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A SPHERE THEOREM FOR MANIFOLDS OF POSITIVE RICCI CURVATURE

BY

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Dedicated to Professor S. Sasaki on his 70th birthday

ABSTRACT. Instead of injectivity radius, the contractibility radius is estimated for a class of complete manifolds such that $\mathrm{Ric}_M \geqslant 1$, $K_M \geqslant -\kappa^2$ and the volume of $M \geqslant$ the volume of the $(\pi - \varepsilon)$-ball on the unit m-sphere, $m = \dim M$. Then for a suitable choice of $\varepsilon = \varepsilon(m, \kappa)$ every M belonging to this class is homeomorphic to S^m.

1. Introduction. An important problem in Riemannian geometry is to investigate relations between curvature and topology of Riemannian manifolds. A classical and beautiful theorem due to Myers [**14**] states that if the Ricci curvature Ric_M of a complete Riemannian manifold M satisfies $\mathrm{Ric}_M \geqslant 1$, then the diameter $d(M)$ of M is not greater than π and hence M is compact and the fundamental group $\pi_1(M)$ of M is finite. After the pioneering work of Rauch [**15**], the so-called rigidity theorem was proved by Berger [**1**] for even dimensional complete simply connected M with the sectional curvature $\frac{1}{4} \leqslant K_M \leqslant 1$, and the so-called sphere theorem was proved by Klingenberg [**13**] which states that a complete simply connected M with $\frac{1}{4} < K_M \leqslant 1$ is homeomorphic to a sphere. Recently the sphere theorem has been generalized by Grove and the author in [**12**].

However very little has been known for the topology of complete manifolds of positive Ricci curvature. A splitting theorem due to Cheeger and Gromoll [**6**] states that if M is complete noncompact with nonnegative Ricci curvature and if M admits a straight line, then M is isometric to the Riemannian product $M' \times R$. Recently Schoen and Yau has proved in [**17**] that a 3 dimensional complete noncompact M with positive Ricci curvature is diffeomorphic to R^3. Making use of the first eigenvalue for the Laplacian operator, Cheng [**7**] has proved that if M is complete and if $\mathrm{Ric}_M \geqslant 1$ and if $d(M) = \pi$, then M is isometric to the standard unit sphere.

Since Cheng's method is not useful for perturbation of metrics, it was expected to obtain an elementary and geometric approach for the proof of the above theorem. An elementary proof will make it possible to relax assumptions to replace isometry by homeomorphism or possibly by diffeomorphism.

The purpose of the present paper is to give first of all an elementary proof of the maximal diameter theorem. This is done by using the monotone property of the volume rate between concentric metric balls on M and on the complete simply

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connected space form of constant sectional curvature which is equal to the infimum of Ric_M. This property was first obtained by Bishop (see [**2**, 11.10]) for balls inside cut locus of their center and has recently been proved by Gromov (without giving details in [**11**]) beyond the cut locus of the center. Gromov used this property to obtain a uniform upper bound for the sum of Betti numbers for certain classes of complete Riemannian manifolds [**11**]. In the next place, an estimate of the radii of contractible metric balls on M is obtained under certain conditions for Ricci and sectional curvatures on M. We then prove the

MAIN THEOREM. *Let m be a positive integer and let $\kappa > 0$ be a constant. Then there exists an $\varepsilon(m, \kappa) > 0$ such that if M is an m-dimensional complete manifold whose Ricci and sectional curvatures satisfy $\mathrm{Ric}_M \geqslant 1$, $K_M \geqslant -\kappa^2$, and if the volume $v(M)$ of M satisfies $v(M) \geqslant \eta(m, \pi - \varepsilon(m, \kappa))$, then M is homeomorphic to S^m, where $\eta(m, r)$ is the volume of the r-ball on $S^m(1)$.*

The author does not know whether the assumption for the sectional curvature is essentially needed. It is a rather technical one from which the radius of contractible metric balls (instead of injectivity radius of exponential map) is estimated.

Note also that $v(M) \geqslant \eta(m, \pi - \varepsilon)$ and $\mathrm{Ric}_M \geqslant 1$ imply that the diameter $d(M)$ of M takes value in $\pi - \varepsilon \leqslant d(M) \leqslant \pi$. However it is not certain whether the assumptions $\mathrm{Ric}_M \geqslant 1$ and $d(M) \geqslant \pi - \varepsilon$ will give a lower bound for the volume of M.

