A NECESSARY AND SUFFICIENT CONDITION
FOR THE ASYMPTOTIC VERSION OF
AHLFORS’ DISTORTION PROPERTY

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Abstract. Let \( f \) be a conformal map of \( R \subset \mathbb{C} \) onto \( S \subset \mathbb{C} \) where the \( \varphi_0, \varphi_1 \) are extended real valued functions defined for \( -\infty < u < +\infty \). For the sake of simplicity we shall require \( \varphi_0 \) and \( \varphi_1 \) to be continuous. Let \( \theta(u) = \varphi_1(u) - \varphi_0(u) \).

Let \( S \subset \mathbb{C} \) be the parallel strip \( S = \{ z = x + iy | 0 < y < 1 \} \). Let \( w = F(z) \) be a one-to-one conformal map of \( S \) onto \( R \) such that \( \text{Re} F(z) \to \pm \infty \) as \( \text{Re} z \to \pm \infty \), respectively. Let \( z = f(w) \) be the inverse mapping.

Inequalities of the form

\[
\begin{align*}
\int_0^u \frac{du}{\theta(u)} & \leq m \leq \text{Re} f(w) - \int_0^u \frac{du}{\theta(u)} \leq M \quad (w = u + iv)
\end{align*}
\]

were first investigated in Ahlfors [1] (the left hand inequality corresponds to his Distortion Theorem; the right hand inequality to his Second Fundamental Inequality). That work stimulated efforts to find other properties of \( R \) which imply the validity of (a).

The problem takes a simpler form when Eke’s theorem [2, Theorem 2] is applied (cf. also [8, Theorem 3]). This theorem shows that (a) holds for \( 0 < u < \infty \) if and only if the center term actually tends to a limit

\[
\begin{align*}
\text{Re} f(w) &= \int_0^u \frac{du}{\theta(u)} + \text{const.} + o(1) \quad (w = u + iv),
\end{align*}
\]

where \( o(1) \to 0 \) as \( \text{Re} w \to +\infty \).

Ahlfors’ original results, as strengthened by Jenkins-Oikawa [4], show that (A) holds if \( R \) contains the real axis and its boundary curves \( \varphi_0, \varphi_1 \) are of bounded
variation and bounded away from zero. A number of other geometric properties of
R which imply (A) have been discovered. Examples of such sufficient conditions are
the bounded $2/3$-variation condition of Jenkins-Oikawa [5], the convergence of the
integral $\int_0^\infty \varphi_j^2(1 + |\varphi'_j|)^{-1} \, du$ for $j = 0, 1$ (Lelong-Ferrand [3],
Rodin-Warschawski [10]), and the convergence of $\int_0^\infty (\varphi_0^2 + \varphi_1^2)\theta^{-1} \, du$ (Warschawski [11],
Rodin-Warschawski [8]). None of these sufficient conditions is also necessary.

In Theorem 1 below we give a complete solution to the problem of finding
geometric conditions on R which are both necessary and sufficient for property (A).

Remark. The sufficient condition of [11] referred to above can be derived directly
from Theorem 1 by taking $a(u, t) = t\varphi_1(u) + (1 - t)\varphi_0(u)$. It is not evident if the
other sufficient conditions can be derived from Theorem 1 in a direct manner.

1. A class $C^1$ real valued function $\alpha(u, t)$ defined for $-\infty < u < +\infty$, $0 < t < 1$
will be called a stratification of R if $u + ia(u, t) \in R$ for all $(u, t)$ in the domain of
$\alpha$. For simplicity we shall also require $\alpha_t > 0$.

Theorem 1. A necessary and sufficient condition for $R$ to have property (A) is that $R$
admit a stratification $\alpha(u, t)$ such that the integrals

$$
\int_0^1 \int_{\alpha'(u,t)}^{\alpha''(u,t)} \left( \frac{1}{\theta(u)} - \frac{1}{\alpha_t(u, t)} \right) \, du \, dt
$$

and

$$
\int_0^1 \int_{\alpha'(u,t)}^{\alpha''(u,t)} \frac{\alpha_t^2(u, t)}{\alpha_t(u, t)} \, du \, dt
$$

remain bounded above and below as $u', u'' \to +\infty$.

