; in any case one would work on exemplariness first, since it is easier. Still it requires either handling blocks of operations (from Theorem 1) or tracing positive existential definitions through a number of theories. The actual work done below involves only

a first theory Str $\mathcal{Q}$ and a wave at the last one ("The necessary and sufficient condition is..."); so the description of the whole list will be informal, and we begin with an amusing irrelevancy. All these theories are abstractions of the facts in the ring R. What about positive existential definitions valid in fact in R? Lots. For $R = Z_2$, multiplication and lattice join $(x + y + xy)$ are positively existentially definable over R-module theory. For instance, multiplication: $y = x_1x_2$ iff $x_1 = y = 0$ or $x_2 = y = 0$ or $x_1 + y = x_2 + y = 0$.

Our question is, when is the ring R a dominant R-module, or *self-dominant*? By Lemma 1 this means merely that it is an exemplary R-module. We shall use R^2 because Str R^2 is smaller, and evidently it lies in the same tripleable subcategories as R. Let ${}_R\mathbf{T}$ denote the theory of R-modules (since comodules, "right modules", appear too). The insertion i of R^2 (over sets, and with its endomorphisms) in the module category gives us Str i: ${}_R\mathbf{T} \to$ Str R^2; we want the stable dominion D^∞. By Lemma 1 it is a positive quotient of ${}_R\mathbf{T}$. Therefore it is contained in the *positive dominion* P^1 of Str i, the subtheory of Str R^2 on those operations definable as in [7] over Str $i({}_R\mathbf{T})$. And one can continue; P^1 is trivially contained in the dominion D^1, $P^\lambda \subset D^\lambda$, but $P^\infty = D^\infty$. (I do not know if there is an ordinal bound to the length of this iteration for rings.)

Str R^2 is identified by

LEMMA 2. *For each cardinal n, there is a bijection between natural n-ary operations α on R^2 and right R-linear morphisms α^*: $R^n \to R$; $\alpha^*((x_i)_{i\in n})$ is the first coordinate of $\alpha(((x_i, 0))_{i\in n})$, and $\alpha(((x_i, y_i)))$ is $(\alpha^*((x_i)), \alpha^*((y_i)))$.*

The proof is routine calculation.

Call an n-ary relation $S \subset R^n$ *projectively definable* if there is a system Σ of linear equations $\lambda_j((y_i)_{i\in n}, z_k) = 0$ such that $(y_i) \in S$ if and only if Σ is solvable. Positive dominions over a theory of modules are given 'nearly' by those operations that have projectively definable graphs; but the definability must be not only true for values of the variables in R, but true in all models of the codomain theory, here Str R^2. There exists machinery, by the way, for transporting the notion of projectively definable relation to models of Str R^2, but we have no present use for it. Call an operation $R^n \to R$ (right R-linear) *projectively definable* if its graph is so. Call it *stably definable* if it belongs to the stable dominion of Str i. Equivalently (Theorem 1 and Lemma 1) it belongs to a family of operations $w_j((x_i))$ jointly definable as in Theorem 1.

LEMMA 3. *A ring R is self-dominant if and only if every stably definable $\aleph_0$-ary operation on R is finitary; therefore, if every projectively definable $\aleph_0$-ary operation on R is finitary.*

PROOF. If R is self-dominant, R^2 is a dominant R-module and the stable dominion of Str i: ${}_R\mathbf{T} \to$ Str R^2 is ${}_R\mathbf{T}$; every stably definable operation $(R^2)^n \to R^2$ is finitary for all n, and hence so is every stably definable operation $R^m \to R$. Conversely, if every stably definable $\aleph_0$-ary operation on R is finitary, then any positive quotient of ${}_R\mathbf{T}$ in Str R^2 has the same n-ary operations for $n \leqslant \aleph_0$, and the corresponding tripleable subcategory contains R^ω. Since R^ω is dominant, this is the category of all modules and the proof is complete.

