BRANCHED COVERINGS. I

BY

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ABSTRACT. This paper analyzes the possible cobordism classes \([M] - (\text{deg } \phi)[N]\) for \(\phi: M \rightarrow N\) a smooth branched covering of closed smooth manifolds. It is assumed that the branch set is a codimension 2 submanifold. The results are a fairly complete description in the unoriented case, a partial description in the oriented case, and a detailed analysis of the case in which \(N\) is a sphere.

1. Introduction. The purpose of this note is to describe the possible cobordism classes \([M] - (\text{deg } \phi)[N]\) where \(\phi: M \rightarrow N\) is a smooth branched covering of closed smooth manifolds.

It is well known that for a genuine covering \(\phi: M \rightarrow N\) one has \([M] = (\text{deg } \phi)[N]\) in unoriented cobordism or in oriented cobordism if \(M\) and \(N\) are oriented manifolds. Thus, the class \([M] - (\text{deg } \phi)[N]\) depends entirely upon the branching behavior. For this definition, the choice here is to follow Berstein and Edmonds [2] including a smoothness hypothesis or more specifically Brand [3] since the differentiable structures will be assumed to satisfy his regularity condition. Briefly then,

**Definition.** A branched covering is a smooth map \(\phi: M^n \rightarrow N^n\) between smooth manifolds which is finite-to-one and an open map. The singular set \(\Sigma_\phi\) is the set of points of \(M\) at which \(\phi\) is not a local homeomorphism, and the branch set \(B_\phi\) is the image under \(\phi\) of the singular set. Assume that the branch set is a smooth codimension 2 submanifold of \(N\).

According to [2], the map \(\phi: \phi^{-1}B_\phi \rightarrow B_\phi\) is then an ordinary covering and looks like a union of maps \(\bigcup B_{ij} \rightarrow B_i\), where \(B_i\) is a component of \(B_\phi\) and each \(B_{ij} \rightarrow B_i\) is a covering of degree \(r_{ij}\). If \(v_{ij}\) is the normal bundle of \(B_{ij}\) in \(M\) and \(v_i\) the normal bundle of \(B_i\) in \(N\), then \(\phi^*v_i|B_{ij}\) looks like a quotient of \(v_{ij}\) by an identification of degree \(d_{ij}\) (the local branching degree) on the fibers; i.e. locally \(\phi\) is the map

\[ R^{n-2} \times \mathbb{C} \rightarrow R^{n-2} \times \mathbb{C}: (x, z) \mapsto (x, zd^{d_{ij}}). \]

Of course, the local degrees add up, so that

\[ \text{degree } \phi = \sum_j r_{ij}d_{ij} \]

(which is constant on each component of \(N\)). Up to cobordism, the specific differential structure on \(M\) is irrelevant, and so additionally one assumes Brand’s conditions hold.

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Note. For the orthogonal 2-plane bundle \( v_{ij} \) over \( B_{ij} \), one may form a quotient \( \mu_{d_{ij}}(v_{ij}) \) by identifying vectors in fibers which differ by an angle which is an integral multiple of \( 2\pi/d_{ij} \). This is again a 2-plane bundle and is identified with \( \phi^*v_{ij}|_{B_{ij}} \). If one observes that two 2-plane bundles over a space having the same first Stiefel-Whitney class have a tensor product, \( \mu_{d_{ij}}(v_{ij}) \) is just the \( d_{ij} \)th tensor power of \( v_{ij} \).

From a cobordism standpoint, one should first observe that there is a cobordism group of branched coverings of degree \( d \) for closed manifolds, and that the Conner-Floyd [6] methods can actually be used successfully to analyze branched coverings with a fixed degree almost as if one were working with a group action. Specifically, one has an exact sequence

\[
\cdots \to \Omega_n(\text{d-fold cover}) \overset{i_*}{\to} \Omega_n(\text{d-fold branched cover})
\]

\[
\overset{j_*}{\to} \Omega_n(\text{d-fold branched cover, unbranched } \partial) \overset{\partial}{\to} \Omega_{n-1}(\text{d-fold cover}) \overset{i_*}{\to} \cdots
\]

in either oriented or unoriented cobordism. Clearly, \( \Omega_n(\text{d-fold cover}) \) is just the usual bordism of \( BS_d \) and by the work of Brand [4], \( \Omega(\text{d-fold branched cover}) \) is the usual bordism of Brand’s classifying space \( B_d \). The relative group is \( \Omega_n(B_d, BS_d) \) and is the reduced bordism of a certain wedge of very nasty Thom spaces (i.e. \( B_d \) is obtained from \( BS_d \) by attaching a union of disc bundles of 2-plane bundles by means of maps of their sphere bundles into \( BS_d \)).

In the special case when \( d = 2 \), a branched cover is nothing more than an involution with codimension two fixed point set, and, in fact, one completely understands the Conner and Floyd analysis of this case. One has

**Proposition 1.** Assigning to a 2-fold branched cover \( \phi: M^n \to N^n \) the class of \( N \) and the class of \( v \) over \( \phi^{-1}B_{\phi} \) restricted to the self-intersection of \( \phi^{-1}B_{\phi} \) in \( M \) defines isomorphisms

\[\Omega_n(\text{2-fold branched cover}) \cong \Omega_n \oplus \Omega_{n-4}(BO_2)\]

and

\[\mathcal{R}_n(\text{2-fold branched cover}) \cong \mathcal{R}_n \oplus \mathcal{R}_{n-4}(BO_2).\]

In the unoriented case, the analysis is not overly difficult and one finds

**Proposition 2.** If \( \phi: M^n \to N^n \) is a branched cover of closed manifolds, then \( [M^n, \phi] - (\deg \phi)[N^n, \text{identity}] \) in \( \mathcal{R}_n(N^n) \) is the class of the map

\[\bigcup \{ RP(v_{ij} \oplus 1) | d_{ij} \text{ is even} \} \to B_{\phi} \subset N,\]

i.e., one takes the union of the \( D(v_{ij})/\{x \sim -x \mid x \in S(v_{ij})\} \) for \( d_{ij} \) even, projects onto \( B_{ij} \), composites with \( \phi \) into \( B_i \), and then includes in \( N \).

**Proposition 3.** The set of classes \( [M^n] - d[N^n] \) in \( \mathcal{R}_n \) for \( \phi: M^n \to N^n \) a \( d \)-fold branched covering of closed \( n \) dimensional manifolds is

\[\{ \alpha \in \mathcal{R}_n \mid w_{ij}^{\phi}(\alpha) = 0 \},\]

if \( d \geq 2, n > 0 \).
In the oriented case, one has

**Proposition 4.** If \( \alpha \in \Omega_n, n > 0 \), there is an odd integer \( k \) and a branched covering \( \phi: M^n \to N^n \) of closed oriented manifolds with

\[
[M^n] - (\deg \phi)[N^n] = k\alpha \quad \text{in } \Omega_*. 
\]

**Note.** There is such an odd integer \( k \) for coverings of degree two. If one specifies the degree \( d \geq 2 \), the \( k \) needed may vary.

In the above one cannot take \( k = 1 \) in general. One has

**Proposition 5.** If \( n = 2k(p - 1) \) with \( p \) an odd prime and \( \alpha \in \Omega_n \) is the class \([M^n] - d[N^n]\) for some \( d \)-fold branched covering \( \phi: M^n \to N^n \) of closed oriented manifolds with \( d < 2p \), then one has

\[
s_{k(p-1)/2}(\alpha) \equiv 0 \quad (\text{mod } p)
\]

where \( s_m(\nu) \) is the primitive characteristic class.

**Remarks.** In particular, for \( n = 2k(p - 1) \) and \( n/2 + 1 \) not a power of \( p \), the class \( \alpha \) cannot be indecomposable in \( \Omega_*/p\Omega_* \).

A number of results will be given characterizing the values \( s_m(\nu)[\alpha] \) for \( \alpha \in \Omega_{4m} \) realized by a \( d \)-fold branched covering.

In later work, appearing as part II of this paper, it is shown that \( s_{i_1,((p-1)/2),\ldots,i_r,((p-1)/2)}(\nu)[\alpha] \equiv 0 \mod p \) for all \((i_1,\ldots,i_r)\) without the restriction \( d < 2p \), and precise divisibility of the numbers \( s_m(\nu)[\alpha] \) is obtained.

Recently, Edmonds proved [7] that no simply connected closed Spin 4-manifold of nonzero signature can be a 2-fold branched covering of the 4-sphere. His argument can be extended, and one has

**Proposition 6.** If \( \phi: M^n \to N^n \) is a branched covering of closed oriented manifolds with oriented branch set \( B_\phi \) and \( H^*(N; Q) = 0 \), then \([M^n] - (\deg \phi)[N^n] \in \text{Tor } \Omega_* \).

**Proposition 7.** If \( \phi: M^n \to N^n \) is a branched covering of closed oriented manifolds with \( H^4(N; Q) = 0 \), then \([M^n] - (\deg \phi)[N^n] \in \text{Tor } \Omega_* \).