Proof. Necessity. Since the angle of inclination of any chord of a boundary
component of $R$ is less than $\pi/2$ in magnitude, it follows that $|\text{Arg } F'(z)| < \pi/2$; a
detailed proof is given in Lemma 2 of §2. Hence for each fixed $t \in (0, 1)$ the stream
line $\{w \in R \mid \text{Im } f(w) = t\}$ is the graph of a function. Denote this function by
$u \mapsto \alpha(u, t)$. We shall show that this $\alpha$ is a stratification of $R$ which satisfies the
boundedness conditions of the theorem.

By the definition of $\alpha$ we have

$$
v(x, t) = \alpha(u(x, t), t)
$$

where $F(x + it) = u(x, t) + iv(x, t)$. Take partial derivatives with respect to $x, t$ in
(3) to obtain

$$
\alpha_u(u, t) = -\frac{\text{Im } f'(w)}{\text{Re } f'(w)}, \quad \alpha_t(u, t) = \frac{1}{\text{Re } f'(w)} \quad (w = u + i\alpha(u, t)).
$$
The integral (2) for this $\alpha$ can be estimated as follows, where $u' < u''$:

$$
\int_0^1 \int_{u'}^{u''} \frac{\alpha''_t}{\alpha_t^2} \, du \, dt = \int_0^1 \int_{u'}^{u''} \frac{\operatorname{Im} f'}{R f} \, du \, dt
$$

$$
= \int_0^1 \int_{x(u', \alpha(u', t))}^{x(u'', \alpha(u'', t))} \frac{\operatorname{Im} f'}{R f} (\cos \arg F') \, |F'| \, dx \, dt
$$

$$
= \int_0^1 \int_{x(u', \alpha(u', t))}^{x(u'', \alpha(u'', t))} \sin^2 \arg F' \, dx \, dt
$$

$$
\leq \int_0^1 \int_{x_0}^{+\infty} \operatorname{Arg}^2 F' \, dx \, dt
$$

for suitable $x_0$. By Theorem 5 of Rodin-Warschawski [9] (see Lemma 1 in §2 for a self-contained proof of this fact in the present, less general, context) the last integral is finite under our hypothesis that $R$ satisfies property (A).

The integral (1) for this $\alpha$ can be transformed as follows:

$$
\int_0^1 \int_{u'}^{u''} \left( \frac{1}{\theta(u)} - \frac{1}{\alpha_t(u, t)} \right) \, du \, dt = \int_0^1 \int_{u'}^{u''} \left( \frac{1}{\theta(u)} - \frac{R f'}{\arg f'} \right) \, du \, dt
$$

$$
= \int_{u'}^{u''} \frac{du}{\theta(u)} - \int_0^1 \int_{x(u', \alpha(u', t))}^{x(u'', \alpha(u'', t))} \frac{R f'}{\arg f'} \cos \arg F' \, |F'| \, dx \, dt
$$

$$
= \int_{u'}^{u''} \frac{du}{\theta(u)} - \int_0^1 \int_{x'}^{x''} \cos^2 \arg F' \, dx \, dt
$$

$$
= \int_{u'}^{u''} \frac{du}{\theta(u)} - \int_0^1 \int_{x'}^{x''} \frac{dx}{\theta(u)} \frac{dx}{\theta(u)} - \int \int_{R(u', u'')} \frac{dx}{\theta(u)} \frac{dx}{\theta(u)}
$$

where $x' = x(u', \alpha(u', t))$ and $x'' = x(u'', \alpha(u'', t))$. As already noted, this last integral is bounded under the assumption that property (A) holds. The remaining term

$$
\int_{u'}^{u''} \frac{du}{\theta(u)} - \int_0^1 \int_{x'}^{x''} dx \, dy = \int_{u'}^{u''} \frac{du}{\theta(u)} - \int \int_{f(R(u', u''))} \frac{dx}{\theta(u)} \frac{dx}{\theta(u)}
$$

where $R(u', u'') = \{w \in R \mid u' < \Re w < u''\}$, is also bounded. Indeed, note that property (A) implies that the horizontal oscillation

