THE APPROXIMATION PROPERTY
FOR SOME 5-DIMENSIONAL HENSELIAN RINGS

BY

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ABSTRACT. Let \( k \) be a field of characteristic 0, \( k[[X_1, X_2]] \) the ring of formal power series and \( R = k[[X_1, X_2]][X_3, X_4, X_5] \) the algebraic closure of \( k[[X_1, X_2]][X_3, X_4, X_5] \) in \( k[[X_1, X_2]] \). It is shown that \( R \) has the Approximation Property.

1. Introduction. Let \( R \) be a local ring and \( \hat{R} \) its completion. We say that \( R \) has the Approximation Property if every system of polynomial equations over \( R \), which has a solution in \( \hat{R} \), also has a solution in \( R \). Let \( m \) be the maximal ideal of \( R \), and let \( X = (X_1, \ldots, X_n) \) be variables. We denote the Henselization of \( R[X_1, \ldots, X_n] \) by \( R[X_1, \ldots, X_n] \). For example, if \( k \) is a field, then \( k[X_1, \ldots, X_n] \) is the ring of the formal power series over \( k[X_1, \ldots, X_n] \). Let \( k[X_1, \ldots, X_n] \) be the ring of the formal power series over \( k \) (in the variables \( X_1, \ldots, X_n \)) which converge in some neighborhood of the origin. M. Artin proved [A, A1] that \( k[X_1, \ldots, X_n] \) and \( R[X_1, \ldots, X_n] \) have the Approximation Property if \( R \) is a field or an excellent discrete valuation ring and he conjectured [A2]:

1.1. Conjecture. If \( R \) is an excellent (see [EGA, IV, 7.8.2]) Henselian local ring, then \( R \) has the Approximation Property.

A special case of Conjecture 1.1 is

1.2. Conjecture. Let \( k \) be a field, then \( k[[X_1, \ldots, X_r]][X_{r+1}, \ldots, X_n] \) has the Approximation Property.

It is well known (see Remark 1.5) that Conjecture 1.2 (for particular \( r, n \), with \( r < n \)) implies

1.2'. Conjecture. Let \( k \) be a field. If a system of polynomial equations over \( k[X_1, \ldots, X_n] \) has a solution \( \tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_m) \in k[X_1, \ldots, X_n] \), satisfying

\[
\tilde{y}_1, \ldots, \tilde{y}_n \in k[[X_1]], \\
\tilde{y}_{i+1}, \ldots, \tilde{y}_2 \in k[[X_1, X_2]], \\
\vdots \\
\tilde{y}_{s_i+1}, \ldots, \tilde{y}_{s_j} \in k[[X_1, \ldots, X_j]], \\
0 \leq s_1 \leq s_2 \leq \cdots \leq s_r \leq m,
\]

then it also has a solution \( y = (y_1, \ldots, y_m) \in k[X_1, \ldots, X_n] \) which satisfies the conditions (1).
Gabriélov [Ga] proved that Conjecture 1.2' for \( r = 2, n = 3 \) becomes false if one replaces \( k[X_1, \ldots, X_n] \) by \( C\{X_1, \ldots, X_n\} \). J. Becker [B] proved that Conjecture 1.2' becomes false if one allows disjoint subrings \( k[[X_1]], k[[X_2]] \) in (1), instead of nested subrings \( k[[X_1]] \subseteq k[[X_1, X_2]] \subseteq \cdots \).

Conjecture 1.2 (and hence also 1.2'), for \( r = 1 \) and all \( n \), follows from [A1]. Moreover Conjecture 1.2', for \( r = 1 \) and all \( n \), remains true if one replaces \( k[X_1, \ldots, X_n] \) by \( C\{X_1, \ldots, X_n\} \) (see [DL, §5]). Recently G. Pfister and D. Popescu [PP] proved Conjecture 1.2 when \( r = 2, n = 3 \), and \( \text{Char}(k) = 0 \). In this paper we prove Conjecture 1.2 (and hence also 1.2') when \( r = 2, n = 3, 4 \) or 5, and \( \text{Char}(k) = 0 \).

1.3. THEOREM. Let \( k \) be a field of characteristic zero. Then \( k[[X_1, X_2]][X_3, X_4, X_5] \) has the Approximation Property.

