MEASURABLE REPRESENTATIONS
OF PREFERENCE ORDERS

BY

R. DANIEL MAULDIN[1]

ABSTRACT. A continuous preference order on a topological space Y is a binary relation $\preccurlyeq$ which is reflexive, transitive and complete and such that for each x, $\{y: x \preccurlyeq y\}$ and $\{y: y \preccurlyeq x\}$ are closed. Let T and X be complete separable metric spaces. For each t in T, let B_t be a nonempty subset of X, let $\preccurlyeq_t$ be a continuous preference order on B_t and suppose $E = \{(t, x, y): x \preccurlyeq_t y\}$ is a Borel set. Let $B = \{(t, x): x \in B_t\}$.

THEOREM 1. *There is an $\mathbb{S}(T) \otimes \mathfrak{B}(X)$-measurable map g from B into R so that for each t, $g(t, \cdot)$ is a continuous map of B_t into R and $g(t, x) \preccurlyeq g(t, y)$ if and only if $x \preccurlyeq_t y$. (Here $\mathbb{S}(T)$ forms the C-sets of Selivanovskii and $\mathfrak{B}(X)$ is a Borel field on X.)*

THEOREM 2. *If for each t, B_t is a σ-compact subset of Y, then the map g of the preceding theorem may be chosen to be Borel measurable.*

The following improvement of a theorem of Wesley is proved using classical methods.

THEOREM 3. *Let g be the map constructed in Theorem 1. If μ is a probability measure defined on the Borel subsets of T, then there is a Borel set N such that $\mu(N) = 0$ and such that the restriction of g to $B \cap ((T - N) \times X)$ is Borel measurable.*

1. Introduction. A preference order is a binary relation $\preccurlyeq$ on a set Y which is transitive, reflexive and complete. Such a relation is also known as a linear preorder. One may partition the set Y into equivalence classes via the equivalence relation $x \sim y$ if and only if $x \preccurlyeq y$ and $y \preccurlyeq x$ (one may think of x being equivalent to y as meaning that x and y are equally "preferred"). The natural relation induced by $\preccurlyeq$ on the set of equivalence classes is a linear order. A representation or Paretian utility function of a preference order is a map f from Y into the real numbers such that $x \preccurlyeq y$ if and only if $f(x) \leqslant f(y)$. One of the problems which has been of interest to mathematicians and economists concerns continuous representations of a preference order when the space Y has some underlying topological structure to which the preference order is related [5, 12]. More generally, one may have parametrized versions of this problem. One may consider the space Y and the preference order $\preccurlyeq$ as not being fixed, but the functions of some parameter (say time or "traders"). One may then be concerned with the problem of obtaining (continuous) representations which vary in some reasonable sense with the parameter [16]. This

type of situation is known as "preference orders in markets with a continuum of traders". A discussion and references may be found in [7].

The purposes of this paper are to prove some general representation theorems and to formulate some of the unsolved problems. Some known results will be reproved either for the deeper purposes of this paper or to show these earlier results may be obtained from standard mathematical procedures and constructions without the use of forcing methods from mathematical logic.

Our setting which will be called the "fundamental hypothesis" throughout the paper is the following:

Let T and X be Polish ($=$ complete separable metric) spaces. For each t in T, let B_t be a nonempty subset of X and $\leqslant_t$ a preference order on B_t such that if $x \in B_t$, then both $\{y: x \leqslant_t y\}$ and $\{y: y \leqslant_t x\}$ are closed with respect to B_t. We will also assume that this entire process is "describable" or computable from some parameter. Our assumption in this direction is that

$$E \equiv \{(t, x, y) \in T \times X \times X: x \leqslant_t y\}$$

is a Borel measurable subset of $T \times X \times X$.

In the context of this fundamental hypothesis we will set

$$B = \{(t, x) \in T \times X: x \in B_t\}.$$

Let us note at the outset that B is a Borel measurable subset of $T \times X$ since $B = \pi_{12}(E \cap \Delta)$, where $\Delta = \{(t, x, x): t \in T \text{ and } x \in X\}$ and π_{12} is the canonical projection map of $T \times X \times X$ into the first two coordinate spaces. Of course π_{12} restricted to the Borel set $E \cap \Delta$ is a one-to-one continuous map, so the image of $E \cap \Delta$ under π_{12} is also a Borel set [9, p. 487].