$$
\omega(u) = \sup \{ \Re f(w_2) - \Re f(w_1) \mid w_1, w_2 \in R \text{ and } \Re w_1 = \Re w_2 = u \}
$$

tends to zero as $u \to +\infty$. Let $x'' - x' = \Re f(u'' - iv'') - \Re f(u' + iv')$ where $u' + iv' \in R$, $u'' + iv'' \in R$, and $u' < u''$. Then the assertion of boundedness follows from

$$
x'' - x' - \omega(u') - \omega(u'') \leq \int \int_{f(R(u', u''))} \frac{dx}{\theta(u)} \frac{dx}{\theta(u)} \leq x'' - x' + \omega(u') + \omega(u'')
$$

the consequence of property (A),

$$
x'' - x' = \int_{u'}^{u''} \frac{du}{\theta(u)} + O(1),
$$

and the fact $\omega(u) = o(1)$ mentioned above.
Sufficiency. We now assume that $a(u, t)$ is a stratification of $R$ such that the integrals (1) and (2) are bounded. For given $0 \leq u' < u''$ consider the curve family $\{\gamma(t)\}_{0 < t < 1}$ defined by $u \mapsto \gamma_t(u) = u + i\alpha(u, t)$ for $u' \leq u \leq u''$. By well-known properties of extremal length we have

$$\int_{u'}^{u''} \frac{du}{\theta(u)} \leq \lambda_R(u', u'') \leq \lambda(\{\gamma(t)\}_{0 < t < 1}),$$

where $\lambda_R(u', u'')$ is the extremal distance between the vertical sides in $\{w \in R \mid u' < \Re w < u''\}$. We shall show that $\lambda(\{\gamma(t)\}_{0 < t < 1}) \leq \int_{u''}^{u'} \theta^{-1}(u) \, du + O(1)$ where $O(1)$ is bounded for all $0 \leq u' < u''$. It will then follow that property (A) holds (see Theorem 3 of Rodin-Warschawski [8]; cf. also Eke [2, Theorem 2]).

Since $\{\gamma(t)\}_{0 < t < 1}$ is a 1-parameter curve family one can calculate its extremal length exactly (see, for example, Theorem 14 of [6]). Define a map of $\{0 < u < +\infty, 0 < t < 1\}$ into $R$, denoted $u + it \mapsto c(u, t) = u + iv$, by letting $v = \alpha(u, t)$. Then

$$J(u, t) \equiv \frac{\partial(u, v)}{\partial(u, t)} = \begin{vmatrix} 1 & 0 \\ \alpha_u & \alpha_t \end{vmatrix} = \alpha_t,$$

$$|\frac{\partial c(u, t)}{\partial u}|^2 = 1 + \alpha_u^2,$$

$$l(t) \equiv \int_{u'}^{u''} \frac{|\partial c/\partial u|^2}{J} \, du = \int_{u'}^{u''} \frac{1 + \alpha_u^2}{\alpha_t} \, du.$$

One has

$$\lambda(\{\gamma(t)\}) = \left(\int_0^1 \frac{dt}{l(t)}\right)^{-1} = \int_0^1 l(t) \, dt = \int_0^1 \int_{u'}^{u''} \left(\frac{1 + \alpha_u^2}{\alpha_t}\right) \, du \, dt$$

$$= \int_0^1 \int_{u'}^{u''} \frac{1}{\alpha_t} \, du \, dt + \int_0^1 \int_{u'}^{u''} \frac{\alpha_u^2}{\alpha_t} \, du \, dt.$$

Our hypothesis on the boundedness of integrals (1) and (2) means that the sum of these last two integrals is equal to $\int_{u''}^{u'} \theta^{-1}(u) \, du$ plus bounded terms. Hence $\lambda(\{\gamma(t)\}) \leq \int_{u''}^{u'} \theta^{-1}(u) \, du + O(1)$ as desired.