Theorem 4. *For a ring R to fail to be self-dominant it is necessary that R be uncountable and not right artinian; indeed, there must be a descending sequence of right ideals I_n with intersection* 0 *and a sequence of nonzero elements c_n of I_n such that there is a linear operation α: $R^\omega \to R$ satisfying, for all $(x_i) \in R^\omega$, for all n, $\alpha(x_1, x_2, \ldots) - c_1x_1 - \cdots - c_nx_n \in I_{n+1}$. If this condition holds with projectively definable right ideals I_n then the graph of α is projectively definable.*

Proof. First, given this condition and definitions of $u \in I_n$ by the existence of (z_{nk}) satisfying equations $\lambda_j(u, (z_{nk})) = 0$, then $y = \alpha(x_1, x_2, \ldots)$ iff for all n, for all j,

$$\lambda_j(y - c_1x_1 - \cdots - c_{n-1}x_{n-1}, (z_{nk})) = 0.$$

For the partial converse, let β be a definable nonfinitary operation from R^ω to R and let $\{\lambda_j\}$ be a family of finitary linear operations on $R^\omega \times R \times R^p$ defining β. Let T_n be the subcomodule of R^ω consisting of all sequences (x_i) with $x_1 = x_2 = \cdots = x_{n-1} = 0$, and $H_n = \beta(T_n)$. The H_n are subcomodules of R, i.e. right ideals, and they are nonstrictly decreasing. Suppose $r \neq 0$ to be in all H_n. Since $\beta(0, 0, \ldots) = 0 \neq r$, the point $(0, 0, \ldots; r) \in R^\omega \times R$ is not in the projected kernel of every λ_j. Thus there is a finite subset F of ω such that if $x_i = 0$ for all $i \in F$, $\beta(x_1, x_2, \ldots) \neq r$. But this contradicts $r \in \bigcap H_n$.

Let $c_n^0 = \beta(e^n)$ where $e^n \in R^\omega$ has terms $e_i^n = \delta_{ni}$. Since $e^n \in T_n$, $c_n^0 \in H_n$. We are practically done, for each $x \in R^\omega$ is $x^1 + x^2$ with $x_i^1 = 0$ for $i > n$, and $x_i^2 = 0$ for $i \leqslant n$, $\beta(\alpha^1) = c_n^0x_1 + \cdots + c_n^0x_n$, and $\beta(x^2) \in H_{n+1}$. Not all c_n^0 need be nonzero, but infinitely many must. (If only finitely many, let β' be the corresponding finitary operation. $\beta - \beta'$ must not vanish, so there is (x_i) with $\beta((x_i))$ not the finite sum $b = \beta'((x_i))$. As before, there is finite $F \subset \omega$ such that if $y_i = x_i$ for $i \in F$ then $\beta((y_i)) \neq b$; but such a sequence (y_i) can be constructed as a finite right linear combination of e's, a contradiction.) So there is a subsequence $(c_{n_m}^0) = (c_m)$, all nonzero. Therefore all fail to belong to $\bigcap H_n$, and we can choose this subsequence so that $c_m \notin H_{n_{m+1}}$. Then let s: $R^\omega \to R^\omega$ take (x_i) to (y_i) where $y_{n_j} = x_j$ and $y_i = 0$ otherwise; let $\alpha = \beta s$. The right ideals $\alpha(T_m) = I_m$ are descending, contained in H_m, and c_m is in I_m but not in I_{m+1}. Therefore on the uncountable set $\{0, 1\}^\omega \subset R^\omega$, α is one-to-one; if two zero-one sequences, (x_i), (y_i), first differ at the nth term, then $\alpha((x_i))$ and $\alpha((y_i))$ differ modulo I_{n+1}.

Some Remarks. A "one-line" projective definition of a right ideal has the form $y \in I$ iff $a_0y + \Sigma a_kz_k = 0$ is solvable, which is iff a_0y belongs to the finitely generated ideal generated by $\{a_k\}$. Projectively definable ideals include all intersections of these ideals; but of course they include more, since one parameter z_k may appear in several equations.

Note the case of an annihilator ideal I; $y \in I$ iff $ay = 0$. Since no parameters z_k occur, an operation definable by means of annihilator ideals is stably definable. In power series over any ring in commuting variables $t_1, t_2, \ldots, u_1, u_2, \ldots$, subject to $t_iu_j = 0$ for $i \geqslant j$, the operation $(x_i) \to \Sigma t_ix_i$ is so definable.