Let us make the following terminology. If $\leqslant$ is a preference order on a topological space Y, then $\leqslant$ will be said to be a continuous preference order provided that for each x in Y, both $\{y: y \leqslant x\}$ and $\{y: x \leqslant y\}$ are closed. This definition is reasonable since in order for there to be a continuous representation of $\leqslant$ it is, of course, necessary that these sets be closed. It is a corollary of a result of Debreu [5] that for each t, there is a continuous representation of $\leqslant_t$.

Thus, one question which naturally arises is how these representations can be "pasted together". Heuristically, one could speculate that since the preference orders are "describable", there should be some reasonable manner of putting the individual representations together. Specifically:

Problem. Is there a Borel measurable map f from B into R so that for each t in T, $f(t, \cdot)$ is a continuous representation of $\leqslant_t$?

The answer to this problem is surely negative although we do not have an example. We do prove that the answer is yes if one makes some additional assumptions concerning the T-sections of B.

THEOREM 4.3. *Assume the fundamental hypothesis. If for each t, B_t is a σ-compact subset of X, then there is a Borel measurable map f of B into R such that for each t in T, $f(t, \cdot)$ is a continuous representation of $\leqslant_t$.*

We also show that one can always obtain a universally measurable map f with no additional assumption.

THEOREM 3.4. *Assume the fundamental hypothesis. Then there is a map g of B into R which is $S(T) \otimes \mathcal{B}(X)$-measurable and such that for each t, $g(t, \cdot)$ is a continuous representation of $\leqslant_t$.*

Our method of proof of Theorem 3.4 yields an improvement in a theorem of Wesley [16].

THEOREM 4.1. *Assume the fundamental hypothesis. There is a map g of B into R which is $S(T) \otimes \mathcal{B}(X)$-measurable such that for each t, $g(t, \cdot)$ is a continuous representation of $\leqslant_t$ and such that if μ is a probability measure defined on the Borel subsets of T, then there is a Borel set N such that $\mu(N) = 0$ and g restricted to $B \cap [(T - N) \times X]$ is Borel measurable.*

There are at least two differences between our Theorem 4.1 and Wesley's theorem. First, Wesley's argument involves forcing techniques whereas our argument is purely descriptive set theoretic. Second, for each probability measure μ, Wesley's construction yields a Borel set N with $\mu(N) = 0$ and a Borel measurable function g from $B \cap [(T - N) \times X]$ into R such that for each $t \in T - N$, $g(t, \cdot)$ is a continuous representation of $\leqslant_t$. However, these g's may vary wildly with μ. In our construction, we have a fixed g and only vary the null set N with μ. Of course, our problem raised earlier asked whether it is even necessary to vary N with μ.

We will use the following conventions concerning a preference order $\leqslant$: $x \sim y$ means $x \leqslant y$ and $y \leqslant x$; $x \prec y$ but not $y \leqslant x$. Of course, if $\leqslant$ is a continuous preference order, then each equivalence class is closed.

This paper grew out of and is an improvement upon an earlier unpublished manuscript of the author [11]. The central theme of that manuscript was to first give a number of different constructions of continuous representations of a fixed preference order (Debreu's theorem), and second, to "parametrize" these constructions in the setting of the fundamental hypothesis. The idea was that the more "effective" the construction the better the measurability properties of the parametrized function would be. It was shown that one could take a particular method of construction and obtain Wesley's theorem. This is also carried out in this paper. It was also shown that if one followed this procedure and if one assumed Martin's Axiom (MA), then a universally measurable parametrized representation could be obtained (actually, one could take any of the constructions given and obtain a universally measurable parametrized representation assuming MA). What this result suggested is that Martin's Axiom is not really necessary: there must be an absolute argument for such a representation. This is precisely what happened. This fact was discovered by at least three people who read the original manuscript, John Burgess [2, 3] who used some techinques from "modern" descriptive set theory and by David Fremlin [8] and by the author both of whom used only "classical" descriptive set theoretic techniques (even though the key classical result used here, Lemma 3.2, has only been known for a few years). The proofs given in this paper follow the construction of Fremlin because this single method of construction shows most clearly how all the theorems may be obtained (although one could obtain similar results using any of the constructions given in [11]).