The proof of our main theorem is based on the generalized Schoenflies theorem due to Brown [**3**]. Namely, if M is covered by two open disks, then M is homeomorphic to S^m (for details see Theorem 1.8.4 on p. 49, [**16**]). The proof is achieved by covering M by two contractible metric balls. Thus we prove

THEOREM 3.4. *Let m be a positive integer and let $\kappa > 0$ and $\varepsilon \in (0, \pi/3)$ be given constants. Then there exists for a fixed number $\delta \in [\varepsilon, \pi/3)$ a constant $c_\delta(m, \kappa, \varepsilon) > 0$ such that if M is an m-dimensional complete manifold whose Ricci and sectional curvatures and volume satisfy $\mathrm{Ric}_M \geqslant 1$, $K_M \geqslant -\kappa^2$, and $v(M) \geqslant \eta(m, \pi - \varepsilon)$, then every point x on M has a contractible metric ball $B_r(x)$ around it with $r \geqslant c_\delta(m, \kappa, \varepsilon)$.*

The rest of the paper is organized as follows. In §2 an elementary proof of the Cheng maximal diameter theorem is given by using the basic lemma by Gromov [**11**]. With the aid of the Toponogov theorem, Theorem 3.4 is proved in §3. Finally the proof of our main theorem is stated in §4.

An interesting problem is if a finiteness of homotopy types of the class of complete manifolds whose curvatures and volume fulfill $\mathrm{Ric}_M \geqslant 1$, $K_M \geqslant -\kappa^2$ and $v(M) \geqslant \eta(m, \pi - \varepsilon)$ can be proved, where m, $\kappa > 0$ and $\varepsilon \in (0, \pi/2)$ are constants. If M satisfies the above conditions, then M is covered by at most N contractible metric balls, where N depends only on m, κ and ε. The problem reduces to the following: Let M_1 and M_2 be m-dimensional manifolds each M_i of which is covered by N' $(\leqslant N)$ topological disks $B_1^i, \ldots, B_{N'}^i$ which are all metric balls of the same radius such that the number of components $B_{j_1}^1 \cap \cdots \cap B_{j_k}^1$ is equal to that of $B_{j_1}^2 \cap \cdots \cap B_{j_k}^2$ for all $j_1, \ldots, j_k = 1, \ldots, N'$. Then are they homotopy equivalent to each other? In the general case where they are not metric balls but topological disks, the answer is negative, as stated at the end of §4.

The author would like to express his thanks to T. Kaneto who let him know such a simple example.

2. An elementary proof of the Cheng theorem. Let M be a complete Riemannian manifold of dimension m and let $\mathrm{Ric}_M \geqslant -\kappa^2$, where κ is a real or a pure imaginary number. Let $M(-\kappa^2)$ be the complete simply connected m-dimensional space form of constant sectional curvature $-\kappa^2$. For a point $x \in M$ and for an $r > 0$ let $B_r(x)$ be the metric r-ball centered at x. A metric r-ball in $M(-\kappa^2)$ is denoted by $\tilde{B}_r$. With these notations the basic lemma due to Gromov is stated as

LEMMA (GROMOV [**11**]). *For any fixed $x \in M$, let f: $[0, \infty) \to R$ be defined as*

$$f(r) := v(B_r(x))/v(\tilde{B}_r),$$

where $v(A)$ is by definition the volume of a set A. Then f is monotone nonincreasing.

Making use of this lemma, we shall establish an elementary proof of the maximal diameter theorem due to Cheng [**7**], which is a generalization of the Toponogov theorem for sectional curvature [**18**].

THEOREM (S. Y. CHENG [**7**]). *Let M be a complete Riemannian m-manifold whose Ricci curvature is bounded below by* 1. *If the diameter of M is equal to π, then M is isometric to $S^m(1)$.*

PROOF. Take a pair of points p, q at maximal distance, $d(p, q) = d(M) = \pi$, where d is the distance function induced from the Riemannian metric. Define the volume rate function f_p, f_q: $[0, \pi] \to R$ by $f_p(r) = v(B_r(p))/v(\tilde{B}_r)$ and $f_q(r) = v(B_r(q))/v(\tilde{B}_r)$. It follows from assumption that $\bar{B}_{\pi/2}(p) \cup \bar{B}_{\pi/2}(q) \subset M$ and $B_{\pi/2}(p) \cap B_{\pi/2}(q) = \varnothing$,where $\bar{A}$ is by definition the closure of a set $A \subset M$. Thus $v(M) \geqslant v(\bar{B}_{\pi/2}(p)) + v(\bar{B}_{\pi/2}(q))$.