Of course the simplest case is power series in one variable t; the operation α: $(x_i) \mapsto \Sigma t^{i-1}x_i$ has remainders in the principal ideals generated by powers of t, and the parameter values z_k (in $\alpha(x_1, x_2, \ldots) - x_1 - tx_2 - \cdots - t^{k-1}x_k = t^k z_k$) are $\alpha(x_{k+1}, x_{k+2}, \ldots)$, so α is stably definable.

One step further, consider first a ring R_1 of power series in commuting variables t_i, u_i, v_i, w_i ($i \in \omega$) subject to $t_i u_j = v_i w_j$ for $i \geqslant j$. The operation α: $(x_i) \mapsto \Sigma t_i x_i$ is projectively definable by means of the ideals $H_n = \{y\colon u_n y \in w_n R_1\}$. The remainders $r_n \Sigma\{t_i x_i\colon i > n\}$ satisfy, to be precise, $u_n r_n = w_n \Sigma\{v_i x_i\colon i > n\}$; hence α and β: $(x_i) \mapsto \Sigma v_i x_i$ are both stably definable. The particular ring R_1 has more simply stably definable infinitary operations, but it seems likely that those can be killed off by using a suitable subring. The subring R_2 of power series in which finitely many variables generate only finitely many nonzero terms may do it, though it is not the smallest subring closed under α and β.

One last step; instead of v_i ($i \in \omega$), use variables v_i^j, with $t_i u_j = v_i^j w_j$ for $i \geqslant j$ (to begin with). Then the parameters for the definition of α are given by many operations β^n which are not, yet, definable. But just as u's, v's and w's make α definable, so some further variables and relations can make the β^n definable. Evidently running this construction out an infinite sequence will stabilize all the definitions. What is missing in this sketch is assurance that the operations have no simpler stable definitions.

EXAMPLE. *Relevance of the number of operations in Theorem* 2.

Let G be a group of cardinal $m > \aleph_0$. Consider the variety of pointed G-sets (pointed, to avoid the annoyance of an empty algebra). If F is free on $n < m$ generators, then as a G-set it is just a coproduct of $(n + 1)$ copies of G; for infinite n, each endomorphism of F is at most n-to-one. Therefore we may decompose F^m as $A \cup B$ where A is the set of m-tuples having at most n different coordinates and B is the nonempty remainder. For each endomorphism e of F, $e^m(A) \subset A$ and $e^m(B) \subset B$. Then define h: $F^m \to F$, $h(x)$ to be the first coordinate of x for $x \in A$, the second coordinate for $x \in B$. h commutes with endomorphisms; F is not canonical.

EXAMPLE. *A free algebra on $\aleph_0$ generators, $\aleph_0$ being the greatest actual arity, which is not exemplary.*

For this, start with the rankless varietal theory $\mathbf{T}$ of complete atomic Boolean algebras. Let $\mathbf{T}_0$ be the subtheory on the operations of countable actual arity. The free algebra A on $\aleph_0$ generators is still the algebra of subsets of a set of power c, and every $\mathbf{T}_0$-endomorphism of it is a $\mathbf{T}$-endomorphism. By the way, if there is no measurable cardinal, $\mathbb{S}^{\mathbf{T}}$ is a tripleable full subcategory of $S^{\mathbf{T}_0}$ containing A. But without assumptions, one can write a positive existential definition of the join of $\aleph_1$ elements y_j over the theory $\mathbf{T}_0$, using only $\aleph_0$ additional variables. Identify the set J of indices of the y_j with a proper subset of the set of atoms of A. Every element of A is a word in the generators $x_1, x_2, \ldots$; in particular, $\{j\colon j \leqslant k\}$ is a word $\alpha(x_1, x_2, \ldots)$, and the join J is a word $\alpha^*(x_1, x_2, \ldots)$. Then $z = \bigvee y_j$ (in A) if and only if there exist $t_1, t_2, \ldots,$ such that each join of y_j ($j \leqslant k$) is $\alpha_k(t_1, t_2, \ldots)$ and z is $\alpha^*(t_1, t_2, \ldots)$. For there is a unique endomorphism of A taking all x_i to t_i; it takes the atoms j to $y_j - \bigvee\{y_m\colon m < j\}$, and it takes J to z.

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