We wish to thank the referee for his numerous corrections and suggestions concerning the original manuscript [11] and for pointing out that the map g constructed in Theorem 3.4 is $\mathcal{S}(T) \otimes \mathcal{B}(X)$-measurable. We also wish to thank John Burgess for his comments and David Fremlin for his comments and permission to use his construction.

2. Basic construction of a continuous representation. Our proofs of the general theorems discussed in the introduction follow a basic strategy. First, one must have a method of proving the existence of a (continuous) representation of a given preference order. Then one "parametrizes" the proof. The idea being that if one is sufficiently clever in constructing a representation then the parametrized construction should retain some degree of measurability.

So, here we investigate representations of a fixed preference order. The first two theorems demonstrate a method of constructing a continuous representation of a given preference order. Theorem 2.2 is due to Debreu [5]. It is these proofs which will be parametrized in the next section.

THEOREM 2.1. *Let X be a second countable topological space and $\preccurlyeq$ a continuous preference order on X. Let $\{V_n\}_{n=1}^{\infty}$ be a base for the topology of X. For each k, let*

$$R_k = \{ y : y \in V_k \text{ and } [x \in V_k \to y \preccurlyeq x] \}$$

and let

$$L_k = \{ y : y \in V_k \text{ and } [x \in V_k \to x \preccurlyeq y] \}.$$

Let $D = \{x_n : n \in N\}$ be a countable dense subset of X such that for each k, if $R_k \neq \varnothing$, then $D \cap R_k \neq \varnothing$ and if $L_k \neq \varnothing$, then $D \cap L_k \neq \varnothing$. Define the function h on D as follows.
    Set

$$h(x_1) = 1/2.$$

Suppose h has been defined on $x_1, \ldots, x_n$. Then:
    (a) *If $\exists i \leqslant n \, [x_i \sim x_{n+1}]$, set $h(x_{n+1}) = h(x_i)$.*
    (b) *If $x_i \prec x_{n+1}, \forall i \leqslant n$, set $h(x_{n+1}) = \frac{1}{2}(1 + \max\{h(x_i) : i \leqslant n\})$.*
    (c) *If $x_{n+1} \prec x_i, \forall i \leqslant n$, set $h(x_{n+1}) = \frac{1}{2}(\min\{h(x_i) : x_i \preccurlyeq n\})$.*
    (d) *If $x_{i_0} \prec x_{n+1} \prec x_{j_0}$ for some i_0 and j_0, $1 \leqslant i_0, j_0 \leqslant n$ and $\forall i \leqslant n$, $x_i \preccurlyeq x_{i_0}$ or $x_{j_0} \preccurlyeq x_i$, set*

$$h(x_{n+1}) = \frac{1}{2}\big(h(x_{i_0}) + h(x_{j_0})\big).$$

The function h has the following properties:
    (1) *$x_i \preccurlyeq x_j \leftrightarrow h(x_i) \leqslant h(x_j)$, and*
    (2) *if $\{x_{n_i}\}_{i=1}^{\infty}$ and $\{x_{m_i}\}_{i=1}^{\infty}$ are sequences of elements of D such that*
        (α) *$x_{n_1} \preccurlyeq x_{n_2} \preccurlyeq x_{n_3} \preccurlyeq \cdots \preccurlyeq \cdots \preccurlyeq x_{m_3} \preccurlyeq x_{m_2} \preccurlyeq x_{m_1}$;*
        (β) *either $\{x_{n_i}\}_{i=1}^{\infty}$ is strictly increasing or $\{x_{m_i}\}_{i=1}^{\infty}$ is strictly decreasing; and*
        (γ) *$\forall j \exists p \, [x_j \preccurlyeq x_{n_p} \text{ or } x_{m_p} \preccurlyeq x_j]$,*
    then $\lim_{i \to \infty} h(x_{n_i}) = \lim_{i \to \infty} h(x_{m_i})$.