On the other hand since the basic lemma implies that f_p and f_q are monotone nonincreasing, $f_p(\pi/2) = v(B_{\pi/2}(p))/(c_m/2) \geqslant f_p(\pi) = v(M)/c_m$. Hence $v(B_{\pi/2}(p)) \geqslant v(M)/2$, and similarly $v(B_{\pi/2}(q)) \geqslant v(M)/2$. This fact implies that $f_p(\pi/2) = f_p(\pi) = f_q(\pi/2) = f_q(\pi)$. It follows from the monotone property of f_p that $f_p(r) = f_p(\pi)$ for all $r \in [\pi/2, \pi]$. From $v(M) \geqslant v(B_r(p)) + v(B_{\pi-r}(q))$ for $r \in [\pi/2, \pi]$ it follows that $f_q(\pi - r) = v(B_{\pi-r}(q))/v(\tilde{B}_{\pi-r}) \geqslant f_q(\pi/2) = f_p(\pi)$. Therefore $v(M) \geqslant v(B_r(p)) + v(B_{\pi-r}(q)) \geqslant f_p(\pi) \cdot c_m = v(M)$, and $f_q(\pi - r) = f_q(\pi/2)$ for any $r \in [\pi/2, \pi]$. Since $f_p(0) = f_q(0) = 1$, this fact means that $f_p = f_q = 1$, and hence $C_p = C_q =$ the sphere of radius π, where $C_p \subset M_p$ is by definition the tangent cut locus of p.

If γ: $[0, \pi] \to M$ is a geodesic with $\gamma(0) = p$, $\|\dot{\gamma}\| = 1$, then $\gamma(\pi) \in C(p)$ and from $\mathrm{Ric}_M \geqslant 1$, $\gamma(\pi)$ is conjugate to p along γ. It follows from the index comparison theorem (see [**10**, p. 178]) together with $\mathrm{Ind}_{\gamma|[0,\pi)} \geqslant 0$ that if E is a unit parallel field along γ with $\langle E, \dot{\gamma}\rangle = 0$, then $\sin tE(t)$ is a Jacobi field along γ. Thus the multiplicity of the first conjugate point $\gamma(\pi)$ to p along γ is $m - 1$. This fact implies that $C(p) = \{q\}$. Because the sectional curvature determined by the plane section

$(\dot\gamma(t), E(t))$ is 1, M is isometric to the standard unit sphere by the composed exponential mappings.

3. Estimate of contractibility radius. Let N be a complete manifold of dimension n. For a fixed point $x \in N$ consider the distance function $d_x\colon N \to R$, $d_x(y) = d(x, y)$. A point $y \in N$ is by definition a *noncritical* point of d_x if the set of all unit vectors tangent to minimizing geodesics from y to x is contained in an open half space of N_x. A point $y \in N$ is by definition a *critical* point of d_x if for any nonzero tangent vector $u \in N_x$ there is a minimizing geodesic from y to x whose tangent vector at y makes an angle with u not greater than $\pi/2$. Obviously a critical point y of d_x belongs to the cut locus $C(x)$ of x.

If $\bar B_r(x)$ contains no critical point of d_x except at the origin x of the ball, then $d_x \mid \bar B_r(x)$ can be approximated by smooth functions which have no critical point except at the origin x. In fact, let $2r_0$ be the radius of a strongly convex ball around x. The smooth approximations of d_x are obtained as follows: Let $\Phi\colon R \to R$ be a smooth function whose support is in $[-1, 1]$ such that it is constant 1 around 0 and such that $\int_{v \in R^n} \Phi(\|v\|) = 1$. For a sufficiently small positive ρ, define $(d_x)_\rho\colon N \to R$ by

$$(d_x)_\rho(q) := \rho^{-n} \int_{v \in N_q} d_x(\exp_q v)\Phi(\|v\|/\rho)\, dv,$$

where dv is the Riemannian volume on N_q. It has been proved in **[8]** that $(d_x)_\rho$ is smooth and $\{(d_x)_\rho\}_\rho$ converge uniformly to d_x as $\rho \to 0$. Moreover, it has been proved in **[12]** that $(d_x)_\rho \mid \bar B_r(x) - \bar B_{r_0}(x)$ has nonvanishing gradient and the gradient vectors are transversal to $\partial B_{r_0}(x)$. And hence $\bar B_r(x)$ is contractible to x.