2. We now prove the two lemmas referred to in the necessity part of the proof of Theorem 1.

**Lemma 1.** Suppose $f: R \to S$ satisfies property (A). Then the inverse function $F: S \to R$ satisfies

$$\int_0^1 \int_{0 < x < \infty} \left(\text{Arg}^2 F'(z) \right) \, dx \, dy < \infty.$$
Proof. Let \(R(a, b) = R \cap \{w \mid a < \Re w < b\}\). Let \(l(u)\) be the length of \(f(\theta_u)\), where \(\theta_u = \{w \mid \Re w = u, \varphi_0(u) < \Im w < \varphi_1(u)\}\). We have

\[
0 \leq \iint_{R(0, u)} \left[ \frac{1}{\theta(u)} - |f'(w)| \right]^2 \, du \, dv
= \int_0^u \frac{du}{\theta(u)} - 2 \int_0^u \frac{l(u)}{\theta(u)} \, du + \iint_{R(0, u)} |f'(w)|^2 \, du \, dv
= \left\{ \iint_{R(0, u)} |f'(w)|^2 \, du \, dv - \int_0^u \frac{du}{\theta(u)} \right\} - 2 \int_0^u \frac{l(u)}{\theta(u)} - 1 \, du.
\]

We have already seen that the term in braces is uniformly bounded for \(0 < u < \infty\) (see the last paragraph of the Necessity part of the proof of Theorem 1). The last integral above is nonnegative since \(l(u) \geq 1\). Hence

\[
\int_0^u \frac{l(u)}{\theta(u)} - 1 \, du = O(1).
\]

(\textbf{Remark.} With more work one can show that \(\int_0^u (l^2(u) - 1)/\theta(u) \, du = O(1)\); see Theorem 1 of [9].)

For \(0 < t < 1\) let \(\gamma_t\) be the part of the stream line \(\{w \mid \Im f(w) = t\}\) which lies in \(R(0, u)\). \(\gamma_t\) is a connected set since \(|\text{Arg } F(z)| < \pi/2\). We have

\[
\int_0^u \frac{du}{\theta(u)} \leq \int_{\gamma_t} \frac{|dw|}{\theta(u)} = \int_{f(\gamma_t)} \frac{|F'(z)| \, dx}{\theta(u(z))}.
\]

After integrating for \(t \in (0, 1)\) we obtain

\[
\int_0^u \frac{du}{\theta(u)} \leq \int_0^1 \int_{\gamma_t} \frac{|dw|}{\theta(u)} \, dt = \int \int_{f(R(0, u))} \frac{|F'|}{\theta} \, dx \, dy = \int \int_{R(0, u)} \frac{|F'|}{\theta} \, du \, dv.
\]

The last integral can be rewritten as \(\int_0^u l(u) \theta^{-1}(u) \, du\) which, in view of (6), is equal to \(\int_0^u \theta^{-1}(u) \, du + O(1)\). We conclude that

\[
\int_0^1 \int_{\gamma_t} -\frac{du}{\theta(u)} \, dt = O(1).
\]

Replace \(du\) by \(dw|\cos \text{Arg } f'(w)\) and transform the above integral to

\[
\int_0^1 \int_{\gamma_t} \frac{1 - \cos \text{Arg } f'(w)}{\theta} \, |dw| \, dt = \int \int_{f(R(0, u))} (1 - \cos \text{Arg } f'(w(z))) \frac{|F'|}{\theta} \, dx \, dy.
\]

Thus

\[
\iint_{f(R(0, u))} (1 - \cos \text{Arg } F'(z)) \frac{|F'|}{\theta} \, dx \, dy = O(1).
\]
A change of variables in (5) leads to

\[ \iint_{f(R(0, u))} \left( \frac{F'}{\theta} - 1 \right)^2 \, dx \, dy = O(1). \tag{8} \]

It follows from (7) and (8) that

\[ \iint_{f(R(0, u))} (1 - \cos \text{Arg } F'(w)) \, dx \, dy = O(1); \tag{9} \]

indeed, (8) shows that the set \( E_1 = \{ z \mid |F'(z)|/\theta(u(z)) \leq \frac{1}{2} \} \) has finite area and hence

\[ \iint_{E_1} (1 - \cos \text{Arg } F'(z)) \, dx \, dy < \infty. \]

