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THE APPROXIMATION PROPERTY FOR SOME 5-DIMENSIONAL HENSELIAN RINGS

BY

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ABSTRACT. Let k be a field of characteristic 0, $k[[X_1, X_2]]$ the ring of formal power series and $R = k[[X_1, X_2]][X_3, X_4, X_5]^{\sim}$ the algebraic closure of $k[[X_1, X_2]][X_3, X_4, X_5]$ in $k[[X_1, \dots, X_5]]$. It is shown that R has the Approximation Property.

1. Introduction. Let R be a local ring and $\hat{R}$ its completion. We say that R has the *Approximation Property* if every system of polynomial equations over R, which has a solution in $\hat{R}$, also has a solution in R. Let $\mathfrak{m}$ be the maximal ideal of R, and let $X = (X_1, \dots, X_n)$ be variables. We denote the Henselization of $R[X_1, \dots, X_n]_{(\mathfrak{m}, X)}$ by $R[X_1, \dots, X_n]^{\sim}$. For example, if k is a field, then $k[X_1, \dots, X_n]^{\sim}$ is the ring of the formal power series over k which are algebraic over $k[X_1, \dots, X_n]$. Let $\mathbf{C}\{X_1, \dots, X_n\}$ be the ring of the formal power series over $\mathbf{C}$ (in the variables $X_1, \dots, X_n$) which converge in some neighborhood of the origin. M. Artin proved [**A**, **A1**] that $\mathbf{C}\{X_1, \dots, X_n\}$ and $R[X_1, \dots, X_n]^{\sim}$ have the Approximation Property if R is a field or an excellent discrete valuation ring and he conjectured [**A2**]:

1.1. CONJECTURE. If R is an excellent (see [**EGA**, IV, 7.8.2]) Henselian local ring, then R has the Approximation Property.

A special case of Conjecture 1.1 is

1.2. CONJECTURE. Let k be a field, then $k[[X_1, \dots, X_r]][X_{r+1}, \dots, X_n]^{\sim}$ has the Approximation Property.

It is well known (see Remark 1.5) that Conjecture 1.2 (for particular r, n, with $r < n$) implies

1.2′. CONJECTURE. Let k be a field. If a system of polynomial equations over $k[X_1, \dots, X_n]^{\sim}$ has a solution $\bar{y} = (\bar{y}_1, \dots, \bar{y}_m) \in k[[X_1, \dots, X_n]]$, satisfying

$$(1)\qquad \begin{gathered} \bar{y}_1, \dots, \bar{y}_{s_1} \in k[[X_1]], \\ \bar{y}_{s_1+1}, \dots, \bar{y}_{s_2} \in k[[X_1, X_2]], \\ \vdots \\ \bar{y}_{s_{r-1}+1}, \dots, \bar{y}_{s_r} \in k[[X_1, \dots, X_r]], \qquad 0 \leq s_1 \leq s_2 \leq \cdots \leq s_r \leq m, \end{gathered}$$

then it also has a solution $y = (y_1, \dots, y_m) \in k[X_1, \dots, X_n]^{\sim}$ which satisfies the conditions (1).

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Gabriélov [**Ga**] proved that Conjecture 1.2′ for $r = 2$, $n = 3$ becomes false if one replaces $k[X_1,\dots,X_n]\tilde{}$ by $\mathbf{C}\{X_1,\dots,X_n\}$. J. Becker [**B**] proved that Conjecture 1.2′ becomes false if one allows disjoint subrings $k[[X_1]]$, $k[[X_2]]$ in (1), instead of nested subrings $k[[X_1]] \subset k[[X_1, X_2]] \subset \cdots$.

Conjecture 1.2 (and hence also 1.2′), for $r = 1$ and all n, follows from [**A1**]. Moreover Conjecture 1.2′, for $r = 1$ and all n, remains true if one replaces $k[X_1,\dots,X_n]\tilde{}$ by $\mathbf{C}\{X_1,\dots,X_n\}$ (see [**DL**, §5]). Recently G. Pfister and D. Popescu [**PP**] proved Conjecture 1.2 when $r = 2$, $n = 3$, and $\mathrm{Char}(k) = 0$. In this paper we prove Conjecture 1.2 (and hence also 1.2′) when $r = 2$, $n = 3$, 4 or 5, and $\mathrm{Char}(k) = 0$.