The contractibility radius $c(x)$ at $x \in N$ is defined as

$$c(x) := \sup\{r;\ \bar B_r(x) \text{ is contractible to } x\}.$$

LEMMA 3.1. *Let N be a complete manifold. The contractibility radius function c: $N \to R$ has the following properties*: (1) *For every $x \in N$ $c(x)$ is not less than the positive minimum critical value $=: c_1(x)$ of d_x*; (2) *The positive minimum critical value is lower semicontinuous on N.*

PROOF. If $\bar B_r(x)$ contains no critical point of d_x, then the above argument shows that $c(x) \geq r$. This proves (1). Let $\{x_i\}$ be a sequence of points on N which tends to $x \in N$, and for each i let y_i be a critical point of d_{x_i} with $d(x_i, y_i) = c_1(x_i)$. If $\{y_i\}$ tends to $y \in N$, then y is a critical point of d_x. Therefore $\liminf c_1(x_i) \geq c_1(x)$.

It follows from Lemma 3.1 that a lower bound of the contractibility radius on N is obtained by the infimum of the positive minimum critical values of distance functions.

Now let M be a complete manifold of dimension m whose Ricci curvature and volume satisfy $\mathrm{Ric}_M \geq 1$, $v(M) \geq c_m - \eta(m, \varepsilon)$. The first observation to give an estimate of contractibility radius for a certain class of manifolds with the above properties is this: Under the assumptions for Ricci curvature and volume of M as stated above, each point $x \in M$ has the property that

$$l(x) := \max\{d(x, y);\ y \in M\} \geq \pi - \varepsilon.$$

Indeed, it follows from the basic lemma together with $\bar{B}_{l(x)}(x) = M$ that $v(\tilde{B}_{l(x)}) \geqslant v(M) \geqslant c_m - \eta(m, \varepsilon)$. Since $v(\tilde{B}_{l(x)}) = \eta(m, l(x))$, $c_m - \eta(m, \varepsilon) = \eta(m, \pi - \varepsilon)$ and since $\eta(m, \varepsilon)$ is strictly increasing with ε, $l(x) \geqslant \pi - \varepsilon$.

As a straightforward consequence of the basic lemma, we have

LEMMA 3.2. *If M is a complete manifold of dimension m whose Ricci curvature and volume satisfy* $\mathrm{Ric}_M \geqslant 1$, $v(M) > c_m/2$, *then M is simply connected.*

The next observation is to see what will happen when $c(x)$ is small roughly speaking. Since a critical point of d_x belongs to $C(x)$, it will turn out that the appearance of such a critical point will give less contribution to the volume of concentric balls around x which contain critical points of d_x. But if $v(M)$ is bounded below, then the total volume of the compact set in M_x which is star-shaped with respect to the origin and whose boundary consists of C_x is also bounded below. This phenomenon will be interpreted as follows.

PROPOSITION 3.3. *Let $\varepsilon \in (0, \pi)$ be a given constant. Assume that* $\mathrm{Ric}_M \geqslant 1$ *and* $v(M) \geqslant c_m - \eta(m, \varepsilon)$. *For every point $x \in M$ and a number $\theta \in (0, \pi)$ and for every $v \in S_x$ let $V(v; \theta) := \{w \in M_x;\ \sphericalangle(v, w) \leqslant \theta\}$. Then there exists a positive smooth function $r \to \theta(r, m, \varepsilon)$, $0 < r < \pi - \varepsilon$, such that if every $w \in V(v; \theta) \cap C_x$ has norm $\|w\| \leqslant r$, then $\theta \leqslant \theta(r, m, \varepsilon)$. $\theta(r, m, \varepsilon)$ is obtained as the solution of*

$$c_{m-2}\int_0^{\theta(r,m,\varepsilon)} \sin^{m-2} t\, dt = \eta(m, \varepsilon)\Big/\int_r^{\pi} \sin^{m-1} t\, dt.$$