The proof of Theorem 1.3 has two parts. The first part (§2) consists of a global form of Néron \( p \)-desingularization and is the same as in [PP]. However, for the sake of completeness, we have included proofs. The second part (§3) is different from the method in [PP] and consists of Lemma 3.1.

In [BDLV] (in the remark following Theorem 4.3) we proved that Conjecture 1.2', for particular \( r, n \), implies the corresponding

**Strong Approximation Theorem.** Let \( k \) be a field and let \( f(Y) = 0 \) be a system of polynomial equations over \( k[X] \), where \( Y = (Y_1, \ldots, Y_m) \) and \( X = (X_1, \ldots, X_n) \). There is a function \( \beta: \mathbb{N} \to \mathbb{N} \) (depending on \( f \)) such that for any \( \alpha \in \mathbb{N} \), if there is a \( \bar{y} = (\bar{y}_1, \ldots, \bar{y}_m) \in k[X] \), satisfying conditions (1) of Conjecture 1.2' and \( f(\bar{y}) \equiv 0 \mod (X)^{\beta(\alpha)} \), then there is a solution \( y = (y_1, \ldots, y_m) \in k[X_1] \) of \( f(Y) = 0 \) also satisfying conditions (1) and \( y \equiv \bar{y} \mod (X)^\alpha \).

We conclude this Introduction with a well-known lemma which we need in §3, but for which we could not find a good reference.

1.4. LEMMA. Let \( R \) be a local Noetherian ring which has the Approximation Property. Let \( T = (T_1, \ldots, T_n) \) be variables. Then every system of polynomial equations over \( R[T] \), which has a solution in \( \hat{R}[T] \), also has a solution in \( R[T] \).

**Proof.** We give a proof using the ultraproduct construction (see e.g. [CK or BDLV, §1]), although a classical proof would be as easy. Since \( R \) has the Approximation Property, for every subring \( S \) of \( \hat{R} \) which is finitely generated over \( R \), there exists an \( R \)-algebra homomorphism \( \phi_S: S \to R \).

Let \( I \) be the set of all subrings of \( \hat{R} \) which are finitely generated over \( R \). Choose an ultrafilter \( D \) on \( I \) such that for every \( S_0 \in I \) we have \( \{ S \in I: S_0 \subseteq S \} \in D \). The maps \( \phi_S \) induce an \( R \)-algebra homomorphism

\[
\phi^*: \prod_{S \in I} S/D \to R^* = \prod_{S \in I} R/D.
\]

Consider the map

\[
\theta: \hat{R} \to \prod_{S \in I} S/D: a \mapsto (a_S)_{S \in I} \mod D
\]
where
\[ a_S = a, \quad \text{if } a \in S, \]
\[ a_S = 0, \quad \text{if } a \notin S. \]

It is easy to verify that \( \theta \) is an \( R \)-algebra homomorphism. Thus we have an \( R \)-algebra homomorphism \( \psi = \phi^* \circ \theta: \hat{R} \to R^* \). The ultraproduct \( R^* \) is a local ring (not Noetherian), and \( \psi \) is a local homomorphism (because the maximal ideal of \( \hat{R} \) is generated by the maximal ideal \( m \) of \( R \), and \( \psi \) is an \( R \)-algebra map).

There is a canonical map
\[ R^* \to (R[T])^* = \prod_{S \in I} (R[T])/D. \]

Thus \( \psi: \hat{R} \to R^* \) extends to a local \( R[T] \)-algebra homomorphism
\[ \psi: \hat{R}[T]_{(m,T)} \to (R[T])^*. \]

But \((R[T])^*\) is a local Henselian ring (see [BDLV, §1]), thus, by the universal property of Henselization [EGA, IV, 18.6.6], \( \psi \) extends to an \( R[T] \)-algebra (and in fact an \( R[T] \)-algebra) homomorphism
\[ \tilde{\psi}: \hat{R}[T] \to (R[T])^*. \]

Thus every system of polynomial equations over \( R[T] \), which has a solution in \( \hat{R}[T] \), has a solution in \((R[T])^*\), and hence also in \( R[T] \). Q.E.D.