**Proposition 8.** Let \( \phi: M^n \to S^n \) be a branched covering with \( M^n \) closed and if \( n = 4 \) assume \( B_\phi \) orientable. Then \( M^n \) is orientable and \([M^n] \in \text{Tor } \Omega_* \). If \( M^n \) is a Spin manifold or if \( B_\phi \) is orientable, then \([M^n] = 0 \) in \( \Omega_* \).

By calculating Stiefel-Whitney numbers one then has

**Proposition 9.** If \( \phi: M^n \to S^n \) is a branched covering then \( M^n \) bounds (as oriented manifold) if \( n \) is even and greater than 4 or if \( n \) is odd and \( n + 1 \) does not have exactly two ones in its dyadic expansion.

**2. Involutions.** First, one considers 2-fold branched covers \( \phi: M^n \to N^n \) and defines an involution \( \iota: M \to M \) by the condition that \( \phi^{-1}\phi(m) = \{m, tm\} \). The fixed point set of \( \iota \) is \( \phi^{-1}B_\phi = B_\phi \). If in the oriented case, one insists that the orientation of \( N \) on \( N - B_\phi \) lift back to the orientation of \( M - \phi^{-1}B_\phi \), and with that choice, \( \iota \) is an orientation preserving involution.
For the unoriented case, one has the Conner and Floyd [6, (28.1)] exact sequence

\[ 0 \to \mathcal{R}_n \xrightarrow{F} \bigoplus_k \mathcal{R}_{n-k}(BO_k) \xrightarrow{\partial} \mathcal{R}_{n-1}(BZ_2) \to 0 \]

and restricting to codimension 2 fixed point set, a corresponding exact sequence

\[ \cdots \to \mathcal{R}_n \xrightarrow{F} \mathcal{R}_{n-2}(BO_2) \xrightarrow{\partial} \mathcal{R}_{n-1}(BZ_2) \to \cdots \]

where \( F \) takes the fixed point set with its normal bundle, \( \partial \) assigns to \( \xi^2 \to F \) the antipodal involution on \( S(\xi^2) \) or the double cover \( S(\xi^2) \to RP(\xi^2) \), and \( i \) assigns to a double covering the class of the free involution on the total space.

For an involution with codimension 2 fixed point set \( (M^n, t) \), the quotient \( M^n/t \) is again a manifold giving a homomorphism

\[ q: \mathcal{R}_n^{Z_2}(\text{cod } 2) \to \mathcal{R}_n \]

and the composite

\[ qi: \mathcal{R}_n(BZ_2) \to \mathcal{R}_n \]

is the augmentation \( \epsilon[N \to BZ_2] = [N] \), and so image \( \partial \) is contained in \( \mathcal{R}_n(BZ_2) = \ker \epsilon \). Alternatively, \( RP(\xi^2) \) is an \( S^1 \) bundle over \( F \) and hence bounds.

From [6, (26.4)], one has a commutative diagram

\[
\begin{array}{ccc}
\mathcal{R}_{n-2}(BO_2) & \xrightarrow{\partial} & \mathcal{R}_{n-1}(BZ_2) \\
\uparrow I_* & & \downarrow \Delta \\
\mathcal{R}_{n-2}(BO_1) & \xrightarrow{\partial} & \mathcal{R}_{n-2}(BZ_2) \\
\end{array}
\]

where \( I_* \) is the Whitney sum with a trivial line bundle, \( \Delta \) is the Smith homomorphism, and the lower homomorphism \( \partial \) is an isomorphism. Now \( \Delta: \mathcal{R}_{n-1}(BZ_2) \to \mathcal{R}_{n-2}(BZ_2) \) has kernel given by the classes of the trivial double covers (the class of \( P[S^0, A] \)), and maps \( \mathcal{R}_{n-1}(BZ_2) \) isomorphically to \( \mathcal{R}_{n-2}(BZ_2) \) (on bases \( \Delta([S^k, A] - RP^k[S^0, A]) = [S^{k-1}, A] \)).

Thus the Conner-Floyd exact sequence becomes

\[ 0 \to \mathcal{R}_n \xrightarrow{i} \mathcal{R}_n^{Z_2}(\text{cod } 2) \xrightarrow{F} \mathcal{R}_{n-2}(BO_2) \xrightarrow{\partial} \mathcal{R}_{n-1}(BZ_2) \to 0 \]

with \( q: \mathcal{R}_n^{Z_2}(\text{cod } 2) \to \mathcal{R}_n \) splitting \( i \) and with \( \ker \partial \) being identified with the cokernel of

\[ I_*: \mathcal{R}_{n-2}(BO_1) \to \mathcal{R}_{n-2}(BO_2). \]

One has the cofibration \( BO_1 \to BO_2 \to MO_2 \) from which \( \ker I_* \cong \mathcal{R}_{n-2}(MO_2) \cong \mathcal{R}_{n-4}(BO_2) \), and recognizing the composite

\[ \mathcal{R}_n^{Z_2}(\text{cod } 2) \xrightarrow{F} \mathcal{R}_{n-2}(BO_2) \to \mathcal{R}_{n-2}(MO_2) \cong \mathcal{R}_{n-4}(BO_2) \]
as being the selfintersection of \( F \) with the restriction of the normal bundle, one obtains

**Proposition 1'.** \( \Omega_n^{Z_2}(\text{cod } 2) \cong \Omega_n \oplus \Omega_{n-4}(BO_2) \) with the isomorphism assigning to \((M^n, t)\) the class of \( M^n/t \) and of \( F_{n-2} \cap F_{n-2} \) with bundle \( \nu^2 \big|_{F \cap F} \).

The oriented case is slightly more difficult with the analogue of the Conner-Floyd sequence being

\[
\cdots \to \Omega_n^{Z_2} \xrightarrow{F} \bigoplus_k \check{\Omega}_n(MO_{2k}) \xrightarrow{\delta} \Omega_{n} - (BZ_2) \xrightarrow{\iota} \cdots
\]

with the relative group of orientation preserving involutions which are free on the boundary being identified with \( \bigoplus_k \check{\Omega}_n(MO_{2k}) \) by assigning to \((V^n, t)\) the class of the map \( V^n \to \bigvee_k MO_{2k} \) sending a tubular neighborhood of \( F_{n-2k} \); i.e., \( D(v^{2k}) \), to \( MO_{2k} \) by \( F_{n-2k} \to BO_{2k} \) and extending to the bundles \( D(v^{2k}) \to D(\gamma_{2k}) \to MO_{2k} \) where \( c \) is the collapse, and sending the complement of these tubular neighborhoods to the common basepoint. (Orientation preserving involutions were first analyzed by Rosenzweig [15], but this description is due to Lee and Wasserman [12, p. 206].)

Restricting to a codimension 2 fixed point set gives

\[
\cdots \to \Omega_n^{Z_2}(\text{cod } 2) \xrightarrow{F} \check{\Omega}_n(MO_{2}) \xrightarrow{\delta} \Omega_{n} - (BZ_2) \xrightarrow{\iota} \cdots
\]

and as before one has \( q: \Omega_n^{Z_2}(\text{cod } 2) \to \Omega_n \) sending \((M^n, t)\) to the class of \( M^n/t \), with \( q_! = e: \Omega_n(BZ_2) \to \Omega_n \) so that image \( \delta \subset \check{\Omega}_n(BZ_2) \cong \Omega_{n-2} \).

**Note.** This isomorphism is due to Atiyah [1] and assigns to \( f: P^{n-1} \to BZ_2 \) with \( \partial P \) mapping to the base point, with \((BZ_2, e)\) being thought of as \((MO_1, \infty)\) the submanifold \( Q^{n-2} \subset P^{n-1} \) obtained by making the map transverse to \( BO_1 \subset MO_1 \), \( Q \) is an unoriented manifold and its normal line bundle in \( P \) is just the orientation bundle.