1.3. THEOREM. *Let k be a field of characteristic zero. Then* $k[[X_1, X_2]][X_3, X_4, X_5]\tilde{}$ *has the Approximation Property.*

The proof of Theorem 1.3 has two parts. The first part (§2) consists of a global form of Néron p-desingularization and is the same as in [**PP**]. However, for the sake of completeness, we have included proofs. The second part (§3) is different from the method in [**PP**] and consists of Lemma 3.1.

In [**BDLV**] (in the remark following Theorem 4.3) we proved that Conjecture 1.2′, for particular r, n, implies the corresponding

STRONG APPROXIMATION THEOREM. *Let k be a field and let* $f(Y) = 0$ *be a system of polynomial equations over* $k[X]\tilde{}$, *where* $Y = (Y_1,\dots,Y_m)$ *and* $X = (X_1,\dots,X_n)$. *There is a function* $\beta\colon \mathbf{N} \to \mathbf{N}$ (*depending on* f) *such that for any* $\alpha \in \mathbf{N}$, *if there is a* $\bar{y} = (\bar{y}_1,\dots,\bar{y}_m) \in k[X]$, *satisfying conditions* (1) *of Conjecture* 1.2′ *and* $f(\bar{y}) \equiv 0 \bmod (X)^{\beta(\alpha)}$, *then there is a solution* $y = (y_1,\dots,y_m) \in k[X]\tilde{}$ *of* $f(Y) = 0$ *also satisfying conditions* (1) *and* $y \equiv \bar{y} \bmod (X)^{\alpha}$.

We conclude this Introduction with a well-known lemma which we need in §3, but for which we could not find a good reference.

1.4. LEMMA. *Let R be a local Noetherian ring which has the Approximation Property. Let* $T = (T_1,\dots,T_n)$ *be variables. Then every system of polynomial equations over* $R[T]\tilde{}$, *which has a solution in* $\hat{R}[T]\tilde{}$, *also has a solution in* $R[T]\tilde{}$.

PROOF. We give a proof using the ultraproduct construction (see e.g. [**CK** or **BDLV**, §1]), although a classical proof would be as easy. Since R has the Approximation Property, for every subring S of $\hat{R}$ which is finitely generated over R, there exists an R-algebra homomorphism $\phi_S\colon S \to R$.

Let I be the set of all subrings of $\hat{R}$ which are finitely generated over R. Choose an ultrafilter D on I such that for every $S_0 \in I$ we have $\{S \in I\colon S_0 \subseteq S\} \in D$. The maps ϕ_S induce an R-algebra homomorphism

$$\phi^*\colon \prod_{S\in I} S/D \to R^* = \prod_{S\in I} R/D.$$

Consider the map

$$\theta\colon \hat{R} \to \prod_{S\in I} S/D\colon a \mapsto (a_S)_{S\in I} \mod D$$

where

$$a_S = a, \quad \text{if } a \in S,$$
$$a_S = 0, \quad \text{if } a \notin S.$$

It is easy to verify that θ is an R-algebra homomorphism. Thus we have an R-algebra homomorphism $\psi = \phi^* \circ \theta$: $\hat{R} \to R^*$. The ultraproduct R^* is a local ring (not Noetherian), and ψ is a local homomorphism (because the maximal ideal of $\hat{R}$ is generated by the maximal ideal $\mathfrak{m}$ of R, and ψ is an R-algebra map).

There is a canonical map

$$R^* \to (R[T]^\sim)^* = \prod_{S \in I} (R[T]^\sim)/D.$$

Thus ψ: $\hat{R} \to R^*$ extends to a local $R[T]$-algebra homomorphism

$$\psi: \hat{R}[T]_{(\mathfrak{m},T)} \to (R[T]^\sim)^*.$$

But $(R[T]^\sim)^*$ is a local Henselian ring (see [**BDLV**, §1]), thus, by the universal property of Henselization [**EGA**, IV, 18.6.6], ψ extends to an $R[T]$-algebra (and in fact an $R[T]^\sim$-algebra) homomorphism

$$\tilde{\psi}: \hat{R}[T]^\sim \to (R[T]^\sim)^*.$$

Thus every system of polynomial equations over $R[T]^\sim$, which has a solution in $\hat{R}[T]^\sim$, has a solution in $(R[T]^\sim)^*$, and hence also in $R[T]^\sim$. Q.E.D.