PROOF. As is seen in the first observation, $l(x) \geqslant \pi - \varepsilon$ holds for every $x \in M$. The area of $V(v; \theta) \cap S_x$ is $\eta(m-1, \theta)$. Thus an upper bound for $v(M)$ is given by

$$\eta(m-1, \theta)\int_0^r \sin^{m-1} t\, dt + \eta(m-1, \pi - \theta)\int_0^{\pi} \sin^{m-1} t\, dt \geqslant v(M).$$

Since $v(M) \geqslant c_m - \eta(m, \varepsilon)$, the above inequality reduces to

$$\eta(m-1, \theta)\int_r^{\pi} \sin^{m-1} t\, dt \leqslant \eta(m, \varepsilon).$$

The desired $\theta(r, m, \varepsilon)$ is obtained by solving

$$c_{m-2}\int_0^{\theta} \sin^{m-2} t\, dt \int_r^{\pi} \sin^{m-1} t\, dt = \eta(m, \varepsilon).$$

The function $\theta(r, m, \varepsilon)$ has the following properties: (1) For every fixed $r \in (0, \pi - \varepsilon)$ $\lim_{\varepsilon \downarrow 0} \theta(r, m, \varepsilon) = 0$ and $\varepsilon \to \theta(r, m, \varepsilon)$ is monotone increasing. (2) For every fixed $\varepsilon \in (0, \pi)$ $0 < \lim_{r \downarrow 0} \theta(r, m, \varepsilon) < \eta^{-1}[\eta(m, \varepsilon)/\int_0^{\pi} \sin^{m-1} t\, dt]$.

Let $\delta \in [\varepsilon, \pi/3)$ be a fixed number and set $r_1 = \pi - 3\delta$. It follows from Proposition 3.3 together with the continuity of the map $u \in S_x \to$ the distance from x to the cut point of x along the goedesic $t \to \exp_x tu$ that for every $x \in M$ and for every $u \in S_x$ there exists a $w \in S_x$ with the properties that $\sphericalangle(u, w) \leqslant \theta(r_1, m, \varepsilon)$ and that the cut point $\exp_x t_1 w$ to x along the geodisic $t \to \exp_x tw$ appears at $t_1 \geqslant r_1$. Then the Toponogov comparison theorem will be applied to obtain a lower bound for c_1: $M \to R$. Thus an additional assumption for the sectional curvature will be needed.

THEOREM 3.4. *Let m be a positive integer and let $\kappa > 0$ and $\varepsilon \in (0, \pi/3)$ be given constants. Then there exists for a fixed number $\delta \in [\varepsilon, \pi/3)$ a constant $c_\delta(m, \kappa, \varepsilon) > 0$ such that if M is a complete manifold of dimension m whose curvatures and volume satisfy*

$$\mathrm{Ric}_M \geqslant 1, \quad K_M \geqslant -\kappa^2, \quad v(M) \geqslant \eta(m, \pi - \varepsilon),$$

then $c_1(x) \geqslant c_\delta(m, \kappa, \varepsilon)$ for every point $x \in M$. The constant is given by

$$c_\delta(m, \kappa, \varepsilon) = \min\left[\pi - 3\delta, \kappa^{-1}\tanh^{-1}\{\tanh(\pi - 3\delta)\kappa \cdot \cos\theta(\pi - 3\delta, m, \varepsilon)\}\right].$$

PROOF. Let $x \in M$ be a fixed point and let $y \in M$ be a critical point of d_x with the positive minimum critical value $r_0 = c_1(x)$. Let $u \in S_x$ be the unit vector tangent to a minimizing geodesic γ_u: $[0, r_0] \to M$ with $\gamma_u(0) = x$, $\gamma_u(r_0) = y$. The above argument shows that there is a $w \in S_x$ with the properties: $\sphericalangle(u, w) \leqslant \theta(r_1, m, \varepsilon)$ and γ_w has the cut point to x along it at $\gamma_w(t_1)$ with $t_1 \geqslant r_1$. If $r_0 \geqslant r_1$ then nothing is left to prove.