On the complementary set \( E_2 = \{ z \mid |F'(z)|/\theta(u(z)) > \frac{1}{2} \} \) equation (7) shows that

\[ \iint_{E_2} (1 - \cos \text{Arg } F'(z)) \, dx \, dy < \infty. \]

Therefore

\[ \iint_{0 < x < \infty \atop 0 < y < 1} (1 - \cos \text{Arg } F'(z)) \, dx \, dy < \infty. \tag{10} \]

The estimate \( 1 - \cos \beta \geq (4/\pi^2)\beta^2 \) is valid in the range \( |\beta| \leq \pi/2 \). When this is applied to (10) we obtain \( \iint \text{Arg}^2 F'(z) \, dx \, dy < \infty \) as asserted. This completes the proof of Lemma 1.

**Lemma 2.** The map \( F: S \to R \) satisfies \( |\text{Arg } F'(z)| < \pi/2 \) for all \( z \in S \).

**Proof.** The proof is modeled in part after the argument in [7, pp. 102–104]. Let \( R_a = R \cap \{ w = u + iv \mid \text{Re } u > a \} \) for some fixed \( a \) and let \( G \) map the half-strip \( S_1 = \{ 0 < x < \infty, 0 < y < 1 \} \) conformally and one-to-one onto \( R_a \) such that \( 0 \) and \( i \) correspond to \( w = a + i\varphi_0(a) \) and \( w = a + i\varphi_1(a) \), respectively, and

\[ \lim_{x \to -\infty} \text{Re } G(z) = +\infty. \]

We show first that

\[ |\text{Arg } G'(z)| \leq \frac{\pi}{2} \text{ for } z \in S_1. \tag{11} \]

For \( b > a \) we consider the quadrilateral

\[ Q = \{ a < u < b, \varphi_0(u) < v < \varphi_1(u) \}. \]

Then there exists a unique \( \beta > 0 \) and a one-to-one conformal map \( g \) of the rectangle \( T = \{ 0 < x < \beta, 0 < y < 1 \} \) onto \( Q \) such that the vertices \( 0, \beta, i\beta \) and \( i \) of \( T \) correspond to the vertices \( a + i\varphi_0(a), b + i\varphi_0(b), b + i\varphi_1(b), a + i\varphi_1(a) \), respectively. We reflect \( T \) in the line \( x = \beta \) and obtain a symmetrical rectangle \( T' \) and an analytic extension of \( g \) which maps \( T' \) onto a quadrangle \( Q' \) symmetrical to \( Q \) with respect to the line \( u = b \). For fixed \( h > 0 \) \( (h < \beta) \) we define now

\[ P(z, h) = P(z, h; g) = \arg \frac{g(z + h) - g(z)}{h}, \tag{12} \]
where the branch of the argument is determined to coincide with the principal branch at \( z = 0 \). The geometry of the situation shows that \( |P(0, h)| < \pi/2 \) and that \( P(z, h) \) extends continuously to \( \partial T \). As \( z \) describes the boundary of \( T \), \( |P(z, h)| \) remains bounded by \( \pi/2 \). Since \( P \) is harmonic in \( T \) and continuous in \( \partial T \), \( |P(z, h)| < \pi/2 \) for all \( z \in T \). Thus the continuous argument function in (12) is actually the principal branch everywhere.

We choose now a sequence \( \{b_n\} \) with \( b_n \nearrow +\infty \) as \( n \to \infty \) and determine a corresponding sequence \( \{\beta_n\} \) such that the rectangle \( T_n = \{0 < x < \beta_n, 0 < y < 1\} \) is mapped conformally onto the quadrilateral \( Q_n = \{a < u < b_n, \psi_0(u) < v < \psi_1(u)\} \) with vertices of \( T_n \) corresponding to those of \( Q_n \) as indicated above. If \( g_n \) denotes the mapping function, it follows as in [7, p. 303] that \( \lim_{n \to \infty} g_n(z) = G(z) \), uniformly in any compact subset of \( S_1 \). Hence, uniformly in any compact subset of \( S_1 \)

\[
P(z, h; g_n) \to P(z, h; G) = \operatorname{Arg} \left( \frac{G(z + h) - G(z)}{h} \right) \quad \text{as } n \to \infty,
\]

and then

\[
|P(z, h; G)| < \pi/2 \quad \text{for } z \in S_1.
\]

Letting \( h \to 0 \) we obtain (11).