PROOF. It can be checked that h satisfies condition (1). Assume the hypotheses of condition (2) hold and

$$\lim_{i \to \infty} h(x_{n_i}) = b - \varepsilon < b + \varepsilon = \lim_{i \to \infty} h(x_{m_i}).$$

Notice that condition (γ) guarantees us that for all k, either $h(x_k) \leqslant b - \varepsilon$ or $b + \varepsilon \leqslant h(x_k)$. Let i and j be the first integers such that $b - 2\varepsilon < h(x_i) \leqslant b - \varepsilon$ and $b + \varepsilon \leqslant h(x_j) < b + 2\varepsilon$. Let us suppose that $i < j$. Let s be an integer less than j such that $b - 2\varepsilon < h(x_s) \leqslant b - \varepsilon$ and if $k \leqslant j$ and $h(x_k) \leqslant b - \varepsilon$, then $x_k \leqslant x_s$. So, if $k \leqslant j$, either $x_k \leqslant x_s$ or $x_j \leqslant x_k$. Condition (γ) guarantees us that there is a first positive integer p such that $x_s < x_p < x_j$. Of course, $p > j$. Now, by part (d) of the definition of h, $h(x_p) = \frac{1}{2}(h(x_s) + h(x_j))$. But, $|h(x_p) - b| < \varepsilon$. This contradiction establishes part (2) of the theorem in case $i < j$. The argument in case $j < i$ is similar. Q.E.D.

THEOREM 2.2. *Let X be a second countable topological space, $\leqslant$ a continuous preference order on X and let D and h be as constructed in Theorem 2.1. Define the function g from X into $[0, 1]$ by setting*

$$g(x) = \inf\{h(x_i): x \leqslant x_i\}.$$

Then g is a continuous representation of $\leqslant$.

PROOF. Certainly, if $x \leqslant y$, then $g(x) \leqslant g(y)$ and $g \,|\, D = h$.

Suppose $g(x) \leqslant g(y)$ and $y \prec x$. If there were elements x_j and x_i of D such that $y \leqslant x_i \prec x_j \leqslant x$, then $g(y) \leqslant g(x_i) = h(x_i) < h(x_j) = g(x_j) \leqslant g(x)$. From this contradiction it follows that there is an open set U such that $y \in U$ and if $z \in U$, then $z \leqslant y$. Thus, there is some k_1 such that $y \in R_{k_1}$. Therefore, there is some n_1 such that $y \sim x_{n_1}$. Similarly, there is k_2 such that $x \in L_{k_2}$ and some n_2 such that $x \sim x_{n_2}$. But, this implies that $g(y) \prec g(x)$. This last contradiction leads to the conclusion that if $g(x) \leqslant g(y)$, then $x \leqslant y$.

In order to prove that g is continuous, we will demonstrate that g is left continuous at some fixed point y. A similar argument can be given to show that g is right continuous at y, and of course, g is continuous at y if and only if g is both right and left continuous at y. So, let $\{y_n\}_{n=1}^{\infty}$ be a sequence of elements of X converging to y such that $y_1 \leqslant y_2 \leqslant y_3 \leqslant \cdots$ and suppose $\lim_{n \to \infty} g(y_n) = l < g(y)$. Notice that for each n, $y_n \prec y$ and the sequence $\{y_n\}_{n=1}^{\infty}$ must have a strictly increasing subsequence. From this fact, one can construct a sequence $\{x_{n_i}\}_{i=1}^{\infty}$ of elements of D which is strictly increasing such that $\{x_{n_i}\}$ converges to y and $\lim_{i \to \infty} g(x_{n_i}) = \lim_{i \to \infty} h(x_{n_i}) = l$. Now, if there is some element x_j of D such that $x_j \sim y$, let $x_{m_i} = x_j$, for $i = 1, 2, 3, \ldots$. Otherwise, let $\{x_{m_i}\}_{i=1}^{\infty}$ be a decreasing sequence of elements of D such that $g(y) = \lim_{i \to \infty} h(x_{m_i})$. The sequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$ satisfy the hypotheses of (2) in Theorem 2.1, but violate the conclusion of (2). This contradiction demonstrates that g is left continuous at y. Q.E.D.