Assume that $r_0 < r_1$. The Toponogov comparison theorem implies that if $\alpha = \sphericalangle(u, w)$ and if $r_2 = d(y, z)$, where $z = \gamma_w(t_1)$, then

$$\cosh r_2\kappa \leqslant \cosh t_1\kappa \cdot \cosh r_0\kappa - \sinh t_1\kappa \cdot \sinh r_0\kappa \cdot \cos\alpha.$$

Since y is a critical point of d_x, there exists for a minimizing geodesic from y to z, a minimizing geodesic from y to x (possibly different from γ_u) whose angle at y is not greater than $\pi/2$. Thus the Toponogov theorem again implies for this triangle to get

$$\cosh t_1\kappa \leqslant \cosh r_0\kappa \cdot \cosh r_2\kappa.$$

Eliminate r_2 from the above inequalities to obtain

$$\cosh t_1\kappa \cdot \tanh r_0\kappa \geqslant \cos\alpha.$$

Insert $\alpha \leqslant \theta(r_1, m, \varepsilon)$ and $t_1 \geqslant r_1 = \pi - 3\delta$ to complete the proof.

It should be noted that for every fixed $\delta \in [\varepsilon, \pi/3)$, $\lim_{\varepsilon \downarrow 0} c_\delta(m, \kappa, \varepsilon) = \pi - 3\delta$.

4. The proof of Main Theorem and remarks. By means of the generalized Schoenflies theorem, it suffices for the proof of our main theorem to exhibit M as a union of two open disks. In fact the open disks are obtained as contractible metric balls.

PROOF OF MAIN THEOREM. It follows from Theorem 3.4 that if $r \in (0, \pi)$ is arbitrarily given, then there exists an $\varepsilon \in (0, \pi/3)$ and a $\delta \in [\varepsilon, \pi/3)$ such that $r \leqslant c_\delta(m, \kappa, \varepsilon)$.

Let $p, q \in M$ be a pair of points such that $d := d(p, q) = d(M)$. Then $\pi - \varepsilon \leqslant d \leqslant \pi$. The minimal radius R of closed balls around p and q by which M is covered satisfies $d/2 \leqslant R \leqslant d$ and $R = \max\{d(p, y);\, y \in M, d(p, y) = d(q, y)\}$. It follows from the basic lemma that if x is a point with $d(p, x) = d(q, x) = R$, then

$$\begin{aligned} v(M) &\geqslant v\big(B_{d/2}(p)\big) + v\big(B_{d/2}(q)\big) + v\big(B_{R-d/2}(x)\big) \\ &\geqslant \big(v(M)/c_m\big) \times \left[2v\big(\tilde{B}_{d/2}\big) + v\big(\tilde{B}_{R-d/2}\big)\right]. \end{aligned}$$

Thus

$$c_m \geqslant c_{m-1}\left[2\int_0^{(\pi-\varepsilon)/2} \sin^{m-1} t\, dt + \int_0^{R-\pi/2} \sin^{m-1} t\, dt\right].$$

Let $R_0 = R(m, \varepsilon)$ be the solution of

$$\int_{(\pi-\varepsilon)/2}^{(\pi+\varepsilon)/2} \sin^{m-1} t\, dt = \int_0^{R_0 - \pi/2} \sin^{m-1} t\, dt.$$