1.5. Remark. Observe that it follows from the above proof that if in Lemma 1.4 some of the coordinates of the solution are in the subrings \( \hat{R}[T_i, T_j] \), \( i > 0 \), then the new solution can be chosen so that the corresponding coordinates are in the corresponding subrings \( R[T_i, T_j] \). Conjecture 1.2' can be derived from Conjecture 1.2 as follows: Assume the hypothesis of 1.2'. Use 1.2 to get a solution in \( k[[X_1, \ldots, X_r]][X_{r+1}, \ldots, X_n] \) satisfying (1), by fixing \( \bar{y}_1, \ldots, \bar{y}_s \). Now use the above-mentioned strengthened version of Lemma 1.4 \( r \) times in succession to get down to a solution in \( k[[X_1, \ldots, X_n]] \) satisfying (1). (In the \( j \)th use of 1.4 take \( R = k[[X_1, \ldots, X_r]][X_{r-j+1}] \) and \( T = (X_{r-j+1}, \ldots, X_n) \), and fix \( \bar{y}_1, \ldots, \bar{y}_{s-r} \). These rings \( R \) have the Approximation Property by 1.2.)

2. Global Néron \( p \)-desingularization. Let \( B \) be a finitely generated \( A \) algebra and \( \mathfrak{p} \) a prime ideal of \( B \). We say that \( B \) is smooth over \( A \) at \( \mathfrak{p} \) if \( \text{Spec } B \) is smooth over \( \text{Spec } A \) at \( \mathfrak{p} \in \text{Spec } B \) (see e.g. [A3, pp. 80–81]).

2.1. Theorem (Néron \( p \)-desingularization). Let \( \Lambda \subset \Lambda' \) be discrete valuation rings, and let \( p \) be a local parameter of \( \Lambda \). Suppose that \( \Lambda' \) is unramified over \( \Lambda \) (i.e. \( p \) is also a local parameter of \( \Lambda' \)) and suppose that the residue field of \( \Lambda' \) is separable over the residue field of \( \Lambda \). Let \( B \) be a subring of \( \Lambda' \) which is finitely generated over \( \Lambda \), such that \( \text{Frac}(B) \) is separable over \( \text{Frac}(\Lambda) \). (Frac denotes the fraction field.) Then there exists a subring \( C \) of \( \Lambda' \), containing \( B \), such that \( C \) is finitely generated over \( \Lambda \) and smooth over \( \Lambda \) at the prime ideal \( C \cap p\Lambda' \), and such that \( C \subset S^{-1}B \), where \( S = \{ p^e : e \in \mathbb{N} \} \).
This is an immediate consequence of Néron’s $p$-desingularization \([N]\) (see \([A1, \S 4]\)).

The next theorem is a global version of Néron’s $p$-desingularization and is due to Pfister and Popescu \([PP]\).

2.2. Theorem (Global Néron $p$-desingularization). Let $A \subset A'$ be Noetherian Unique Factorisation Domains. Suppose for every prime element $p$ of $A$, that $p$ remains prime in $A'$ and that $A \cap pA' = pA$. Suppose that $\text{Frac}(A')$ is separable over $\text{Frac}(A)$ and that $\text{Frac}(A'/qA')$ is separable over $\text{Frac}(A/A \cap qA')$, for every prime element $q$ of $A'$. Suppose that there exists an infinite set of units of $A'$ which are algebraically independent over $A$. Let $B$ be a subring of $A'$ which is finitely generated over $A$ and smooth over $A$ at $C \cap qA'$ for every prime element $q$ of $A'$.

Proof. It follows from separability that $B$ is smooth over $A$ at the prime ideal $(0)$. Hence there are only a finite number of prime ideals of the form $qA'$, such that $B$ is not smooth over $A$ at $B \cap qA'$. Hence, by the transitivity of smoothness, it is sufficient to prove that for every subring $B$ of $A'$, which is finitely generated over $A$, and for every prime element $q$ of $A'$, there exists a subring $C$ of $A'$, containing $B$, such that (i) $C$ is finitely generated over $A$, (ii) $C$ is smooth over $A$ at $C \cap qA'$, and (iii) $C$ is smooth over $B$ at $C \cap q'A'$ for every prime element $q'$ of $A'$ with $q'A' \neq qA'$.