3. Parametrized constructions. In this section, we shall parametrize the constructions of §2. We shall assume the fundamental hypothesis. In addition, let $\{V_n\}_{n=1}^{\infty}$ be a base for the topology of X and for each k, let

$$R_k = \{(t, y): y \in V_k \cap B_t \text{ and } [x \in V_k \cap B_t \to y \leqslant_t x]\},$$

and let
$$L_k = \{(t, y): y \in V_k \cap B_t \text{ and } [x \in V_k \cap B_t \to x \leqslant_t y]\}.$$

Let $\mathcal{C}$ be the family of all subsets of $T \times X$ which can be obtained from operation (A) applied to sets of the form $C \times D$, where $C \in \mathcal{S}(T)$ and $D \in \mathcal{B}(X)$. Here $S(Z)$ denotes the C-sets of Selivanovskii, the smallest family of subsets of Z containing the open sets which is closed under complementation and operation (A) [1].

LEMMA 3.1. *Each of the sets R_k and L_k is in $\mathcal{C}$.*

PROOF. We have
$$(t, y)R_k \leftrightarrow (t, y) \in (T \times V_k) \cap B$$

and (if $V_j \subseteq V_k$ and $V_j \cap B_t \neq \varnothing$, then $\exists z[z \in V_j \cap B_t$ and $y \leqslant_t z]$). Thus,

(3.1) $R_k = B \cap (T \times V_k)$
$$\cap \bigcap_{V_j \subseteq V_k} \left[\left(\left(T - \pi_{12}((T \times V_j) \cap B) \right) \times X \right) \cup \pi_{12}\left(E \cap (T \times X \times V_j) \right) \right].$$

Q.E.D.

LEMMA 3.2. *If $W \in \mathcal{C}$, then there is an $\mathcal{S}$-measurable map f from T into X such that if $W_t \neq \varnothing$, then $f(t) \in W_t$.*

A proof of this theorem has been given by Dellacherie [6, p. 217]. This theorem seems to have been known to a number of people.

The next theorem is a parametrized version of Theorem 2.1.

THEOREM 3.3. *Assume the fundamental hypothesis. There is a sequence $\{f_n\}_{n=1}^{\infty}$ of maps of T into X such that (1) for each n, f_n is $\mathcal{S}(T)$-measurable, (2) for each t, $D_t = \{f_n(t): n \in N\}$ is a dense subset of B_t and (3) if $R_{kt} \neq \varnothing$, then $D_t \cap R_{kt} \neq \varnothing$ and if $L_{kt} \neq \varnothing$, then $D_t \cap L_{kt} \neq \varnothing$.*

PROOF. Since B is an analytic subset of $T \times X$, it follows from the von Neumann-Yankov theorem (see [15]) that there is a sequence $\{g_n\}_{n=1}^{\infty}$ of maps of T into X each of which is $\mathcal{B}\mathcal{C}(T)$-measurable and such that for each t, $\{g_n(t): n \in N\}$ is dense in B_t.

For each k, let h_k be an $\mathcal{S}(T)$-measurable map of T into X such that if $R_{kt} \neq \varnothing$, then $h_k(t) \in R_{kt}$. Let $H_k = \{t: (t, h_k(t)) \in R_k\}$. Then $H_k \in \mathcal{S}(T)$. Let $\hat{h}_k(t) = h_k(t)$, if $t \in H_k$ and let $\hat{h}_k(t) = g_1(t)$, if $t \notin H_k$. Then $\hat{h}_k$ is an $\mathcal{S}(T)$-measurable map of T into X and for each t, $\hat{h}_k(t) \in B_t$ and if $R_{kt} \neq \varnothing$, $\hat{h}_k(t) \in R_{kt}$. For each k, one can construct a similar map p_k for L_k. Finally, arranging the sequences $\{g_k\}$, $\{\hat{h}_k\}$, and $\{p_k\}$ into one sequence $\{f_k\}_{n=1}^{\infty}$, we obtain the conclusion of the theorem.