Now consider the bundle \( \pi^*\gamma_2 \to D(\gamma_2) \) where \( \gamma_2 \) is the universal 2-plane bundle over \( BO_2 \), for which one has a cofibration sequence

\[
T\left(\pi^*\gamma_2 \mid S(\gamma_2)\right) \to T(\pi^*\gamma_2) \to D(\pi^*\gamma_2)/S(\pi^*\gamma_2) \cup D\left(\pi^*\gamma_2 \mid S(\gamma_2)\right)
\]

where \( T \) denotes the Thom space. The projection \( \pi: D(\gamma_2) \to BO_2 \) is a homotopy equivalence and so \( T(\pi^*\gamma_2) \cong MO_2 \). The sphere bundle \( S(\gamma_2) \) may be identified with \( BO_1 \) with the projection onto \( BO_2 \) pulling \( \gamma_2 \) back to \( \gamma_1 \oplus 1 \), so that \( T(\pi^*\gamma_2 \mid S(\gamma_2)) \) may be identified with \( T(\gamma_1 \oplus 1) \cong \Sigma MO_1 \) and so that the map \( \Sigma MO_1 \to MO_2 \) is induced by \( BO_1 \to BO_2 \) classifying the Whitney sum of \( \gamma_1 \) with a trivial line. Finally, the disc bundle \( D(\pi^*\gamma_2) \) is the disc bundle of \( \gamma_2 \oplus \gamma_2 \) over \( BO_2 \) and collapsing \( S(\pi^*\gamma_2) \cup D(\pi^*\gamma_2 \mid S(\gamma_2)) \cong S(\gamma_2 \oplus \gamma_2) \) makes the cofiber just \( M(\gamma_2 \oplus \gamma_2) \). This one has a cofibration

\[
\Sigma MO_1 \to MO_2 \to M(\gamma_2 \oplus \gamma_2).
\]
Applying the function \( \tilde{\Omega}_n \), one has an exact sequence
\[
\cdots \to \tilde{\Omega}_n(\Sigma MO_1) \to \tilde{\Omega}_n(M O_2) \to \tilde{\Omega}_n(M(\gamma_2 \oplus \gamma_2)) \to \tilde{\Omega}_{n-1}(\Sigma MO_1) \to \cdots
\]
and since \( \gamma_2 \oplus \gamma_2 \) is an oriented vector bundle, one has a Thom isomorphism
\[
\tilde{\Omega}_n(M(\gamma_2 \oplus \gamma_2)) \cong \Omega_{n-4}(BO_2),
\]
while \( \tilde{\Omega}_n(\Sigma MO_1) \cong \tilde{\Omega}_{n-1}(MO_1) \cong \mathcal{R}_{n-2} \). One may easily check that the composite
\[
\mathcal{R}_{n-2} \cong \tilde{\Omega}_n(\Sigma MO_1) \to \tilde{\Omega}_n(M O_2) \xrightarrow{\partial} \tilde{\Omega}_{n-1}(BO_2) \cong \mathcal{R}_{n-2}
\]
is the identity (one quick way to see this is to compare with the unoriented case with \( \tilde{\Omega}_n(\Sigma MO_1) \to \tilde{\Omega}_n(BO_2) \) being the monomorphism \( \mathcal{R}_{n-2} \to \mathcal{R}_{n-2}(BO_2) \) which takes the orientation cover. One has a commutative diagram
\[
\begin{array}{ccc}
\mathcal{R}_{n-2} & \xrightarrow{\partial} & \tilde{\Omega}_n(M O_2) & \xrightarrow{\partial} & \mathcal{R}_{n-2} \\
\downarrow \text{mono} & & \downarrow & & \downarrow \text{mono} \\
\mathcal{R}_{n-2}(BO_2) & \to & \tilde{\Omega}_n(M O_2) & \to & \mathcal{R}_{n-2}(BO_2)
\end{array}
\]
and the composite along the bottom is the identity.

One then has
\[
0 \to \Omega_n \xrightarrow{i_*} \Omega^{Z_2}(\text{cod} 2) \overset{F}{\to} \tilde{\Omega}_n(M O_2) \xrightarrow{\partial} \mathcal{R}_{n-2} \to 0
\]
with \( q : \Omega^{Z_2}(\text{cod} 2) \to \Omega_n \) splitting \( i_* \) and with kernel \( \partial \) being identified with \( \Omega_{n-4}(BO_2) \) via the exact sequence
\[
0 \to \mathcal{R}_{n-2} \xrightarrow{j} \tilde{\Omega}_n(M O_2) \to \Omega_{n-4}(BO_2) \to 0
\]
with \( j \) split by \( \partial \).

This gives

**Proposition 1'**. Assigning to \( (M^n, t) \) the class of \( M^n/t \) and \( F^{n-2} \cap F^{n-2} \) with normal bundle \( \nu^2 \mid F \cap F \) gives an isomorphism
\[
\Omega^{Z_2}(\text{cod} 2) \cong \Omega_n \oplus \Omega_{n-4}(BO_2).
\]

One could modify this argument by using \( BSO_2 \) rather than \( BO_2 \) for involutions preserving orientation and with oriented codimension 2 fixed point set. It is, however, more reasonable to consider actions of \( Z_m, \) the cyclic group of order \( m \) simultaneously with the orientation hypothesis being automatic except for \( m = 2 \).

If one considers semifree \( Z_m \) actions preserving orientation with codimension 2 fixed point set (assumed orientable if \( m = 2 \), and in fact, oriented) then one has an exact sequence of Conner-Floyd type
\[
\cdots \to \Omega_n^{Z_2}(\text{semifree}) \overset{F}{\to} \bigoplus \Omega_{n-2}(BSO_2) \xrightarrow{\partial} \Omega_{n-1}(BZ_m) \to \cdots
\]
where the sum on \( j \) is for \( 1 \leq j \leq (m - 1)/2 \) and \( (j, m) = 1 \). This indexing by \( j \) corresponds to the classification of the nontrivial irreducible real representations,
which are of the form multiplication by \( \exp(2\pi i j/m) \) on \( \mathbb{C} \) with \( 1 \leq j \leq (m - 1)/2 \), with \((j, m) = 1\) giving the semifree representations. This choice of \( j \)'s gives the normal bundle to \( F^{n-2} \) a complex structure or orientation and hence orients \( F \) (see Conner and Floyd [6, §38] for \( m > 2 \), while for \( m = 2 \) the orientation is chosen on \( F \)).

One also has a cofibration for the \( m \)th tensor power \( \tilde{y}^m_2 \) of the bundle \( \tilde{y}_2 \) over \( BSO_2 \)

\[
\begin{align*}
S(\tilde{y}^m_2) & \to D(\tilde{y}^m_2) \\
BZ_m & \to BSO_2
\end{align*}
\]

and applying \( \Omega_* \), one obtains an exact sequence

\[
\cdots \to \Omega^Z_n(\text{semifree, special}) \xrightarrow{\partial} \Omega_{n-2}(BSO_2) \to \Omega_{n-1}(BZ_m) \to \cdots
\]

where "special" means that \( Z_m \) is to act by the standard representation in the (codim 2 assumed) normal bundle to the fixed set. This "special"-sequence maps into the above, and corresponding to a different choice of generator for \( Z_m \) can be mapped in once for each \( j \).

*Note.* Because image \( \partial \) is finite, it follows that

\[
\theta: \bigoplus_j \Omega_n(BSO_2) \to \Omega^Z_n(\text{semifree})
\]

has image of finite index, or induces a rational isomorphism

\[
\theta: \bigoplus_j \Omega_n(BSO_2) \xrightarrow{\text{identify } \Omega_* \text{'s}} \Omega^Z_n(\text{semifree})
\]

by identifying the copies of \( \Omega_n \approx \Omega_n(\text{point}) \) for the different \( j \)'s. This says that some multiple of every semifree action is cobordant to a sum of actions, each having the same representation in the normal bundle to each component of \( F \).

For \( \Omega^Z_n(\text{semifree, special}) \approx \Omega_n(BSO_2) \approx \Omega_n(D(\tilde{y}^m_2)) \), the classifying space for the appropriate ramified coverings is \( BSO_2 = D(\tilde{y}^m_2) \). The universal ramified covering is given by the infinite \( m \)-dric \( \{ z \in CP^\infty \mid \sum z_i^m = 0 \} \) ramified over \( CP^\infty = BSO_2 \) (see [16, §4, and particularly p. 308]). The standard basis over \( \Omega_* \) of \( \Omega_*(BSO_2) \) is given by the inclusions \( CP^r \to CP^\infty = BSO_2 \) classifying the Hopf bundles and the induced \( m \)-fold ramified cover of the \( m \)-dric in \( CP^{r+1} \), \( \tilde{Q}_m^{2r} = \{ z \in CP^{r+1} \mid \sum_0^{i+1} z_i^m = 0 \} \), over \( CP^r \).

*Note.* These ramified coverings were studied by Hirzebruch [9] and Hattori [8]. Both incorrectly indicate that the \( BSO_2 \) classifies semifree \( Z_m \) actions, but one needs a single normal representation. The error is on line \( - 2 \), p. 260 of [9]; there is more than one way to include \( G_n \) in \( \mathbb{C}^* \) corresponding to the different \( j \) values.
If one wishes to consider these semifree \( Z_m \) actions as \( m \)-fold branched coverings with a single local branching degree \( m \), i.e. \( \bigcup_j B_{ij} \cong B_i \), and with \( B_\phi \) oriented, one has a corresponding exact sequence

\[
\cdots \to \Omega_n(m\text{-fold, special}) \to \Omega_{n-2}(BSO_2) \to \Omega_{n-1}(B\Sigma_m) \to \cdots
\]

where "special" refers to the local degree only. The map \( \partial \) factors through \( \Omega_{n-1}(BZ_m) \), but one cannot distinguish a generator of \( Z_m \) and hence has no dependence on the representation \( j \). One has \( \Omega_*(m\text{-fold special}) \cong \Omega_*(X_m) \) where \( X_m \) is obtained by sewing \( D(\tilde{\gamma}_m^2) \) to \( B\Sigma_m \) along \( S(\tilde{\gamma}_m^2) \cong BZ_m \), and is a special Brand classifying space.