1.5. Remark. Observe that it follows from the above proof that if in Lemma 1.4 some of the coordinates of the solution are in the subrings $\hat{R}[T_1,\ldots,T_i]^\sim$, $i \geq 0$, then the new solution can be chosen so that the corresponding coordinates are in the corresponding subrings $R[T_1,\ldots,T_i]^\sim$. Conjecture 1.2′ can be derived from Conjecture 1.2 as follows: Assume the hypothesis of 1.2′. Use 1.2 to get a solution in $k[[X_1,\ldots,X_r]][X_{r+1},\ldots,X_n]^\sim$ satisfying (1), by fixing $\bar{y}_1,\ldots,\bar{y}_{s_r}$. Now use the above-mentioned strengthened version of Lemma 1.4 r times in succession to get down to a solution in $k[X_1,\ldots,X_n]^\sim$ satisfying (1). (In the jth use of 1.4 take $R = k[[X_1,\ldots,X_{r-j}]][X_{r-j+1}]^\sim$ and $T = (X_{r-j+2},\ldots,X_n)$, and fix $\bar{y}_1,\ldots,\bar{y}_{s_{r-j}}$. These rings R have the Approximation Property by 1.2.)

2. Global Néron p-desingularization. Let B be a finitely generated A algebra and $\mathcal{P}$ a prime ideal of B. We say that B is smooth over A at $\mathcal{P}$ if $\operatorname{Spec} B$ is smooth over $\operatorname{Spec} A$ at $\mathcal{P} \in \operatorname{Spec} B$ (see e.g. [**A3**, pp. 80–81]).

2.1. Theorem (Néron p-desingularization). *Let $\Lambda \subset \Lambda'$ be discrete valuation rings, and let p be a local parameter of Λ. Suppose that Λ' is unramified over Λ (i.e. p is also a local parameter of Λ') and suppose that the residue field of Λ' is separable over the residue field of Λ. Let B be a subring of Λ' which is finitely generated over Λ, such that* $\operatorname{Frac}(B)$ *is separable over* $\operatorname{Frac}(\Lambda)$. (Frac *denotes the fraction field.*) *Then there exists a subring C of Λ', containing B, such that C is finitely generated over Λ and smooth over Λ at the prime ideal $C \cap p\Lambda'$, and such that $C \subset S^{-1}B$, where $S = \{p^e: e \in \mathbf{N}\}$.*

This is an immediate consequence of Néron's p-desingularization **[N]** (see **[A1**, §4]).

The next theorem is a global version of Néron's p-desingularization and is due to Pfister and Popescu **[PP]**.

2.2. THEOREM (GLOBAL NÉRON p-DESINGULARIZATION). *Let $A \subset A'$ be Noetherian Unique Factorisation Domains. Suppose for every prime element p of A, that p remains prime in A' and that $A \cap pA' = pA$. Suppose that* $\operatorname{Frac}(A')$ *is separable over* $\operatorname{Frac}(A)$ *and that* $\operatorname{Frac}(A'/qA')$ *is separable over* $\operatorname{Frac}(A/A \cap qA')$, *for every prime element q of A'. Suppose that there exists an infinite set of units of A' which are algebraically independent over A. Let B be a subring of A' which is finitely generated over A. Then there exists a subring C of A', containing B, such that C is finitely generated over A and smooth over A at $C \cap qA'$ for every prime element q of A'.*

PROOF. It follows from separability that B is smooth over A at the prime ideal (0). Hence there are only a finite number of prime ideals of the form qA', such that B is not smooth over A at $B \cap qA'$. Hence, by the transitivity of smoothness, it is sufficient to prove that for every subring B of A', which is finitely generated over A, and for every prime element q of A', there exists a subring C of A', containing B, such that (i) C is finitely generated over A, (ii) C is smooth over A at $C \cap qA'$, and (iii) C is smooth over B at $C \cap q'A'$ for every prime element q' of A' with $q'A' \neq qA'$. Let q be a fixed prime element of A'. There are two cases:

Case 1. $A \cap qA' \neq (0)$. Then there exists a prime element p of A such that $p \in qA'$. Since p remains prime in A', we have $pA' = qA'$. Thus we may as well suppose that $q \in A$, and q is a prime element in both A and A'. Moreover we have $A \cap qA' = qA$ and $A_{qA} \subset A'_{qA'}$ are discrete valuation rings. Let $U = A \backslash qA$. The conditions of Theorem 2.1 are satisfied for $\Lambda = A_{qA} \subset U^{-1}B \subset \Lambda' = A'_{qA'}$. Thus there exists a subring D of $A'_{qA'}$, containing $U^{-1}B$, such that D is finitely generated over A_{qA} and smooth over A_{qA} at $D \cap qA'_{qA'}$, and such that $D \subset S^{-1}U^{-1}B$, where $S = \{q^e \colon e \in \mathbf{N}\}$. Let $y_1, \ldots, y_s$ be generators for D over A_{qA}. Then there are $e \in \mathbf{N}$ and $u \in U$ such that $q^e u y_i \in B$, for $i = 1, \ldots, s$. Since $q^e u y_i \in A'$ and $u y_i \in A'_{qA'}$, we have $u y_i \in A'$. Let $C = B[u y_1, \ldots, u y_s] \subset A'$. We have $C \subset S^{-1}B$, thus C is smooth over B at $C \cap q'A'$ for every prime element q' of A' with $q'A' \neq qA'$. Moreover $U^{-1}C = D$ is smooth over $U^{-1}A = A_{qA}$ at $D \cap qA'_{qA'}$. Hence **[EGA**, IV, 17.7.1], C is smooth over A at $C \cap (D \cap qA'_{qA'}) = C \cap qA'$. This completes the treatment of Case 1.

Case 2. $A \cap qA' = (0)$. We may suppose that q is transcendental over B. (Otherwise multiply q with a unit which is transcendental over B.) Then $A[q]$ is a Noetherian UFD, and $A[q]_{qA[q]}$ is a discrete valuation ring. We have $A[q] \cap qA' = qA[q]$. Indeed if $x \in A[q]$ and $x \in qA'$, then $x - a \in qA[q]$ for some $a \in A$, hence $a \in qA'$; thus $a = 0$ (since we are in Case 2) and $x \in qA[q]$. Thus we have $\Lambda = A[q]_{qA[q]} \subset \Lambda' = A'_{qA'}$. Let $U = A[q] \backslash qA[q]$. The conditions of Theorem 2.1 are satisfied for $\Lambda \subset U^{-1}B[q] \subset \Lambda'$. By the same argument as in Case 1 we obtain a subring C of A', containing $B[q]$, such that (i) C is finitely generated over $A[q]$, (ii) C is smooth over $A[q]$ at $C \cap qA'$, and (iii) C is smooth over $B[q]$ at $C \cap q'A'$, for every prime element q' of A' with $q'A' \neq qA'$. Since q is transcendental over B, we have that $B[q]$ is smooth over B and $A[q]$ is smooth over A. The theorem now follows by the transitivity of smoothness. Q.E.D.

2.3. COROLLARY. *Let*

$$A_0 = k[[X_1,\dots,X_r]][X_{r+1},\dots,X_n],$$
$$A = k[[X_1,\dots,X_r]][X_{r+1},\dots,X_n]^{\sim} \quad \text{and} \quad \hat{A} = k[[X_1,\dots,X_n]],$$

where k is a field of characteristic zero. Let B be a subring of $\hat{A}$ which is finitely generated over A_0. Then there exists a subring C of $\hat{A}$, containing B, such that C is finitely generated over A_0 and smooth over A_0 at $C \cap q\hat{A}$, for every prime element q of $\hat{A}$.