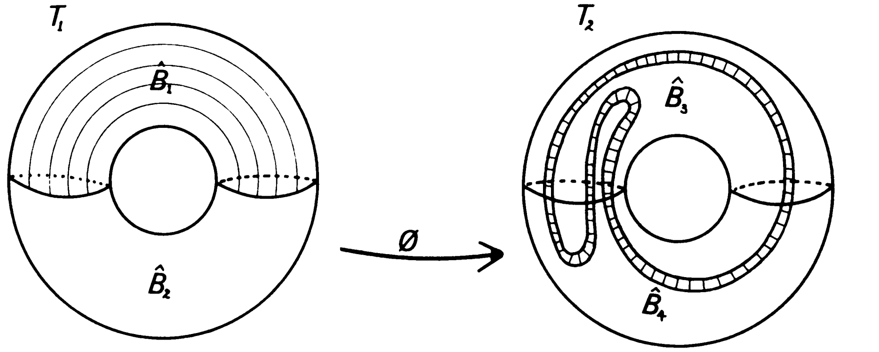
Then $R_0 \geqslant R$ and $\lim_{\varepsilon \downarrow 0} R(m, \varepsilon) = \pi/2$ and $\varepsilon \to R(m, \varepsilon)$ is strictly monotone increasing. The desired $\varepsilon = \varepsilon(m, \kappa)$ is obtained as follows. Let $\delta \in (0, \pi/6)$ be a fixed number and let $\varepsilon \in (0,\delta]$. Then $\lim_{\varepsilon \downarrow 0} c_\delta(m, \kappa, \varepsilon) = \pi - 3\delta > \pi/2 = \lim_{\varepsilon \downarrow 0} R(m, \varepsilon)$. Because $c_\delta(m, \kappa, \varepsilon)$ is monotone decreasing with $\varepsilon \in (0,\delta]$ and because $R(m, \varepsilon)$ is monotone increasing with ε, there exists a unique $\varepsilon(m, \kappa)$ such that if $\varepsilon \in (0, \varepsilon(m, \kappa))$ then $R(m, \varepsilon) < c_\delta(m, \kappa, \varepsilon)$. If $r := (R(m, \varepsilon) + c_\delta(m, \kappa, \varepsilon))/2$, then $r < c_\delta(m, \kappa, \varepsilon)$ implies that $B_r(p)$ and $B_r(q)$ are open disks and $r > R(m, \varepsilon)$ implies that they cover M.

REMARK 1. If we apply Gromov's technique developed in [**11**] to our case, then the sum of Betti numbers is bounded as follows. For given constants m, $\kappa > 0$ and $\varepsilon \in (0, \pi/3)$ consider the class of all complete manifolds of dimension m satisfying $\mathrm{Ric}_M \geqslant 1$, $K_M \geqslant -\kappa^2$, $v(M) \geqslant c_m - \eta(m, \varepsilon)$. Let $2c := c_\varepsilon(m, \kappa, \varepsilon)$. Then each M in the class has the property that every point x on M has a contractible metric ball of radius at least $2c$. The minimal covering argument (see [**4, 11** and **19**]) implies that M is covered by at most $N(m, \kappa, \varepsilon) := [c_m/v(B_c)] + 1$ contractible balls. Since every contractible ball has content 1, $\mathrm{cont}(M) = \Sigma^m b_i(M; F) \leqslant (m + 1)2^{N(m,\kappa,\varepsilon)}$ follows from the topological lemma in [**11**].

REMARK 2. In the proof of finiteness theorems due to Weinstein [**19**] and Cheeger [**4**], it is essential to find a uniform positive lower bound for convexity radius of a certain class of complete manifolds. The assumption for an upper bound of sectional curvature of the class plays an important role to obtain such a uniform positive lower bound for convexity radius. However in our case it is not easy to find such a lower bound because there is no assumption for the upper bound of sectional curvature. It is difficult to see when a contractible metric ball becomes a convex ball. Also it is hard to control the intersections of contractible metric balls. And this point makes it difficult to prove finiteness of homotopy types of the class.

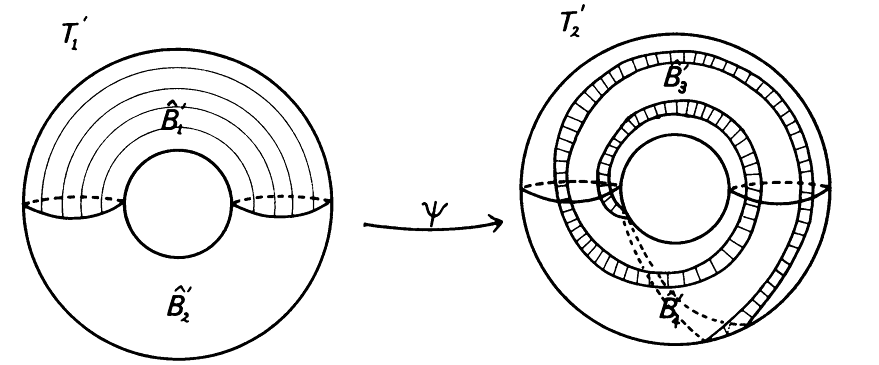
Let us assume that two manifolds M and M' of the same dimension have the same number of *topological* disks $B_1, \ldots, B_N \subset M$ and $B'_1, \ldots, B'_N \subset M'$ such that they satisfy: (1) $\bigcup B_i = M$ and $\bigcup B'_i M'$, (2) for every $1 \leqslant i_1, \ldots, i_k \leqslant N$ the number of components of the intersection $B_{i_1} \cap B_{i_2} \cap \cdots \cap B_{i_k}$ is equal to that of the intersection $B'_{i_1} \cap B'_{i_2} \cap \cdots \cap B'_{i_k}$. The question is if M and M' have the same homotopy type. There is a counterexample of 3-dimensional manifolds M and M' with the covers by the same number of disks which satisfy the above properties and the homotopy type of M is different from that of M'. Such an example was first discovered by Tsukui for the lens space L^3 (3; 1) of type (3; 1) and a connected sum $P^3 \# P^3$ of real projective spaces. Then a simpler example has been furnished by T. Kaneto for S^3 and P^3 as follows.