Next we observe that \( \psi \), the inverse of \( F \), maps \( R_a \) onto a subregion \( f(R_a) \subset S \) (pictured in the \( \xi = \xi + i\eta \) plane). If \( \theta_a \) denotes a crosscut \( \{u = a, \varphi_0(a) < \xi < \varphi_1(a)\} \) of \( R \) which determines \( R_a \), then \( f(R_a) \) is bounded by the arc \( \gamma = f(\theta_a) \) and the two half-lines on \( \eta = 0 \) and \( \eta = 1 \) which extend from the endpoints of \( \gamma \) to \( \infty \). Let \( \psi: S_1 \to f(R_a) \) be the one-to-one conformal map of \( S_1 \) onto \( f(R_a) \) such that \( z = 0 \) and \( z = i \) correspond to the endpoints of \( \gamma \) and \( \lim_{\eta \to +\infty} \Re \psi(z) = +\infty \). Then \( G(z) = F(\psi(z)) \) and thus \( G'(z) = F'(\psi(z)) \cdot \psi'(z) \). It is an elementary fact that \( \lim_{k \to \infty} \psi'(z) \) exists for unrestricted approach in \( S_1 \) and is positive. Hence given any \( \epsilon > 0 \) there exists an \( \xi_0 = \xi_0(\epsilon) \) such that (by (11))

\[
|\operatorname{Arg} F'(\psi(z))| < \epsilon \quad \text{for } \Re z = \xi_0, 0 < \eta < 1.
\]

Returning to \( f(R_a) \) in the \( \xi \)-plane we can, given \( \epsilon \), determine a \( \xi_0 = \xi_0(\epsilon) \) such that

\[
|\operatorname{Arg} F'(\xi)| < \pi/2 + \epsilon \quad \text{for } \xi = \xi_0(\epsilon) \text{ and } 0 < \eta < 1.
\]

In an analogous manner—by choosing \( R_a \) as the subregion of \( R \) determined by \( \theta_a \) to the left of \( \theta_a \)—we can establish that for every \( \epsilon > 0 \) there exists a \( \xi_1 = \xi_1(\epsilon) \) such that

\[
|\operatorname{Arg} F'(\xi)| < \pi/2 + \epsilon \quad \text{for } \xi = \xi_1(\epsilon) \text{ and } 0 < \eta < 1.
\]

To complete the proof we consider the rectangle

\[
\{\zeta = \xi + i\eta \mid \xi \leq \xi_1(\epsilon), 0 \leq \eta \leq 1\}
\]

for fixed \( h > 0 \)

\[
P(\zeta, h; F) = \arg \left( \frac{F(\zeta + h) - F(\zeta)}{h} \right).
\]
Again, we see from the geometry that for \( \xi \) on the horizontal sides of (15) we have by choosing the principal view

\[
|P(\xi, h; F)| < \pi/2.
\]

We can continue \( P(\xi) \) as a harmonic function into \( S \). Since for \( \xi \in S \)

\[
\arg \frac{F(\xi + h) - F(\xi)}{h} = \arg F'(\xi + \alpha h), \quad 0 < \alpha < 1,
\]

where the same determination of the argument is taken on both sides, we see from (13) and (14) that the continuation of \( P \) along the two vertical sides of (15) remain the principal value and that

\[
|P(\xi, h; F)| \leq \pi/2 + \epsilon.
\]

Hence we have on the boundary and therefore in the interior of the rectangle

\[
\left| \arg \frac{F(\xi + h) - F(\xi)}{h} \right| \leq \pi/2 + \epsilon.
\]

Letting \( h \to 0 \) we obtain \( |\arg F'(\xi)| \leq \pi/2 + \epsilon \) for \( \xi \) in (15). Since \( \epsilon \) is arbitrary we obtain \( |\arg F'(s)| \leq \pi/2 \) for \( \xi \in S \). By the maximum principal the strict inequality holds.

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