Let $q$ be a fixed prime element of $A'$. There are two cases:

Case 1. $A \cap qA' \neq (0)$. Then there exists a prime element $p$ of $A$ such that $p \in qA'$. Since $p$ remains prime in $A'$, we have $pA' = qA'$. Thus we may as well suppose that $q \in A$, and $q$ is a prime element in both $A$ and $A'$. Moreover we have $A \cap qA' = qA$ and $A_qA \subset A_qA'$ are discrete valuation rings. Let $U = A \setminus qA$. The conditions of Theorem 2.1 are satisfied for $\Lambda = A_qA \subset U^{-1}B \subset \Lambda' = A_qA'$. Thus there exists a subring $D$ of $A_qA'$, containing $U^{-1}B$, such that $D$ is finitely generated over $A_{qA}$ and smooth over $A_{qA}$ at $D \cap qA'_q$, and such that $D \subset S^{-1}U^{-1}B$, where $S = \{q^e : e \in \mathbb{N}\}$. Let $y_1, \ldots, y_s$ be generators for $D$ over $A_{qA}$. Then there are $e \in \mathbb{N}$ and $u \in U$ such that $q^eu_i \in D$, for $i = 1, \ldots, s$. Since $q^eu_i \in A'$ and $u \in A_qA'$, we have $u_i \in A'$. Let $C = B[y_1, \ldots, y_s] \subset A'$. We have $C \subset S^{-1}B$, thus $C$ is smooth over $B$ at $C \cap qA'$ for every prime element $q'$ of $A'$ with $q'A' \neq qA'$. Moreover $U^{-1}C = D$ is smooth over $U^{-1}A = A_{qA}$ at $D \cap qA'_q$. Hence \([EGA, IV, 17.7.1]\), $C$ is smooth over $A$ at $C \cap (D \cap qA'_q) = C \cap qA'$. This completes the treatment of Case 1.

Case 2. $A \cap qA' = (0)$. We may suppose that $q$ is transcendental over $B$. (Otherwise multiply $q$ with a unit which is transcendental over $B$.) Then $A[q]$ is a Noetherian UFD, and $A[q]_{qA[q]}$ is a discrete valuation ring. We have $A[q] \cap qA' = qA[q]$. Indeed if $x \in A[q]$ and $x \in qA'$, then $x - a \in qA[q]$ for some $a \in A$, hence $a \in qA$; thus $a = 0$ (since we are in Case 2) and $x \in qA[q]$. Thus we have $\Lambda = A[q]_{qA[q]} \subset \Lambda' = A_qA'$. Let $U = A[q] \setminus qA[q]$. The conditions of Theorem 2.1 are satisfied for $\Lambda \subset U^{-1}B[q] \subset \Lambda'$. By the same argument as in Case 1 we obtain a subring $C$ of $A'$, containing $B[q]$, such that (i) $C$ is finitely generated over $A[q]$, (ii) $C$ is smooth over $A[q]$ at $C \cap qA'$, and (iii) $C$ is smooth over $B[q]$ at $C \cap q'A'$. for
every prime element \( q' \) of \( A' \) with \( q'A' \neq qA' \). Since \( q \) is transcendental over \( B \), we have that \( B[q] \) is smooth over \( B \) and \( A[q] \) is smooth over \( A \). The theorem now follows by the transitivity of smoothness. Q.E.D.

2.3. COROLLARY. Let

\[
A_0 = k[[X_1, \ldots, X_r]][X_{r+1}, \ldots, X_n],
A = k[[X_1, \ldots, X_r]][X_{r+1}, \ldots, X_n], \quad \text{and} \quad \hat{A} = k[[X_1, \ldots, X_n]],
\]

where \( k \) is a field of characteristic zero. Let \( B \) be a subring of \( \hat{A} \) which is finitely generated over \( A_0 \). Then there exists a subring \( C \) of \( \hat{A} \), containing \( B \), such that \( C \) is finitely generated over \( A_0 \) and smooth over \( A_0 \) at \( C \cap q\hat{A} \), for every prime element \( q \) of \( \hat{A} \).