**Curiosity.** In the case \( m = 2 \), the quadric \( Q_2^{2r} \subset CP^{r+1} \) may be identified as the Grassmannian of oriented 2-planes in \( R^{r+2} \) (see [11]). One may also observe that in the case \( r \) even, \( [Q_2^{2r}] = 2[HCP^r] \) in \( \Omega_4 \), while for \( r \) odd, both \( Q_2^{2r} \) and \( CP^r \) bound.

**Note.** It would appear that Brand’s classifying space for 2-fold branched covers might be identifiable with \( CP^\infty/\text{conjugation} \). Inside \( CP^\infty \) one has the quadric \( BSO_2 \) and \( RP^\infty \), with the normal bundle of \( BSO_2 \) being \( \tilde{\gamma}_2^2 \), and with \( CP^\infty \) being the union of tubular neighborhoods of these subsets. Conjugation fixes \( RP^\infty \) and acts on \( BSO_2 \) as the standard free involution reversing orientation. Thus inside \( CP^\infty/\text{conjugation} \) one has copies of \( BZ_2 = RP^\infty \) and \( BO_2 = BSO_2/Z_2 \) with the complement of \( RP^\infty \) being the disc bundle of a 2-plane bundle over \( BO_2 \).

**Comment.** The ideas about \( m \)-fold covers above derive from my joint efforts with Larry Smith on cobordism of ramified covers.

### 3. Unoriented branchings

In order to describe the classes \([M^n] - (\deg \phi)[N^n]\) in cobordism, one must be able to compute the characteristic numbers, hence, needs to describe the characteristic classes. For this, one follows Brand [3].

Let \( \phi: M^n \to N^n \) be a branched covering, and let \( B = B_\phi \subset N \) be the branch set with normal bundle \( \nu \). Let \( \phi^{-1}B_\psi \) be written as the disjoint union of the submanifolds \( \tilde{B}_k \), where \( \tilde{B}_k \) is the set of points with local branching degree \( k \), and let \( \tilde{\nu}_k \) be the normal bundle of \( \tilde{B}_k \) in \( M \). If one chooses disjoint tubular neighborhoods \( D(\tilde{\nu}_k) \) of the sets \( \tilde{B}_k \), one may collapse the complement to obtain a map

\[
c: M^n \to \bigvee_k T(\tilde{\nu}_k)
\]

and by classification of \( \tilde{\nu}_k \) one has \( \tilde{B}_k \to BO_2 \) covered by a map \( T(\tilde{\nu}_k) \to MO_2 \), and wedging these maps together, one obtains a composite

\[
g: M^n \to \bigvee_k MO_2.
\]

(Note. Brand’s map is defined using only those terms \( k \geq 2 \), but here the wedge is for \( k \geq 1 \), not that it makes a significant difference.) One also has a composite

\[
g: N^n \to T(\nu) \to MO_2
\]

obtained by collapsing onto the tubular neighborhood of \( B \) and then classifying.
Beginning with the bundle $\gamma_2$ over $BO_2$, one has maps

$$
\begin{array}{ccc}
D(\gamma_2) & \overset{a}{\rightarrow} & D(\mu_k(\gamma_2)) & \overset{b}{\rightarrow} & D(\gamma_2) \\
\downarrow & & \uparrow & & \downarrow \\
BO_2 & \overset{b'}{\rightarrow} & BO_2
\end{array}
$$

where $a$ is the degree $k$ wrapping on fibers and $b$ is the bundle map covering $b'$ to classify $\mu_k(\gamma_2)$. The composite $b \circ a$ then induces a map $b \circ a: MO_2 \to MO_2$ which may be wedged together to give a map $\bigvee_k MO_2 \to MO_2$. If the tubular neighborhoods of the $\tilde{B}_k$ are taken as the inverse images of a small tubular neighborhood of $B$, one obtains a commutative diagram

$$
\begin{array}{ccc}
M^n & \overset{\tilde{g}}{\rightarrow} & \bigvee_k MO_2 \\
\phi \downarrow & & \downarrow \\
N^n & \overset{g}{\rightarrow} & MO_2
\end{array}
$$

(up to homotopy of the classifying maps). (Note. This requires the wedge for $k \geq 1$.)

One then has a certain collection of cohomology classes. One has $U \in H^2(MO_2; Z_2)$, the Thom class, and the Thom class $U_k \in H^2(\bigvee_k MO_2; Z_2)$ coming from the $k$th wedge summand. Rather corrupting notation one has classes $w_i^*U$ and $w_i^*U_k$ obtained by applying the Thom isomorphism to $w_i^* \in H^*(BO_2)$. There is also a unique class $v_1 \in H^4(MO_2; Z)$ mapped to the Pontrjagin class $v_1 \in H^4(BO_2; Z)$ under the map $BO_2 \to MO_2$ including the base space and one lets $v_{1,k} \in H^4(\bigvee_k MO_2; Z)$ by taking the Pontrjagin class in the $k$th wedge summand.

One then has the results of Brand [3]:

**Proposition.** One has

$$\begin{align*}
\omega(\tau(M) - \phi^*\tau(N)) &= 1 + \tilde{g}^* \left( \sum_{k \text{ even}} \left( U_k + w_i U_k + w_i^2 U_k + \cdots \right) \right) \in H^*(M^n; Z_2) \\
\nu(\tau(M) - \phi^*\tau(N)) &= 1 + g^* \left( \sum_{k \text{ even}} \sum_{i=1}^\infty (-1)^i (k^2 - 1) k^{2i-2} v_{1,k}^i \right) \in H^*(M^n; Z).
\end{align*}
$$

**Note.** Brand only refers to the classes $v_{1,k}$ in rational cohomology, and asserts the formula for the Pontrjagin class rationally. This all works integrally. If you consider the cofibration $BO_1 \overset{i}{\rightarrow} BO_2 \overset{j}{\rightarrow} MO_2$, $H^*(BO_1; Z)$ is isomorphic to the polynomial ring on the integral Bockstein of $w_1$ (of order 2), i.e. $Z[\beta w_1]/\{2\beta w_1 = 0\}$ and $i^*(\beta w_1) = \beta w_1$ so $i^*$ is epic, and $j^*$ is monic. Since $i^*(v_1) = 0$, there is a unique integral class hitting $v_1$. Using Brand’s arguments one does the calculation by pulling back to $BO_2$, where he uses the Whitney sum formula for Pontrjagin classes. Thus, the formula for the class $\nu(\tau(M) - \phi^*\tau(N))$ is actually correct in integral cohomology modulo 2-torsion. To see that the formula is correct integrally one must
check in the $BO_2$'s that the purported Pontrjagin class has the correct reduction to mod 2 cohomology and that is sufficient because all torsion in $H_*(BO_2; \mathbb{Z})$ has order 2. However, mod $2\Sigma_k \Sigma_{l=1}^{\infty} (-1)(k^2 - 1)k^{2l-2} p_{1,k}$ is $\Sigma_k$ even $p_{1,k}$ and has mod 2 reduction $\Sigma_k$ even $U_k = (\Sigma_k$ even $U_k)^2$.

One then has, almost trivially

**Proposition 3.** The set of classes $[M^n] - d[N^n]$ in $\mathcal{R}_n$ for $d$-fold branched coverings of closed $n$-dimensional manifolds is

\[ \{ \alpha \in \mathcal{R}_n | w_i^n(\alpha) = 0 \}, \]

if $d \geq 2$, $n > 0$.

**Proof.** If $\phi: M^n \to N^n$ is a $d$-fold branched cover, one has $w_\omega[\alpha] = w_\omega[M] - dw_\omega[N] = w_\omega(M) - w_\omega(\phi(M)) - \phi^*w_\omega(\tau N)[M]$. By Brand's formula, $\phi^*(w_i(N)) = w_i(M)$ and so $w_i(M)^n[M] = (\phi^*w_i(N))^n[M]$, and so $w_i^n[M] - d[N^n] = 0$.

From [17, Proposition 9.2], a class $\alpha \in \mathcal{R}_n$ with $w_i^n(\alpha) = 0$ is the class of a manifold $M^n$ having an involution $T$ with fixed point set $F$ of codimension 2. Letting $\phi: M^n \to N^n = M^n/T$ be the quotient map, one has a branched covering of degree 2 with $[M^n] - 2[N^n] = [M^n] = \alpha$. For $d > 2$, let $\phi': M^n \cup (d-2)N^n \to N^n$ by using $\phi$ on $M$ and the trivial cover for $(d-2)$ copies of $N$ and then $[M^n] - d[N^n] = [M^n] - 2[N^n] = \alpha$. Thus, obtains all classes $\alpha$ with $w_i^n(\alpha) = 0$ from coverings. $\Box$

**Note.** This trick of replacing a branched cover $\phi: M^n \to N^n$ by

$\phi': M^n \cup (d - \deg \phi)N^n \to N^n$

to increase the degree of the cover without changing the class $[M^n] - (\deg \phi)[N^n]$ will be used repeatedly.