PROOF. The pair $A \subset \hat{A}$ satisfies the hypothesis of Theorem 2.2 (see [**EGA**, IV, 18.7.6 and 18.9.2]). Moreover, it follows easily from the definition of Henselization [**EGA**, IV, 18.6.5] that every subring D of A, which is finitely generated over A_0, is contained in a subring A_1 of A such that A is flat over A_1, and A_1 is finitely generated over A_0 and étale over A_0 at $A_1 \cap (X_1,\dots,X_n)\hat{A}$. (Indeed, notice that the maps $\phi_{\mu\lambda}$ in [**EGA**, IV, 18.6.5] are faithfully flat, and hence injective.) Let $B = A_0[y_1,\dots,y_e]$, and let $C' = A[y_1,\dots,y_e,\dots,y_m]$ be a subring of $\hat{A}$ such that C' is smooth over A at every $C' \cap q\hat{A}$ (cf. Theorem 2.2). Let $f_1,\dots,f_r \in A[Y_1,\dots,Y_m]$ be generators for the ideal $\{f \in A[Y_1,\dots,Y_m]\colon f(y_1,\dots,y_m) = 0\}$. Let A_1 be as above and containing the coefficients of $f_1,\dots,f_r$. Let $C = A_1[y_1,\dots,y_m]$; then $C' \cong C \otimes_{A_1} A$. From [**EGA**, IV, 17.7.1] it follows that C is smooth over A_1 at every $C \cap q\hat{A}$. The corollary now follows from the transitivity of smoothness. Q.E.D.

3. Proof of Theorem 1.3. Let k be a field of characteristic zero,

$$A_0 = k[[X_1, X_2]][X_3, X_4, X_5],$$
$$A = k[[X_1, X_2]][X_3, X_4, X_5]^{\sim} \quad \text{and} \quad \hat{A} = k[[X_1, X_2, X_3, X_4, X_5]].$$

We use the following notation: $X_{12} = (X_1, X_2)$, $X_{345} = (X_3, X_4, X_5)$, $X_{1234} = (X_1, X_2, X_3, X_4)$, etc.... We have to prove that every system of polynomial equations over A, which has a solution in $\hat{A}$, also has a solution in A. Since A is algebraic over A_0, we may suppose that the equations have coefficients in A_0 by introducing more equations and congruences if necessary. Thus we have to prove that for every subring B of $\hat{A}$, which is finitely generated over A_0, there exists an A_0-algebra homomorphism $B \to A$. It follows from Corollary 2.3 that we may suppose that B is smooth over A_0 at $B \cap q\hat{A}$, for every prime element q of $\hat{A}$. Let $B = A_0[\bar{y}_1,\dots,\bar{y}_N]$, with $\bar{y}_1,\dots,\bar{y}_N \in \hat{A}$. Let $f_1(Y),\dots,f_m(Y) \in A_0[Y]$ be generators for the ideal $\{f(Y) \in A_0[Y]\colon f(\bar{y}) = 0\}$, where $Y = (Y_1,\dots,Y_N)$ and $\bar{y} = (\bar{y}_1,\dots,\bar{y}_N)$. Thus $f_i(\bar{y}) = 0$ for $i = 1,\dots,m$. We have to prove that there exists $y = (y_1,\dots,y_N) \in A$, such that $f_i(y) = 0$ for $i = 1,\dots,m$. But by Lemma 1.4 and induction, it is sufficient to prove that there exists $y = (y_1,\dots,y_N) \in k[[X_{1234}]][X_5]^{\sim}$, such that $f_i(y) = 0$ for $i = 1,\dots,m$. Choose $\delta_1(Y),\dots,\delta_s(Y) \in A_0[Y]$ such that (i) for every prime ideal $\mathcal{P}$ of B, B is smooth over A_0 at $\mathcal{P}$ if and only if there is an i such that $\delta_i(\bar{y}) \notin \mathcal{P}$, and (ii) the ideal $H_B = (\delta_1(Y),\dots,\delta_s(Y))A_0[Y]$ satisfies the condition in [**E**, 0.2, p. 555]. Since B is smooth over A_0 at $B \cap q\hat{A}$, we have that $(\delta_1(\bar{y}),\dots,\delta_s(\bar{y}))\hat{A} \not\subset q\hat{A}$ for every prime element q of $\hat{A}$. Thus the height of the ideal $(\delta_1(\bar{y}),\dots,\delta_s(\bar{y}))\hat{A}$ is not smaller than two. Thus we have