PROOF. The pair \( A \subset \hat{A} \) satisfies the hypothesis of Theorem 2.2 (see [EGA, IV, 18.7.6 and 18.9.2]). Moreover, it follows easily from the definition of Henselization [EGA, IV, 18.6.5] that every subring \( D \) of \( A \), which is finitely generated over \( A_0 \), is contained in a subring \( A_1 \) of \( A \) such that \( A \) is flat over \( A_1 \), and \( A_1 \) is finitely generated over \( A_0 \) and étale over \( A_0 \) at \( A_1 \cap (X_1, \ldots, X_n)\hat{A} \). (Indeed, notice that the maps \( \phi_{\alpha_k} \) in [EGA, IV, 18.6.5] are faithfully flat, and hence injective.) Let \( B = A_0[y_1, \ldots, y_n] \), and let \( C' = A[y_1, \ldots, y_n] \) be a subring of \( \hat{A} \) such that \( C' \) is smooth over \( A \) at every \( C' \cap q\hat{A} \) (cf. Theorem 2.2). Let \( f_1, \ldots, f_r \in A[Y_1, \ldots, Y_m] \) be generators for the ideal \( \{ f \in A[Y_1, \ldots, Y_m] : f(y_1, \ldots, y_n) = 0 \} \). Let \( A_1 \) be as above and containing the coefficients of \( f_1, \ldots, f_r \). Let \( C = A_1[y_1, \ldots, y_m] \); then \( C' \simeq C \otimes_{A_1} A \). From [EGA, IV, 17.7.1] it follows that \( C \) is smooth over \( A_1 \) at every \( C \cap q\hat{A} \). The corollary now follows from the transitivity of smoothness. Q.E.D.

3. Proof of Theorem 1.3. Let \( k \) be a field of characteristic zero,

\[
A_0 = k[[X_1, X_2]][X_3, X_4, X_5],
A = k[[X_1, X_2]][X_3, X_4, X_5] \quad \text{and} \quad \hat{A} = k[[X_1, X_2, X_3, X_4, X_5]].
\]

We use the following notation: \( X_{12} = (X_1, X_2), X_{345} = (X_3, X_4, X_5), X_{1234} = (X_1, X_2, X_3, X_4) \), etc. We have to prove that every system of polynomial equations over \( A \), which has a solution in \( \hat{A} \), also has a solution in \( A \). Since \( A \) is algebraic over \( A_0 \), we may suppose that the equations have coefficients in \( A_0 \) by introducing more equations and congruences if necessary. Thus we have to prove that for every subring \( B \) of \( \hat{A} \), which is finitely generated over \( A_0 \), there exists an \( A_0 \)-algebra homomorphism \( B \to A \). It follows from Corollary 2.3 that we may suppose that \( B \) is smooth over \( A_0 \) at \( B \cap q\hat{A} \), for every prime element \( q \) of \( \hat{A} \). Let \( B = A_0[\tilde{y}_1, \ldots, \tilde{y}_N] \), with \( \tilde{y}_1, \ldots, \tilde{y}_N \in \hat{A} \). Let \( f_1(Y), \ldots, f_m(Y) \in A_0[Y] \) be generators for the ideal \( \{ f(Y) \in A_0[Y] : f(\tilde{y}) = 0 \} \), where \( Y = (Y_1, \ldots, Y_N) \) and \( \tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_N) \). Thus \( f_i(\tilde{y}) = 0 \) for \( i = 1, \ldots, m \). We have to prove that there exists \( y = (y_1, \ldots, y_N) \in A \), such that \( f_i(y) = 0 \) for \( i = 1, \ldots, m \). But by Lemma 1.4 and induction, it is sufficient to prove that there exists \( y = (y_1, \ldots, y_N) \in k[[X_{1234}]][X_3] \), such that \( f_i(y) = 0 \) for \( i = 1, \ldots, m \). Choose \( \delta(Y), \ldots, \delta(Y) \in A_0[Y] \) such that (i) for every prime ideal \( \mathfrak{p} \) of \( B \), \( B \) is smooth over \( A_0 \) at \( \mathfrak{p} \) if and only if there is an \( i \) such that \( \delta_i(\tilde{y}) \not\in \mathfrak{p} \), and (ii) the
ideal $H_B = (\delta_1(Y), \ldots, \delta_s(Y))A_0[Y]$ satisfies the condition in [E, 0.2, p. 555]. Since $B$ is smooth over $A_0$ at $B \cap qA$, we have that $(\delta_1(\bar{y}), \ldots, \delta_s(\bar{y}))\hat{A} \not\subseteq q\hat{A}$ for every prime element $q$ of $\hat{A}$. Thus the height of the ideal $(\delta_1(\bar{y}), \ldots, \delta_s(\bar{y}))\hat{A}$ is not smaller than two. Thus we have