Now consider

**Proposition 2'.** If $\phi: M^n \to N^n$ is a branched cover of closed manifolds, then $[M^n, \phi] - (\deg \phi)[N^n, \text{identity}]$ in the bordism of $N$ is the class of the map $RP(\tilde{v}_{even} + 1) \to B \subset N$.

**Proof.** One considers a class $x \in H^*(N; \mathbb{Z}_2)$, and Stiefel-Whitney class $w_\omega$, and wishes to compute $w_\omega(\phi^*(x))[M] - (\deg \phi)w_\omega x[N] = (w_\omega(\phi^*(x)) - \phi^*(w_\omega(N) \cdot x))[M]$. For this one uses Brand's formula to write

\[ w(M) = w((\tau(M) - \phi^*(\tau(N))) \oplus \phi^*\tau(N)) \]

\[ = \phi^*(w(N)) \left( 1 + \sum_{k \text{ even}} (U_k + w_1 U_k + w_1^2 U_k + \cdots) \right), \]

where notationally one deletes $\tilde{g}^*$. If one expands out $w_\omega(M)$ one obtains $\phi^*(w_\omega(N))$ + terms involving factors $w_i^kU_k$, and the first term in that expression, when multiplied by $\phi^*(x)$ and evaluated on $[M]$ gives $\phi^*(w_\omega(N) \cdot x)[M]$. Thus the characteristic number remaining is the value on the fundamental class of $[M]$ of the part of $w_\omega(M)\phi^*(x)$ involving the classes $w_i^kU_k$. 

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If you now consider $RP(\tilde{v}_k \oplus 1) \to \tilde{B}_k \to \tilde{B} \subset N$, and let $c \in H^1(RP(\tilde{v}_k \oplus 1); \mathbb{Z}_2)$ be the first Stiefel-Whitney class of the double cover by the sphere bundle, one has

$$w(RP(\tilde{v}_k \oplus 1)) = \pi^*w(\tilde{B}_k) \cdot \left\{(1 + c)^3 + w_1(\tilde{v}_k)(1 + c)^2 + w_2(\tilde{v}_k)(1 + c)\right\}$$

(where actually $w_i(\tilde{v}_k)$ should have a $\pi^*$ with a relation $c^3 + w_1(\tilde{v}_k)c^2 + w_2(\tilde{v}_k)c = 0$.

Also $\phi: \tilde{B}_k \to \tilde{B}$ is a covering so $\pi^*w(\tilde{B}_k) = \pi^*\phi^*w(B) = \pi^*\phi^*(w(N)/w(v)) = \phi^*(w(N)/\pi^*\phi^*w(v)$, and assuming $k$ is even, $\phi^*w(v) = 1 + w_1(\tilde{v}_k)$, so

$$w(RP(\tilde{v}_k \oplus 1)) = \phi^*w(N) \cdot \left\{(1 + c)^2 + w_1(\tilde{v}_k)(1 + c) + w_2(\tilde{v}_k)\right\} \frac{1 + w_1(\tilde{v}_k)}{1 + w_1(\tilde{v}_k)}$$

$$= \phi^*w(N) \cdot \left\{1 + U_k + w_1(\tilde{v}_k)U_k + w_1(\tilde{v}_k)^2U_k + \cdots \right\}(1 + c),$$

where $U_k = c^2 + w_1(\tilde{v}_k)c + w_2(\tilde{v}_k)$.

Note. One has a cofibration $RP(\tilde{v}_k) \to RP(\tilde{v}_k \oplus 1) \to T(\tilde{v}_k)$ and $U_k$ is the pull back of the Thom class in $T(\tilde{v}_k)$. It is the “same” class as the Brand class, but considered in a different space. In homology, $M \to \bigvee_{k \text{ even}} T(\tilde{v}_k)$ sends the fundamental class of $M$ to the same class as the image of the fundamental class under

$$\bigcup_{k \text{ even}} RP(\tilde{v}_k \oplus 1) \to \bigvee_{k \text{ even}} T(\tilde{v}_k).$$

Define

$$\hat{w}(RP(\tilde{v}_k \oplus 1)) = w(RP(\tilde{v}_k \oplus 1))/(1 + c)$$

$$= \phi^*w(N) \cdot \left\{1 + U_k + w_1(\tilde{v}_k)U_k + w_1(\tilde{v}_k)^2U_k + \cdots \right\}.\]$$

Noting that the evaluation of $\hat{w}_\phi((RP(\tilde{v}_k \oplus 1))\phi^*(x)[RP(\tilde{v}_k \oplus 1)]$ annihilates the term $\phi^*w_\phi(N)\phi^*(x)$, which comes from $H^n(\tilde{B}_k; \mathbb{Z}_2) = 0$, only the terms involving classes $w_1(\tilde{v}_k)/U_k$ give nonzero value. One then has

Observation. $w_\phi(x)[M] - (\deg \phi)w_\phi(x)[N] = \hat{w}_\phi(x)[RP(\tilde{v}_\text{even} \oplus 1)]$, where

$$\hat{w} = w(RP(\tilde{v}_\text{even} \oplus 1))/(1 + c).$$

Of course, what this means is that one calculates with the class $\hat{w}$ just as if it were a Stiefel-Whitney class and one was computing a Stiefel-Whitney number. To complete the proof of the proposition one has

Lemma. Let $N$ be a space and $f: P^n \to N$, $g: Q^n \to N$ two maps of closed manifolds into $N$. Suppose there is a class $c \in H^1(P; \mathbb{Z}_2)$ and that $\hat{w} = w(P)/(1 + c)$. If for all $x \in H^1(N; \mathbb{Z}_2)$ and all $\omega$ one has

$$\hat{w}_\omega f^*(x)[P] = w_\omega g^*(x)[Q],$$

then $[P^n, f] = [Q^n, g]$ in $\mathcal{R}_\omega(N)$; i.e. $\hat{w}_\omega f^*(x)[P] = w_\omega f^*(x)[P]$.

Note. What this says is that if the modified Stiefel-Whitney number of a bordism element is again a bordism element, then the modification was irrelevant. The
modification simply does not give the characteristic numbers of a bordism element in general.

PROOF. Let $\bar{w} = 1/w$, $\check{w} = 1/\check{w}$ for the dual Stiefel-Whitney classes. One then has

$$\langle \bar{w}, w_f^* (x), [P] \rangle = \langle \check{w}, w_g^* (x), [Q] \rangle = \langle \chi(\text{Sq}^j) w_g^* (x), [Q] \rangle,$$

and

$$\chi(\text{Sq}^j) \check{w} f^* (x) = \sum_j \chi(\text{Sq}^j) (\check{w}) f^* (\chi(\text{Sq}^{i-j}) x)$$

$$= \sum_j \left( \sum_\omega a^{\omega j}_w \check{w} \right) f^* (\chi(\text{Sq}^{i-j}) x)$$

where $\chi(\text{Sq}^j) \check{w} = \sum_\omega a^{\omega j}_w \check{w}$. is the universal formula for the Steenrod operation on a Stiefel-Whitney class of a bundle, and $\check{w} = w(\tau(p) - 1)$ where $w_1(l) = c$ is the class of a bundle, so

$$\langle \chi(\text{Sq}^j) \check{w} f^* (x), [P] \rangle = \langle \sum_j \left( \sum_\omega a^{\omega j}_w \check{w} \right) f^* (\chi(\text{Sq}^{i-j}) x), [P] \rangle$$

$$= \langle \sum_j \left( \sum_\omega a^{\omega j}_w w \right) f^* (\chi(\text{Sq}^{i-j}) x), [Q] \rangle$$

$$= \langle \chi(\text{Sq}^j) w_g^* (x), [Q] \rangle.$$

Thus one has

$$\langle \bar{w} w f^* (x), [P] \rangle = \langle \chi(\text{Sq}^j) \check{w} f^* (x), [P] \rangle = \langle \bar{w} \bar{w} f^* (x), [P] \rangle$$

or

$$\langle (\bar{w} + \check{w}) \bar{w} f^* (x), [P] \rangle = 0.$$

Summing over all $i$, one has

$$\langle (\bar{w} + \check{w}) w f^* (x), [P] \rangle = 0.$$

Noting that $\check{w} = w/(1 + c)$, $w = \check{w}(1 + c)$ so

$$\bar{w} = \bar{w}(1 + c + c^2 + c^3 + \cdots)$$

and

$$\bar{w} + \check{w} = \bar{w}(c + c^2 + c^3 + \cdots).$$

Thus, one has

$$\langle (c + c^2 + c^3 + \cdots) \bar{w} \bar{w} f^* (x), [P] \rangle = 0$$

for all $\omega$ and $x$, so that powers of $c$ annihilate all expressions $w_\omega f^* (x)$ when evaluated on $[P]$. Since $w = \check{w}(1 + c)$, $w_\omega = \check{w} + \sum_{i>0} c^i b^{\omega i} \check{w}$, in a universal formula, and so

$$\langle w_\omega f^* (x), [P] \rangle = \langle \left( \check{w} + \sum_{i>0} c^i b^{\omega i} \check{w} \right) f^* (x), [P] \rangle$$

$$= \langle \check{w} f^* (x), [P] \rangle = \langle w_\omega g^* (x), [Q] \rangle.$$

Thus, the maps $f$ and $g$ have the same Stiefel-Whitney numbers.  □
**Special Note.** If one reverses this, one sees that $c^i w_\phi(x)[RP(\tilde{\nu}_{even} \oplus 1)] = 0$, which is equivalent to the assertion that $RP(\tilde{\nu}_{even}) = N \times RP^\infty$, with the map induced by $\phi$ and the class $c$, is cobordant to zero. For involutions, this is a crucial feature of Conner and Floyd's work with involutions \cite[(24.1)]{6} and is the observation $S(\tilde{\nu}) \to N$ freely bounds $M$-interior $(D(\tilde{\nu})) \to N$. The above argument shows that the analogue holds for branched covers, but this is certainly not a direct geometric argument.