$$\sqrt{(\delta_1(\bar{y}),\dots,\delta_s(\bar{y}))\hat{A}} = \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_t,$$

where $\mathcal{P}_1,\dots,\mathcal{P}_t$ are prime ideals in $\hat{A}$ with height not smaller than two. Hence if $\mathcal{P}_j \subset (X_1, X_2)\hat{A}$, then $\mathcal{P}_j = (X_1, X_2)\hat{A}$. Hence there exist $g \in \hat{A}$, with $g \notin (X_1, X_2)\hat{A}$, and $r \in \mathbf{N}$, such that

(1) $$X_1^r g \in (\delta_1(\bar{y}),\dots,\delta_s(\bar{y}))\hat{A}, \qquad X_2^r g \in (\delta_1(\bar{y}),\dots,\delta_s(\bar{y}))\hat{A}.$$

After a linear change of coordinates among X_3, X_4 and X_5, we may suppose that g is regular in X_5 (as a formal power series; see e.g. [**ZS**, p. 145]), because $g \notin (X_1, X_2)\hat{A}$. Let $w \in k[[X_{1234}]][X_5]$ be the distinguished pseudopolynomial associated with g (see e.g. [**ZS**, p. 146]). Let $A_1 = k[[X_{1234}]][X_5]^{\sim}$ and $\mathcal{J} = w.(X_1^r, X_2^r)A_1$. Applying Elkik's theorem [**E**, Théorème 2, p. 560] to the Henselian pair $(A_1, \mathcal{J})$, we see that it is sufficient to prove that there exists $y \in A_1^N$ such that

(2) $$f_i(y) \in w^e_.(X_1^{re}, X_2^{re})A_1, \qquad i = 1,\dots,m,$$

and

(3) $$wX_1^r, wX_2^r \in (\delta_1(y),\dots,\delta_s(y))A_1,$$

where $e \in \mathbf{N}$ is big enough.

We are going to use the

3.1. Congruence Lemma. *Let k be a field of characteristic zero. Let $w \in k[[X_{1234}]][X_5]$ be a distinguished pseudopolynomial (with respect to X_5) and $l \in \mathbf{N}$. Every system of polynomial equations over $k[[X_{1234}]][X_5]$ which has a solution in $k[[X_{12345}]]$ also has a solution in $k[[X_{1234}]][X_5]^{\sim}/w.(X_1^l, X_2^l)$.*

We prove Lemma 3.1 later, and proceed first with the proof of Theorem 1.3. Define

$$G(Z, Y) = \sum_{i=1}^{s} Z_i\delta_i(Y) \in A_0[Y, Z], \qquad Z = (Z_1,\dots,Z_s).$$

It follows from (1) that there exist $\bar{z}_1 \in \hat{A}^s$, $\bar{z}_2 \in \hat{A}^s$ such that

$$wX_1^r = G(\bar{z}_1, \bar{y}), \qquad wX_2^r = G(\bar{z}_2, \bar{y}).$$

From Lemma 3.1 it follows that there exist $y \in A_1^N$, $z_1 \in A_1^s$, $z_2 \in A_1^s$, such that

$$f_i(y) \equiv_{A_1} 0, \qquad wX_1^r \equiv_{A_1} G(z_1, y), \qquad wX_2^r \equiv_{A_1} G(z_2, y) \mod w^e_.(X_1^{re}, X_2^{re}),$$

where $\equiv_{A_1}$ denotes congruence in A_1.