$$\sqrt{(\delta_1(\bar{y}), \ldots, \delta_s(\bar{y}))\hat{A}} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r,$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are prime ideals in $\hat{A}$ with height not smaller than two. Hence if $\mathfrak{p}_j \subset (X_1, X_2)\hat{A}$, then $\mathfrak{p}_j = (X_1, X_2)\hat{A}$. Hence there exist $g \in \hat{A}$, with $g \notin (X_1, X_2)\hat{A}$, and $r \in \mathbb{N}$, such that

$$(1) \quad X'_ig \in (\delta_1(\bar{y}), \ldots, \delta_s(\bar{y}))\hat{A}, \quad X'_ig \in (\delta_1(\bar{y}), \ldots, \delta_s(\bar{y}))\hat{A}.$$

After a linear change of coordinates among $X_3, X_4$ and $X_5$, we may suppose that $g$ is regular in $X_5$ (as a formal power series; see e.g. [ZS, p. 145]), because $g \notin (X_1, X_2)\hat{A}$. Let $w \in k[[X_{1234}]]X_5$ be the distinguished pseudopolynomial associated with $g$ (see e.g. [ZS, p. 146]). Let $A_1 = k[[X_{1234}]]X_5$ and $\hat{A} = w(X'_i, X'_j)A_1$. Applying Elkik’s theorem [E, Théorème 2, p. 560] to the Henselian pair $(A_1, \hat{A})$, we see that it is sufficient to prove that there exists $y \in A_1^N$ such that

$$(2) \quad f_i(y) \equiv w^e(X'_i, X'_j)A_1, \quad i = 1, \ldots, m,$$

and

$$(3) \quad wX'_i, wX'_j \in (\delta_1(y), \ldots, \delta_s(y))A_1,$$

where $e \in \mathbb{N}$ is big enough.

We are going to use the

3.1. CONGRUENCE LEMMA. Let $k$ be a field of characteristic zero. Let $w \in k[[X_{1234}]]X_5$ be a distinguished pseudopolynomial (with respect to $X_5$) and $l \in \mathbb{N}$. Every system of polynomial equations over $k[[X_{1234}]]X_5$ which has a solution in $k[[X_{1234}]]X_5$ also has a solution in $k[[X_{1234}]]X_5/w(X'_i, X'_j)$.

We prove Lemma 3.1 later, and proceed first with the proof of Theorem 1.3. Define

$$G(Z, Y) = \sum_{i=1}^s Z_i \delta_i(Y) \in A_0[Y, Z], \quad Z = (Z_1, \ldots, Z_s).$$

It follows from (1) that there exist $\bar{z}_1 \in \hat{A}, \bar{z}_2 \in \hat{A}$ such that

$$wX'_i = G(\bar{z}_1, \bar{y}), \quad wX'_j = G(\bar{z}_2, \bar{y}).$$

From Lemma 3.1 it follows that there exist $y \in A_1^N, z_1 \in A_1^N, z_2 \in A_1^N$, such that

$$f_i(y) \equiv_{A_1} 0, \quad wX'_i \equiv_{A_1} G(z_1, y), \quad wX'_j \equiv_{A_1} G(z_2, y) \mod w^e(X'_i, X'_j),$$

where $\equiv_{A_1}$ denotes congruence in $A_1$.