**Remark.** These results do not agree with Theorem 3.2 of \cite[2]{6}, which is valid only with the additional unstated hypothesis that $w(N) |_{\phi^*} = 1$. In line 1 of the proof, $w(B_\phi)$ is the normal class of $B_\phi$ in $N$, while on line 4 it is the normal class in Euclidean space. In the applications only this special case was used. With the hypotheses given the correct conclusion is $w(M) |_{\phi^*} = w(N) |_{\phi^*}$. One should also remark that the hypothesis that $M^n$ have even Euler characteristic is unnecessary in Corollary 3.5 of \cite[2]{6}, since $w_n(M^n) = (v_n/2(M^n))^2$ and is also a product, where $v$ is the Wu class.

**Corollary.** If $\phi: M^n \to N^n$ is a branched covering of closed manifolds with $w_i(\tilde{\nu}_{even}) \in \text{image}(i^* \phi^*: H^*(N^n, \mathbb{Z}_2) \to H^*(B_{even}; \mathbb{Z}_2))$ then $[A/n] - (deg \phi)[N^n] = 0$ in $\mathcal{H}_n$.

**Note.** This condition is satisfied if $B_\phi$ is orientable, if $\tilde{\nu}_{even}$ is orientable, if $v$ is orientable, or $\tilde{\nu}_{even}$ is orientable, for one has either $w_i(\tilde{\nu}_{even}) = i^* \phi^* w_i(N)$ or $w_i(\tilde{\nu}_{even}) = i^* \phi^*(0)$. In particular, Theorem (4.4) of Hattori \cite[8]{8} is a special case of this.

**Proof.** As noted above, $(\phi \circ \eta) \times c: RP(\tilde{\nu}_{even}) \to N \times B\mathbb{Z}_2$ bounds, and

$$w(RP(\tilde{\nu}_{even})) = w(B_{even}) \left(1 + c\right)^2 + w_1(\tilde{\nu}_{even})(1 + c) + w_2(\tilde{\nu}_{even})$$

with $c^2 + cw_1(\tilde{\nu}_{even}) + w_2(\tilde{\nu}_{even}) = 0$. Letting $w_1(\tilde{\nu}_{even}) = \phi^*(x)$, one has for any $i, j, \omega$ that

$$0 = c \left(1 + c\phi^*(x)^j \right)^i \phi^*(x)^j \left(\frac{w(RP(\tilde{\nu}_{even}))}{1 + \phi^*(x)}\right) \omega [RP(\tilde{\nu}_{even})]$$

$$= cw_2(\tilde{\nu}_{even})^i w_1(\tilde{\nu}_{even})^j \omega_\omega [RP(\tilde{\nu}_{even})]$$

$$= w_2(\tilde{\nu}_{even})^i w_1(\tilde{\nu}_{even})^j \omega [B_{even}] [B_{even}].$$

and hence the map $\tilde{\nu}_{even}: B_{even} \to BO_2$, and so $RP(\tilde{\nu}_{even} \oplus 1)$ bounds. \qed

**Note.** By including a factor $\phi^*(y)$ with $y \in H^*(N; \mathbb{Z}_2)$ one may conclude that $(\phi \circ \eta) \times \tilde{\nu}_{even}: B_{even} \to N \times BO_2$ bounds to see that $[M^n, \phi] - (\deg \phi)[N, \text{identity}] = 0$ in $\mathcal{H}_n(N^n)$.

4. Oriented branched covers. To begin the study of the oriented case, one has

**Lemma 1.** Every class $\alpha \in \text{Tor}(\Omega_n)$ is of the form $[M^n] - (\deg \phi)[N^n]$ for some branched covering of closed oriented manifolds of degree $d$, if $d \geq 2$. 

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PROOF. One has a homomorphism \( \partial : \Omega_{n+1} \to \Omega_n \) assigning to \( P^{n+1} \) the class of the submanifold dual to \( w_1 \). According to Wall [18], \( \partial \) maps onto \( \text{Tor}(\Omega_n) \) and in fact \( \partial : \Omega_{n+1} \to \Omega_n \) maps onto the torsion where \( \Omega_{n+1} \) is the cobordism group of manifolds with \( w_1 \) reduced integral.

Being given \( d > 2 \) and \( \alpha \in \text{Tor}(\Omega_n) \), there is a class \( \beta \in \Omega_{n+1} \) having all numbers divisible by \( w_1^2 \) zero, i.e., coming from \( \Omega_{n+1} \), and so that \( \partial \beta = \alpha \). By Proposition 3, there is a branched covering \( \theta : P^{n+1} \to Q^{n+1} \) of degree \( d \) for which \( [P] - d[Q] = \beta \).

Let \( f : Q^{n+1} \to RP^N \) for some large integer \( N \) with \( f^*(i) = w_i(Q) \) where \( i \in H^1(RP^N, \mathbb{Z}_2) \) is the nonzero class, and deform \( f|_{B_\theta} \) to be transverse to \( RP^{N-1} \) and then deform \( f \) to be the projection

\[
D(v) \to B_\theta \to RP^N
\]

on a tubular neighborhood of \( B_\theta \), \( f \) is then transverse to \( RP^{N-1} \) on a neighborhood of \( B_\theta \) and without changing the map on a smaller tubular neighborhood of \( B_\theta \) one may further deform \( f \) to be transverse to \( RP^{N-1} \subset RP^N \). Thus, one assumes \( f \) has this form; i.e. \( f \) and \( f|_{B_\theta} \) are transverse to \( RP^{N-1} \) and on a tubular neighborhood of \( B_\theta \), \( f \) is given by projection on \( B_\theta \) followed by \( f|_{B_\theta} \). The composite \( f \circ \theta \) is then also transverse to \( RP^{N-1} \) with \( f \circ \theta|_{\theta^{-1}B_\theta} \) being transverse to \( RP^{N-1} \) and being given by \( (f \circ \theta) \circ \text{projection on a tubular neighborhood of } \theta^{-1}B_\theta \), and further \( (f \circ \theta)^*(i) = \theta^*(w_i(Q)) = w_i(P) \).

Letting \( \overline{P}^n \subset P^{n+1} \) and \( \overline{Q}^n \subset Q^{n+1} \) be \( (f \circ \theta)^{-1}(RP^{N-1}) \) and \( f^{-1}(RP^{N-1}) \), \( \overline{\theta} : \overline{P}^n \to \overline{Q}^n \) is then a branched covering of degree \( d \), where \( \overline{\theta} = \theta |_{\overline{P}^n} \). Further \( \overline{P} \) and \( \overline{Q} \) are orientable, being the duals to \( w_i \) and one has \( [\overline{P}] - d[\overline{Q}] = (\partial[P] - d[Q]) = \partial([P] - d[Q]) = \partial\beta = \partial\alpha \). (Note. Identification of the normal bundle of \( \overline{Q} \) in \( Q \) with \( \det \tau(Q) | Q \) and similarly for \( \overline{P} \) gives a choice of compatible orientations by the \( \partial \) process. The only indeterminacy is to completely reverse orientation in the \( \partial \) process, i.e. \( \overline{Q} \equiv -Q \), which does not change \( \alpha \.) Thus the class \( \alpha \) is represented in the desired form.

This reduces the problem of realizing classes in \( \Omega_* \) entirely to a question of possible Pontrjagin numbers and the realization of classes in \( \Omega_*/\text{Tor}(\Omega_*) \).