Thus (2) holds and we prove now that (3) is also satisfied. It follows from the last two congruences that there exist $v_1, v_2, v_3, v_4 \in A_1$ such that

$$wX_1^r = G(z_1, y) + v_1 w^e X_1^{re} + v_2 w^e X_2^{re},$$
$$wX_2^r = G(z_2, y) + v_3 w^e X_1^{re} + v_4 w^e X_2^{re}.$$

This can be written as

$$\left(1 - v_1 w^{e-1} X_1^{r(e-1)}\right)\left(wX_1^r\right) - \left(v_2 w^{e-1} X_2^{r(e-1)}\right)\left(wX_2^r\right) = G(z_1, y),$$
$$-v_3 w^{e-1} X_1^{r(e-1)}\left(wX_1^r\right) + \left(1 - v_4 w^{e-1} X_2^{r(e-1)}\right)\left(wX_2^r\right) = G(z_2, y).$$

We consider this as a system of two linear equations with two unknowns wX_1^r and wX_2^r. The determinant of this system is congruent to $1 \bmod (X_1, X_2)$ (we may suppose $r > 0$, $e > 1$) and hence a unit in A_1.

Solving for wX_1^r, wX_2^r, we obtain $wX_1^r, wX_2^r \in (G(z_1, y), G(z_2, y))A_1$. From the definition of $G(Z, Y)$ we have that

$$(G(z_1, y), G(z_2, y))A_1 \subset (\delta_1(y), \ldots, \delta_n(y))A_1.$$

This proves (3) and the proof of Theorem 1.3 is completed if we prove Lemma 3.1.

PROOF OF CONGRUENCE LEMMA 3.1. Let $\hat{A} = k[[X_{12345}]]$ and $A_1 = k[[X_{1234}]][X_5]^{\sim}$, as before. Let $h_i(Y) \in k[[X_{1234}]][X_5][Y]$, $i = 1, \ldots, m$, $Y = (Y_1, \ldots, Y_N)$.

Suppose there exists $\bar{y} \in \hat{A}^N$ such that $h_i(\bar{y}) = 0$ for $i = 1, \ldots, m$. We have to prove that there exists $y \in A_1^N$ such that

$$h_i(y) \equiv_{A_1} 0 \mod w.\left(X_1^l, X_2^l\right), \qquad i = 1, \ldots, m,$$

where $\equiv_{A_1}$ denotes congruence in the ring A_1.

By the Weierstrass Preparation Theorem we can write

$$\bar{y} = y_0 + w\bar{q} \quad \text{with } y_0 \in k[[X_{1234}]][X_5]^N \quad \text{and} \quad \bar{q} \in \hat{A}^N.$$

Moreover we can write

$$\bar{q} = \tilde{y}_1 + X_1^l \bar{q}_1 + X_2^l \bar{q}_2 \quad \text{with } \tilde{y}_1 \in k[[X_{345}]][X_{12}]^N \quad \text{and} \quad \bar{q}_1, \bar{q}_2 \in \hat{A}^N.$$

Define

$$\tilde{y} = y_0 + w\tilde{y}_1. \tag{4}$$

Thus we have

$$\bar{y} = \tilde{y} + wX_1^l \bar{q}_1 + wX_2^l \bar{q}_2. \tag{5}$$

Let $B = k[[X_{345}]] \cdot k[[X_{1234}]]$ be the compositum of the two rings $k[[X_{345}]]$ and $k[[X_{1234}]]$ in $\hat{A}$. We have $\tilde{y} \in B$. From $h_i(\bar{y}) = 0$ and (5), follows

$$h_i(\tilde{y}) \equiv_{\hat{A}} 0 \mod w.\left(X_1^l, X_2^l\right), \quad \text{for } i = 1, \ldots, m, \tag{6}$$

where $\equiv_{\hat{A}}$ denotes congruence in the ring $\hat{A}$.

We are going to prove that

$$h_i(\tilde{y}) \equiv_B 0 \mod w.\left(X_1^l, X_2^l\right), \tag{6'}$$

where $\equiv_B$ denotes congruence in the ring B.