Thus (2) holds and we prove now that (3) is also satisfied. It follows from the last two congruences that there exist $v_1, v_2, v_3, v_4 \in A_1$ such that

$$wX'_i = G(z_1, y) + v_1 w^eX'_i + v_2 w^eX'_j,$$

$$wX'_j = G(z_2, y) + v_3 w^eX'_i + v_4 w^eX'_j.$$
This can be written as
\[
(1 - v_1 w_{e-1} X_1^{(e-1)})(w X_1') - (v_2 w_{e-1} X_2^{(e-1)})(w X_2') = G(z_1, y),
\]
\[
-v_3 w_{e-1} X_1^{(e-1)}(w X_1') + (1 - v_4 w_{e-1} X_2^{(e-1)})(w X_2') = G(z_2, y).
\]

We consider this as a system of two linear equations with two unknowns \(w X_1'\) and \(w X_2'\). The determinant of this system is congruent to 1 mod \((X_1, X_2)\) (we may suppose \(r > 0, e > 1\)) and hence a unit in \(A_1\).

Solving for \(w X_1'\), \(w X_2'\), we obtain \(w X_1', w X_2' \in (G(z_1, y), G(z_2, y))A_1\). From the definition of \(G(z, y)\) we have that
\[
(G(z_1, y), G(z_2, y))A_1 \subset (\delta_1(y), \ldots, \delta_n(y))A_1.
\]
This proves (3) and the proof of Theorem 1.3 is completed if we prove Lemma 3.1.

**Proof of Congruence Lemma 3.1.** Let \(A = k[[X_{1234}]]\) and \(A_1 = k[[X_{1234}]][X_5]\), as before. Let \(h_i(Y) \in k[[X_{1234}]][X_5][Y], i = 1, \ldots, m, Y = (Y_1, \ldots, Y_m)\).

Suppose there exists \(y \in \hat{A}^N\) such that \(h_i(y) = 0\) for \(i = 1, \ldots, m\). We have to prove that there exists \(v \in A_1^N\) such that
\[
h_i(y) \equiv_A 0 \mod w.(X_1', X_2'), \quad i = 1, \ldots, m,
\]
where \(\equiv_A\) denotes congruence in the ring \(A_1\).

By the Weierstrass Preparation Theorem we can write
\[
y = y_0 + w\bar{q} \quad \text{with } y_0 \in k[[X_{1234}]][X_5]^N \quad \text{and } \bar{q} \in \hat{A}^N.
\]
Moreover we can write
\[
\bar{q} = \tilde{y}_1 + X_1'\tilde{q}_1 + X_2'\tilde{q}_2 \quad \text{with } \tilde{y}_1 \in k[[X_{345}]](X_5)^N \quad \text{and } \tilde{q}_1, \tilde{q}_2 \in \hat{A}^N.
\]
Define
\[
(4) \quad \tilde{y} = y_0 + w\tilde{y}_1.
\]
Thus we have
\[
(5) \quad \bar{y} = \bar{y} = \bar{y} + wX_1'\tilde{q}_1 + wX_2'\tilde{q}_2.
\]
Let \(B = k[[X_{345}]] \cdot k[[X_{1234}]]\) be the compositum of the two rings \(k[[X_{345}]]\) and \(k[[X_{1234}]]\) in \(A\). We have \(\tilde{y} \in B\). From \(h_i(\bar{y}) = 0\) and (5), follows
\[
h_i(\bar{y}) \equiv_A 0 \mod w.(X_1', X_2'), \quad \text{for } i = 1, \ldots, m,
\]
where \(\equiv_A\) denotes congruence in the ring \(A\).

We are going to prove that
\[
h_i(\tilde{y}) \equiv_B 0 \mod w.(X_1', X_2'), \quad \text{where } \equiv_B\text{ denotes congruence in the ring } B.
\]