LEMMA 2. Let \( n = 4m \) and \( s_m(v) \) the primitive Pontrjagin class. If \( \phi : M^n \to N^n \) is a branched covering of closed oriented manifolds, then

\[
s_m(v)[[M^n]] - (\deg \phi)[N^n]] = \sum_{k \geq 2} (1 - k^2m)v_1,k[M^n]
= \sum_{k \geq 2} (1 - k^2m)v_1(\tilde{v}_k)^{m-1}[B_k \cap \tilde{B}_k].
\]

PROOF. Clearly,

\[
s_m(v)[[M^n]] - (\deg \phi)[N^n]] = s_m(v)[M^n] - (\deg \phi)s_m(v)[M^n]
= (s_m(v)(\tau(M)) - \phi^*s_m(v)(\tau(N)))[M^n]
= (s_m(v)(\tau(M) - \phi^*\tau(N)))[M^n]
\]
by primitivity. To compute this class universally, following Brand [4], one has
\[ p = \frac{1 + p_1}{1 + k^2 p_1} \in H^*(BO_2; \mathbb{Z}) \]
to give \( s_m(p) = p_1^m - k^{m+1}p_1^m = (1 - k^{2m})p_1^m \). The rest of the result is the observation that \( \tilde{B}_k \cap \tilde{B}_k \) is the submanifold dual to \( p_{1,k} \) and \( p_{1,k} | \tilde{B}_k \cap \tilde{B}_k = p_{1,k} \).

Combining results, one then has easily

**Proposition 4'.** The set classes \( \alpha \in \Omega_n \) of the form \([M^n] - 2[N^n] \) with \( \phi: M^n \to N^n \) a degree 2 branched covering of closed oriented manifolds is a subgroup of \( \Omega_n \) of odd index, if \( n > 0 \).

**Proof.** If \( \phi: M^n \to N^n \) is a degree 2 branched covering so is \( \phi \times \) (identity): \( M^n \times P^m \to N^n \times P^m \), and so the set of classes \( \alpha \) of the form \([M^n] - 2[N^n] \) in \( \Omega_* \) forms an ideal \( \Lambda_* \) (i.e. \( \Omega_* \) submodule).

By Proposition 1', \( \Omega^Z(\text{cod} 2) \cong \Omega_* \oplus \Omega_{n-4}(BO_2) \) and one has \([\xi^2 \to CP^{2r}] \in \Omega_{4r}(BO_2) \), with \( \xi^2 \) the Hopf bundle, having \( p[(\xi^2)[CP^{2r}] = 1 \). Since this is the cobordism class of the self-intersection \( [\tilde{p}_2 \to \tilde{B}_2 \cap \tilde{B}_2] \) for some 2-fold branched cover \( \theta: P^{4r+4} \to Q^{4r+4} \), one has a class \( \alpha = [P^{4r+4}] - 2[Q^{4r+4}] \in \Lambda_{4r+4} \) for which \( s_{r+1}(\alpha) = (1 - 2^{r+2}) \). This \( \alpha \) is a suitable polynomial generator for \((\Omega_* / Tor \Omega_* ) \otimes \mathbb{Z}_2\).

Since \( \Lambda_* \) contains \( Tor \Omega_* \) and maps onto \((\Omega_* / Tor \Omega_* ) \otimes \mathbb{Z}_2\) in positive dimensions, \( \Lambda_n \subset \Omega_n \) has odd index if \( n > 0 \).

**Remark.** One may actually write down branched coverings for the low dimensional classes in \( \Omega_* \). Specifically, one has \( \pi: CP^2 \to CP^2/\text{conjugation} = S^4 \), with the identification to \( S^4 \) due to Kuiper [10], for which the self-intersection class is the inclusion of a point in \( BO_2 \). One also has \( \pi: P(1,2) \to S^1 \times S^4 \), where \( P(1,2) = S^1 \times CP(2)/(-1 \times \text{conjugation}) \) is the Dold manifold and \( \pi \) is the quotient by dividing out the involution \(-1 \times 1 \sim 1 \times \text{conjugation} \), with the self-intersection being the nonzero element in \( \Omega_1(BO_2) \cong \mathbb{Z}_2 \). From \$2\$, one also has the branched covering \( \phi: Q^{2r} \to CP^{2r} \), where \( Q^{2r} \) is the quadric for which \([Q^{2r}] - 2[CP^{2r}] = 2([HP^r] - [CP^{2r}]) \). Since an odd multiple of this class may also be hit, one has \([HP^r] - [CP^{2r}] \in \Lambda_{4r} \), and this class has \( s \)-number \( 1 - 2^{2r} \), to give an explicit choice of generators for \( \Lambda_* \).

**Proposition 5'.** If \( n = 4m = 2(k(p - 1)) \) with \( p \) an odd prime and \( \alpha = [M^n] - d[N^n] \) is the class of a \( d \)-fold branched cover with \( d < 2p \), then \( s_m(p)[\alpha] \equiv 0 \mod p \).

**Proof.** Let \( \phi: M^n \to N^n \) be the branched covering. By Lemma 2, \( s_m(p)[\alpha] = \Sigma_{j \geq 2} (1 - j^{2m}) p_1^m [M^n] \) where the sum is for \( j \leq d \) only. For \( j \neq 0, j^{2m} = j^{k(p - 1)} \equiv 1 \mod p \), and hence mod \( p \), \( s_m(p)[\alpha] \equiv p_1^m [M^n] = p_1^{m-1}(\tilde{B}_p \cap \tilde{B}_p) \). If one considers a component \( \tilde{B}_p = \tilde{B} \cup \tilde{B}' \), with \( \tilde{B}' = \phi \tilde{B} \) being the corresponding component of the branch set with \( \phi: \tilde{B} \to \tilde{B}' \) being an \( r \)-fold cover, \( 2p > d \geq r \cdot p \) and hence \( r = 1 \), and \( \phi: \tilde{B} \to \tilde{B}' \) is an isomorphism. If \( \tilde{v} \) and \( \tilde{v}' \) are the normal bundles, then \( \tilde{v}' \equiv \mu_p(\tilde{v}) \) and \( \phi: S(\tilde{v}) \to S(\tilde{v}') \) is a \( p \)-fold cover, forming a portion of the \( d \)-fold covering \( \phi: \phi^{-1}(S(\tilde{v}')) \to S(\tilde{v}') \) classified by a map \( S(\tilde{v}') \to BS_{\Sigma_d} \). This local covering actually factors through a map \( S(\tilde{v}') \to BS_{\Sigma_{d-p}} \times BS_{\Sigma_p} \) corresponding to the two parts of the covering \( \phi^{-1}(S(\tilde{v}')) = S(\tilde{v}) \) and \( S(\tilde{v}) \).
Claim. The number \( v_p^{-1}(\tilde{v}_p) \in \mathbb{Z}_p \) is precisely the class \( f_\ast[S(v')] \in p\)-torsion part of \( H_{2k(p-1)-1}(BΣ_d, Z) = \mathbb{Z}_p \).

Since the local branchings for \( \phi: M^n \to N^n \) give rise to the zero element in \( Ω_{n-1}(BΣ_d) \), the total homology class in the \( p\)-torsion part of \( H_{n-1}(BΣ_d; Z) \) is zero, and hence one must have \( v_p^{-1}(\tilde{v}_p) \in \mathbb{Z}_p \).

Note. For the necessary information about \( H_\ast(BΣ_d; Z) \), one should recall the work of Nakaoka [13, 14].

There are undoubtedly many ways to see this claim, and one rather unsophisticated way is to consider the diagram,

\[
\begin{array}{cccccc}
Ω^{* - 4}(BSO_2) & \xleftarrow{b} & \tilde{Ω}_\ast(MSO_2) & \xrightarrow{a} & Ω^{* - 1}(BZ_p) & \xrightarrow{c} & H_\ast(BZ_p; Z) \\
\downarrow{d} & & \downarrow{e} & & \downarrow{f} & & \\
Ω^{* - 4}(BO_2) & \xleftarrow{d} & \tilde{Ω}_\ast(MO_2) & \xrightarrow{a} & Ω^{* - 1}(BΣ_d) & \xrightarrow{c} & H_\ast(BΣ_d; Z) \\
\end{array}
\]

where, for a reduced bordism element, the map to the left takes the self-intersection and to the right one takes the \( p\)-fold ramified cover or local branching, and one takes the usual maps from bordism to homology and the maps of classifying spaces induced by the inclusion of \( Z_p \), the Sylow \( p \) subgroup, in \( Σ_p \) and \( Σ_d \), with any two inclusions being conjugate.

Now \( \tilde{Ω}_\ast(MSO_2) \cong Ω^{* - 2}(BSO_2) \) is the free \( Ω_\ast \) module on the classes \( CP^r \to BSO_2 \) classifying the Hopf bundle. Applying \( a \) takes the class of \( [S^{2r+1}, \exp(2\pi i/p)] \) as free \( Z_p \) action or the map of the standard lens space \( L^{2r+1}(p) \) into \( BZ_p \), giving the standard generator in \( H_{2r+1}(BZ_p; Z) = Z_p \). Further, decomposables in the \( Ω_\ast \) module structure of \( \tilde{Ω}_\ast(MSO_2) \) give zero in homology and hence for \( V^{2r+2} \to MSO_2 \), \( U_\ast^{r+1}(V, ∂V) \in Z_p \) is just the multiple of the standard generator in homology which is hit by the class of \( V \). Now \( U_\ast^{r+1}(V, ∂V) \) is just the Euler class in \( H^2(BSO_2; Z) \) and \( B \to BSO_2 \) is obtained by applying \( b \) to the class of \( V \to MSO_2 \), i.e. taking the appropriate self-intersection.