THEOREM 3.4. *Assume the fundamental hypothesis. Then there is a map g of B into R which is $\mathcal{S}(T) \otimes \mathcal{B}(X)$-measurable and such that for each t, $g(t, \cdot)$ is a continuous representation of $\leqslant_t$.*

PROOF. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{S}(T)$-measurable maps satisfying the conclusion of Theorem 3.3. Let $D = \bigcup \{\text{Gr}(f_n): n = 1, 2, 3, \ldots\}$ and define h on D

by the following procedure. First, let $h(t, f_1(t)) = \frac{1}{2}$, for all t in T. Certainly, h is $\mathbb{S}(T) \otimes \mathcal{B}(X)$-measurable since $h^{-1}(E)$ is either empty or is $\mathrm{Gr}(f_1)$ which is in $\mathbb{S}(T) \otimes \mathcal{B}(X)$. Next, let

$$T_1 = \{t: f_2(t) \prec_t f_1(t)\},$$
$$T_2 = \{t: f_1(t) \sim_t f_2(t)\},$$

and

$$T_3 = \{t: f_1(t) \prec_t f_2(t)\}.$$

Each of the sets T_1, T_2, and T_3 is in $\mathbb{S}(T)$. Now, extend the domain of h to include $\mathrm{Gr}(f_2)$ by setting

$$h(t, f_2(t)) = \begin{cases} \frac{1}{4}, & \text{if } t \in T_1, \\ \frac{1}{2}, & \text{if } t \in T_2, \\ \frac{1}{3}, & \text{if } t \in T_3. \end{cases}$$

Now, continue this process by induction as in Theorem 2.1. It can be checked that the map h is $\mathbb{S}(T) \otimes \mathcal{B}(X)$-measurable. Finally, define g on B by setting

$$g(t, x) = \inf\{h((t, f_n(t))): x \preccurlyeq_t f_n(t)\}.$$

Since $g(t, \cdot)$ is the parametrized version of the function constructed in Theorem 2.2, $g(t, \cdot)$ is a continuous representation of $\preccurlyeq_t$ for each t in T.

In order to see that g is $\mathbb{S}(T) \otimes \mathcal{B}(X)$-measurable, fix a and let $V = g^{-1}((-\infty, a))$. Note that

$$V = \{(t, x): \exists n \, [x \preccurlyeq_t f_n(t) \text{ and } h(t, f_n(t)) < a]\}.$$

Thus,

$$V = \bigcup_n \left[\{(t, x): x \preccurlyeq_t f_n(t)\} \cap \{t: h(t, f_n(t)) < a\} \times X\right].$$

For each n, let $F_n(t, x) = (t, x, f_n(t))$. Then F_n is an $\mathbb{S}(T) \otimes \mathcal{B}(X)$-measurable map of B into $T \times X \times X$. So,

$$\{(t, x): x \preccurlyeq_t f_n(t)\} = F_n^{-1}(E)$$

and is therefore in $\mathbb{S}(T) \otimes \mathcal{B}(X)$. Also, for each n, the map $t \mapsto (t, f_n(t))$ is an $\mathbb{S}(T)$-measurable map of T into D and since $t \mapsto h(t, f_n(t))$ is the composition of two $\mathbb{S}$-measurable maps, it is $\mathbb{S}(T)$-measurable [**14**; **4**, p. 403]. This implies $\{t: h(t, f_n(t)) < a\}$ is in $\mathbb{S}(T)$. Therefore, g is $\mathbb{S}(T) \otimes \mathcal{B}(X)$-measurable.   Q.E.D.

4. Applications and Borel measurable selections of continuous representations. In this section we apply the techniques of §3 to obtain Wesley's theorem and to obtain some sufficient conditions for the existence of a Borel measurable selection of continuous representations. First, we give an improvement of Wesley's theorem [**16**].

THEOREM 4.1. *Assume the fundamental hypothesis. There is a map g of B into R which is $\mathbb{S}(T) \otimes \mathcal{B}(X)$-measurable such that for each t, $g(t, \cdot)$ is a continuous representation of $\preccurlyeq_t$ and such that if μ is a probability measure defined on the Borel subsets of T, then there is a Borel set N such that $\mu(N) = 0$ and g restricted to $B \cap [(T - N) \times X]$ is Borel measurable.*

PROOF. Let $\{f_n\}_{n=1}^\infty$ be a sequence of $\mathbb{S}(T)$-measurable maps as constructed in Theorem 3.3. Since these maps are universally measurable there is a Borel subset N of T with $\mu(N) = 0$ such that for each n, $f_n | T'$ is Borel measurable where $T' = T - N$. Now, note that our constructions of h and g in Theorems 3.4 from the maps f_n will be Borel measurable maps.   Q.E.D.