From (4) we have that
\[
(6) \quad h_i(\tilde{y}) \equiv_B h_i(y_0) \mod w,
\]
and from (7) and (6) that
\[
h_i(y_0) \equiv_A 0 \mod w.
\]
Now \(h_i(y_0)\) and \(w\) are in \(k[[X_{1234}]]\langle X_5\rangle\), and \(w\) is a distinguished pseudopolynomial. Hence by [ZS, p. 146] we have that
\[
h_i(y_0) \equiv c \mod w,
\]
where \(C = k[[X_{1234}]]\langle X_5\rangle\). Combining this with (7) we obtain
\[
h_i(y) \equiv b \mod w.
\]
Thus there exist \(a_i \in B\) with \(h_i(y) = wa_i\). It follows from (6) that \(a_i \equiv \delta \mod (X'_1, X'_2)\). This implies \(a_{i_j} \equiv 0 \mod (X'_1, X'_2)\) and (6') follows. Indeed, suppose \(a \in B\) and \(a \equiv 0 \mod (X'_1, X'_2)\), we will prove that \(a \equiv 0 \mod (X'_1, X'_2)\). Every element in \(k[[X_{1234}]]\) is congruent in \(B\) to an element of \(k[[X_{1234}]]\langle X_{12}\rangle \mod (X'_1, X'_2)\). Thus there exists \(c \in k[[X_{345}]]\langle X_{12}\rangle\) with \(a \equiv c \mod (X'_1, X'_2)\). Hence \(c \equiv 0\). Thus \(c \in (X'_1, X'_2)k[[X_{345}]]\langle X_{12}\rangle\). Hence \(a \equiv 0\). This finishes the proof of (6').

Congruence Lemma 3.1 now follows at once from (6'), and the following:

**Claim.** Every system of polynomial equations over \(k[[X_{1234}]]\langle X_5\rangle\), which has a solution in \(B\), also has a solution in \(A_1\).

**Proof of the Claim.** Let \(F(Z) \in k[[X_{1234}]]\langle X_5\rangle[Z]^m, Z = (Z_1, \ldots, Z_N)\). Suppose there exists \(\tilde{z} \in B^N\) with \(F(\tilde{z}) = 0\). We have to prove that there exists \(z \in A_1^N\) with \(F(z) = 0\). Now, \(\tilde{z} \in B^N\) can be written as \(\tilde{z} = E(\tilde{u})\), with \(E(U) \in k[[X_{1234}]]\langle U \rangle^N, U = (U_1, \ldots, U_s)\), and \(\tilde{u} \in k[[X_{345}]]^s\). Thus \(F(E(\tilde{u})) = 0\). We can write
\[
F(E(U)) = \sum_{i,j} C_{ij}(U) X_i X_j,
\]
with
\[
C_{ij}(U) \in k[[X_{34}]]\langle X_5\rangle[U]^m.
\]
We have \(C_{ij}(\tilde{u}) = 0\), for all \(i, j \in \mathbb{N}\). By Noetherianess, there is a finite set \(S \subset \mathbb{N}\) such that the equations \(C_{ij}(U) = 0\) for all \(i, j\), are implied by the finite set of equations \(C_{ij}(U) = 0, i, j \in S\).

First we prove the Claim in the special case that \(X_3\) and \(X_4\) do not appear. Then, by Greenberg’s theorem [G], there exists \(u \in (k[X_5])^s\) such that \(C_{ij}(u) = 0\) for \(i, j \in S\), and hence also for all \(i, j \in \mathbb{N}\). Thus \(F(E(u)) = 0\) and \(E(u) \in (k[[X_{12}]]\langle X_5\rangle)^N\).

This proves the Claim, and hence Lemma 3.1 and Theorem 1.3, in the special case that \(X_3\) and \(X_4\) do not appear (the 3-dimensional case). Thus \(k[[X_1, X_2]][X_3]\) has the Approximation Property. Thus also \(k[[X_{34}]][X_5]\) has the Approximation Property. Thus also in the general case, there exists \(u \in (k[[X_{34}]]\langle X_5\rangle)^s\) such that \(C_{ij}(u) = 0\) for \(i, j \in S\), and hence also for all \(i, j \in \mathbb{N}\). Let \(z = E(u)\). Then \(F(z) = 0\) and \(z \in (k[[X_{1234}]]\langle X_5\rangle)^N\). This proves the claim. Q.E.D.

**Added in proof.** Theorem 1.3 is also true when \(k\) is a field of nonzero characteristic. This follows by using a generalization of Theorem 2.2 as in D. Popescu, *Global forms of Néron’s p-desingularization and approximation*, Teubner Texte Bd. 40, Teubner, Leipzig, 1981.
References