From (4) we have that

$$h_i(\tilde{y}) \equiv_B h_i(y_0) \mod w, \tag{7}$$

and from (7) and (6) that

$$h_i(y_0) \equiv_{\hat{A}} 0 \mod w.$$

Now $h_i(y_0)$ and w are in $k[[X_{1234}]][X_5]$, and w is a distinguished pseudopolynomial. Hence by [**ZS**, p. 146] we have that

$$h_i(y_0) \equiv_C 0 \mod w,$$

where $C = k[[X_{1234}]][X_5]$. Combining this with (7) we obtain

$$h_i(\tilde{y}) \equiv_B 0 \mod w.$$

Thus there exist $a_i \in B$ with $h_i(\tilde{y}) = wa_i$. It follows from (6) that $a_i \equiv_{\hat{A}} 0 \mod (X_1^l, X_2^l)$. This implies $a_i \equiv_B 0 \mod (X_1^l, X_2^l)$ and (6′) follows. Indeed, suppose $a \in B$ and $a \equiv_{\hat{A}} 0 \mod (X_1^l, X_2^l)$, we will prove that $a \equiv_B 0 \mod (X_1^l, X_2^l)$. Every element in $k[[X_{1234}]]$ is congruent in B to an element of $k[[X_{34}]][X_{12}] \mod (X_1^l, X_2^l)$. Thus there exists $c \in k[[X_{345}]][X_{12}]$ with $a \equiv_B c \mod (X_1^l, X_2^l)$. Hence $c \equiv_{\hat{A}} 0$. Thus $c \in (X_1^l, X_2^l)k[[X_{345}]][X_{12}]$. Hence $a \equiv_B 0$. This finishes the proof of (6′).

Congruence Lemma 3.1 now follows at once from (6′), and the following:

Claim. Every system of polynomial equations over $k[[X_{1234}]][X_5]$, which has a solution in B, also has a solution in A_1.

PROOF OF THE CLAIM. Let $F(Z) \in k[[X_{1234}]][X_5][Z]^m$, $Z = (Z_1, \ldots, Z_N)$. Suppose there exists $\tilde{z} \in B^N$ with $F(\tilde{z}) = 0$. We have to prove that there exists $z \in A_1^N$ with $F(z) = 0$. Now, $\tilde{z} \in B^N$ can be written as $\tilde{z} = E(\tilde{u})$, with $E(U) \in k[[X_{1234}]][U]^N$, $U = (U_1, \ldots, U_s)$, and $\tilde{u} \in k[[X_{345}]]^s$. Thus $F(E(\tilde{u})) = 0$. We can write

$$F(E(U)) = \sum_{i,j} C_{ij}(U) X_1^i X_2^j,$$

with

$$C_{ij}(U) \in k[[X_{34}]][X_5][U]^m.$$

We have $C_{ij}(\tilde{u}) = 0$, for all $i, j \in \mathbf{N}$. By Noetherianess, there is a finite set $S \subset \mathbf{N}$ such that the equations $C_{ij}(U) = 0$ for all i, j, are implied by the finite set of equations $C_{ij}(U) = 0$, $i, j \in S$.

First we prove the Claim in the special case that X_3 and X_4 do not appear. Then, by Greenberg's theorem [**G**], there exists $u \in (k[X_5]^{\sim})^s$ such that $C_{ij}(u) = 0$ for $i, j \in S$, and hence also for all $i, j \in \mathbf{N}$. Thus $F(E(u)) = 0$ and $E(u) \in (k[[X_{1,2}]][X_5]^{\sim})^N$.

This proves the Claim, and hence Lemma 3.1 and Theorem 1.3, in the special case that X_3 and X_4 do not appear (the 3-dimensional case). Thus $k[[X_1, X_2]][X_5]^{\sim}$ has the Approximation Property. Thus also $k[[X_{34}]][X_5]^{\sim}$ has the Approximation Property. Thus also in the general case, there exists $u \in (k[[X_{34}]][X_5]^{\sim})^s$ such that $C_{ij}(u) = 0$ for $i, j \in S$, and hence also for all $i, j \in \mathbf{N}$. Let $z = E(u)$. Then $F(z) = 0$ and $z \in (k[[X_{1234}]][X_5]^{\sim})^N$. This proves the claim. Q.E.D.

ADDED IN PROOF. Theorem 1.3 is also true when k is a field of nonzero characteristic. This follows by using a generalization of Theorem 2.2 as in D. Popescu, *Global forms of Néron's p-desingularization and approximation*, Teubner Texte Bd. 40, Teubner, Leipzig, 1981.

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