EXAMPLE. Let T_1 and T_2 be 3-dimensional solid tori. It follows from the Hopf fibration that S^3 is obtained by attaching ∂T_1 and ∂T_2 with an attaching map ϕ: $\partial T_1 \to \partial T_2$ as $S^3 \approx T_1 \cup_\phi T_2$. T_1 and T_2 are decomposed into $\hat{B}_1$, $\hat{B}_2$ and $\hat{B}_3$, $\hat{B}_4$ such that $\hat{B}_1 \cap \hat{B}_2$ and $\hat{B}_3 \cap \hat{B}_4$ consist of two closed 2-disks, as indicated in the figure.



For each $i = 1,\ldots,4$ let B_i be a neighborhood of B_i in S^3 which is homeomorphic to an open 3-disk. Choose an attaching map ϕ in such a way that ϕ maps ∂B_1 into ∂T_2 whose image is indicated by the shaded region and ∂B_2 onto the unshaded region of ∂T_2. Then $\{B_1, B_2, B_3, B_4\}$ forms a covering of S^3 by disks and satisfies: (1) for every $i, j = 1,\ldots,4$ with $i \neq j$, $B_i \cap B_j$ consists of two 3-cells, (2) for every $i, j, k = 1,\ldots,4$ with $i \neq j \neq k \neq i$, $B_i \cap B_j \cap B_k$ consists of four 3-cells, (3) $B_1 \cap B_2 \cap B_3 \cap B_4$ consists of eight 3-cells.

Let $B \subset R^3$ be the unit ball around the origin. P^3 is obtained by identifying the boundary $\partial B = S^2$ via the antipodal map. For a fixed number $r \in (0, 1)$ and for a fixed straight line L passing through the origin, let U be an r-tubular neighborhood of L in R^3. P^3 is then decomposed as a union of two solid tori T_1' and T_2' which are obtained by identifying $\partial B \cap U$ and $\partial B - U$ by the antipodal map. Obviously $\partial U \cap B$ is homeomorphic to T^2 after the identification. Every circle on $\partial T_2'$ which is obtained by a line segment in $\partial U \cap B$ parallel to L turns around $\partial T_2'$ twice.



T_1' and T_2' are decomposed into $\hat{B}_1'$, $\hat{B}_2'$ and $\hat{B}_3'$, $\hat{B}_4'$ respectively such that $\hat{B}_1' \cap \hat{B}_2'$ and $\hat{B}_3' \cap \hat{B}_4'$ consist of two closed 2-disks, as indicated in the figure. For each $i = 1,\ldots,4$ let B_i' be a neighborhood of $\hat{B}_i'$ in P^3 which is homeomorphic to an open 3-disk. Choose an attaching map ψ: $\partial T_1' \to \partial T_2'$ which sends $\partial B_1'$ onto the shaded region of $\partial T_2'$ and $\partial B_2'$ onto the unshaded region of $\partial T_2'$. Then $\{B_1', B_2', B_3', B_4'\}$ is a covering of P^3 by disks and satisfies: (1) for every $i, j = 1,\ldots,4$ with $i \neq j$, $B_i' \cap B_j'$ consists of two 3-cells, (2) for every $i, j, k = 1,\ldots,4$ with $i \neq j \neq k \neq i$, $B_i' \cap B_j' \cap B_k'$ consists of four 3-cells, (3) $B_1' \cap B_2' \cap B_3' \cap B_4'$ consists of eight 3-cells.

Thus S^3 and P^3 have the same type of covering by disks, however they do not have the same homotopy type.

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