In fact, the preceding argument can be used (with $N = \varnothing$) to prove the following theorem.

THEOREM 4.2. *Assume the fundamental hypothesis. Suppose there is a sequence $\{f_n\}_{n=1}^\infty$ of Borel measurable maps of T into X such that conditions (2) and (3) of Theorem 3.3 hold. Then there is a Borel measurable map g of B into $[0, 1]$ such that for each t, $g(t, \cdot)$ is a continuous representation of $\leqslant_t$.*

THEOREM 4.3. *Assume the fundamental hypothesis. Furthermore assume that for each t, B_t is σ-compact. Then there is a Borel measurable g from B into $[0, 1]$ such that for each t, $g(t, \cdot)$ is a continuous representation of $\leqslant_t$.*

PROOF. First, notice that there is a sequence $\{d_n: n \in N\}$ of Borel measurable maps of T into X such that for each t, $\{d_n(t): n \in N\}$ is a dense subset of B_t [13]. Next, notice that for each k, R_{kt} is closed with respect to B_t. So, each R_{kt} is σ-compact. Also, each set R_k is a Borel set. This follows from the expression for R_k given in (3.1) and the facts that if V is open in X, then $(T \times X) \cap B$ is a Borel set with each T-section σ-compact and $W = E \cap (T \times X \times V)$ is a Borel set in $T \times X \times X$ such that for each $(t, x) \in T \times X$, $W_{(t,x)} = \{z: (t, x, z) \in W\}$ is σ-compact. Thus, the projections of these sets are Borel sets [8] and therefore R_k is a Borel set. Thus, for each k, there is a Borel measurable map h_k of T into X such that for each t, $h_k(t) \in B_t$ and if $R_{kt} \neq \varnothing$, then $h_k(t) \in R_{kt}$. Similarly, one can construct Borel maps for the sets L_k. Thus, there is a sequence $\{f_n\}_{n=1}^\infty$ of Borel measurable maps of T into X satisfying properties (2) and (3) of Theorem 3.3. The corollary now follows. Q.E.D.

COROLLARY 4.4. *Assume the fundamental hypothesis. Furthermore, assume that for each t, $\leqslant_t$ is the linear order and assume that there is a sequence $\{g_n\}_{n=1}^\infty$ of Borel measurable maps of T into X such that for each t, $\{g_n(t): n \in N\}$ is a dense subset of B_t. Then there is a Borel measurable map g of B into $[0, 1]$ such that for each t, $g(t, \cdot)$ is a continuous representation of $\leqslant_t$.*

PROOF. All that needs to be shown is that each of sets R_k (L_k) is a Borel set, since for each t, R_{kt} (L_{kt}) is σ-compact being either empty or consisting of a single point.
But,

$$(t, y) \in R_k \leftrightarrow y \in V_k \cap B_t \quad \text{and} \quad \forall n\,[\,g_n(t) \leqslant_t y \text{ or } g_n(t) \notin V_k\,].$$

This means each set R_k (and similarly each L_k) is a Borel set.   Q.E.D.

In closing we would like to raise two questions.

Question 1. Assume the fundamental hypothesis. Is there a map $f: B \to R$ which is $\mathcal{B}\mathcal{C}(T) \otimes \mathcal{B}(X)$-measurable and for each t, $f(t, \cdot)$ is a continuous representation of $\leqslant_t$?

Question 2. Assume the fundamental hypothesis and assume that for each t, $\preccurlyeq_t$ is a pre-wellordering of B_t. Is there a Borel measurable map f: $B \to R$ such that for each t, $f(t, \cdot)$ is a continuous representation of $\preccurlyeq_t$?

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DEPARTMENT OF MATHEMATICS, NORTH TEXAS STATE UNIVERSITY, DENTON, TEXAS 76203