If one ignores the prime \( 2 \), \( d \) is an isomorphism, for the third term in the exact sequence with \( d \) is \( Ω_\ast(ΣMO_1) \cong Ω_\ast \) which is a 2 group. Also ignoring the prime \( 2 \) and taking \( * \equiv 0 \) mod 4, \( c \) becomes an isomorphism. (Ignoring 2, \( Ω_\ast \) is entirely concentrated in dimensions a multiple of 4 and the \( CP^{2r} \to BO_2 \) form a base of \( Ω_\ast(BO_2) \) ignoring 2. Similarly, the \( CP^{2r} \to BSO_2 \) form a base for \( Ω_\ast(BSO_2) \) and the \( CP^{2r+1} \to BSO_2 \) form an \( Ω_\ast \) base for \( Ω_\ast(BO_2) \).)

Commutativity of the diagram then gives the claim, since \( e \) and \( f \) are epimorphisms on the \( p\)-primary part of the homology.

One may obtain fairly precise information about the possible \( s \)-numbers with

**Proposition.** The set of possible \( s \)-numbers \( s_m(\nu)[[M^{4m}] - d[N^{4m}]] \) for \( d \)-fold branched coverings of closed oriented manifolds is the subgroup \( s_m^d \mathbb{Z} \) of the integers where

\[
s_m^d = a \cdot \gcd\{(1 - 2^2m), (1 - 3^2m), \ldots, (1 - d^{2m})\}
\]
and $a = p_1 \cdot p_2 \cdots p_r$, $p_1 < p_2 < \cdots < p_r$, is a product of odd primes with $p_i \leq d$ and $p_i - 1$ dividing $2m$. If $p$ is an odd prime with $p \leq d$ and $p - 1$ dividing $2m$, then $p$ occurs in $a$ if either $2m + 1$ is a power of $p$ or $d < 2p$.

**Proof.** By taking the disjoint union of $d$-fold covers and by reversing orientation one sees that the set of $s_m(p)[a]$ forms a subgroup of $\mathbb{Z}$, and so is $s_m^dZ$ for some integer $s_m^d$.

Let $h_m^d = \gcd\{(1 - 2^2m), (1 - 3^2m), \ldots, (1 - d^2m)\}$, and

$$g_m^d = \gcd\{(1 - 2^2m), (1 - 3^2m), \ldots, (1 - d^2m)\}.$$ 

For any $d$-fold branched cover, Lemma 2 gives

$$s_m(p)[a] = \sum_{d=k+2}^{d^2} (1 - k^2m)p_{1,k}[M4m]$$

and since each $p_{1,k}[M4m]$ is integral, $h_m^d$ divides $s_m(p)[a]$, and hence $h_m^d | s_m^d$. One also has a 2-fold, and hence $d$-fold, branched covering with $s_m(p)[a] = (1 - 2^2m)$, and for $j < d$ one has the $j$-fold covering $Q_j^2m \to CP_{2m}$ by the $j$-diric having $s_m(p)[a] = j(1 - j^2m)$, hence also a $d$-fold branched covering with the same number. Thus $s_m^d$ divides $(1 - 2^2m)$ and $j(1 - j^2m)$ for $3 \leq j < d$, and hence their greatest common divisor, so $s_m^d$ divides $g_m^d$.

Now, for $p$ dividing $h_m^d$, one has $p > d$ since for $p \leq d$, $p$ does not divide $1 - p^2m$. Then $p$ divides $g_m^d$ and if $p'$ is the power of $p$ dividing $g_m^d$, then $p' | (1 - j^2m)$ and $j < d < p$, $p | j$ so $p | (1 - j^2m)$, and so $p'$ divides $h_m^d$. Thus $g_m^d = bh_m^d$ where $b$ is divisible only by primes less than or equal to $d$ and $h_m^d$ only by primes larger than $d$.

For $p \leq d$, $p^2$ is not a factor of $p(1 - p^2m)$ and so $b$ cannot be divisible by $p^2$. If $p - 1$ divides $2m$, then for $j \equiv 0 (p)$, $p$ divides $1 - j^2m$, while for $j \equiv 0 (p)$, $p$ divides $j$ and so $p$ divides each $j(1 - j^2m)$, $3 \leq j < d$, and also $1 - 2^2m$ and hence $p$ divides $b$. If $p - 1$ does not divide $2m$, then taking $j$ to be a primitive root for $p$, $j < p \leq d$, one has that $p$ does not divide $j(1 - j^2m)$, and so does not divide $b$.

Thus $b = g_m^d/h_m^d$ is the product of those odd primes $p$ with $p \leq d$ and $p - 1$ dividing $2m$. Since $h_m^d | s_m^d | g_m^d = b \cdot h_m^d$, $s_m^d = a \cdot h_m^d$ for some $a$ dividing $b$, giving the desired form for $a$.

For an odd prime $p$ with $p \leq d$ and $p - 1$ dividing $2m$, and either $2m + 1 = p^r$ or $d < 2p$, one must have $p$ dividing $a$. For the case $2m + 1 = p^r$, $s_m(p)[M4n] \equiv 0 \mod p$ for all manifolds and hence for all classes $a$. For the case $d < 2p$, Proposition 5 gives the divisibility.

**Note.** I am indebted to Gordon Keller for the argument using primitive roots in the above. For the following comments I am indebted to my son, Richard Stong.

**Comment 1.** If $p$ divides $h_m^d = \gcd\{(1 - 2^2m), \ldots, (1 - d^2m)\}$, which is true for example if $p > d$ and $p - 1$ divides $2m$, and if $p < 3 \times 10^9$ then

$$v_p(h_m^d) = \begin{cases} 1 + v_p(m) & \text{for } d \geq 3 \text{ or } d = 2 \text{ and } p \neq 1093, 3511, \\ 2 + v_p(m) & \text{for } d = 2 \text{ and } p = 1093 \text{ or } 3511. \end{cases}$$

**Proof.** If $p$ is an odd prime dividing $1 - j^2m$, then $p$ divides $1 - j^{p-1}$ and so $1 - j^l$ where $l = \gcd(p - 1, 2m)$. Letting $j^l = x = 1 + sp^r$, $r > 0$ and $s \equiv 0 \mod p$,
one has
\[ x^p = 1 + \left\{ s + \sum_{i=2}^{p} p\left(\frac{s}{i}\right)\right\} p^{r+1} \]
and for \( q \equiv 0 \mod p \),
\[ x^q = 1 + \left\{ qs + \sum_{i=2}^{q} q\left(\frac{s}{i}\right)\right\} p^{r+1} \]
and so \( x^{pq} = 1 + tp^{r+1} \) with \( t \equiv qs \equiv 0 \mod p \).

**Comment 2.** The argument is also valid for \( p > 3 \times 10^9 \) if \( p^2 \) does not divide \( 2^{2m} - 1 \). In order that \( p^2 \) divide \( 2^{2m} - 1 \), \( m \) must be large, and in fact \( m \geq 63 \).

**Proof.** If \( p^2 \) divides \( 2^{2m} - 1 = (2^m - 1) \cdot (2^m + 1) \), the two factors are relatively prime and so \( p^2 \) divides either \( 2^m - 1 \) or \( 2^m + 1 \). Thus \( 2^m + 1 \geq p^2 > 9 \times 10^{18} \). Now \( 2^{63} > 9 \times 10^{18} > 2^{62} \) so one must have \( m \geq 63 \) at the minimum.

**Comment 3.** There are examples of primes \( p \) dividing both \( (1 - 2^{2m}) \) and \( (1 - 3^{2m}) \) without having \( (p - 1) \mid 2m \). Specifically, 2 and 3 are both quadratic residues of 73, so 73 divides \( 1 - 2^{36} \) and \( 1 - 3^{36} \).

5. **Edmonds' theorem.** The arguments given by Edmonds actually prove considerably more, and this section will show how these arguments work.

**Proposition 6.** If \( \phi: M^n \to N^n \) is a branched covering of closed oriented manifolds with oriented branch set \( B_\phi \) and \( H^2(N; \mathbb{Q}) = 0 \), then \( [M^n] - (\deg \phi)[N^n] \in \text{Tor}(\Omega_\ast) \).

**Proof.** If \( B \) is oriented, then its covering \( \phi^{-1}B_\phi \) is also orientable. Thus one has a factorization for \( g \) and \( \tilde{g} \).

\[
\begin{array}{cccc}
M^n & \xrightarrow{\tilde{h}} & \bigvee_k MSO_2 & \xrightarrow{\tilde{u}} & \bigvee_k MO_2 \\
\downarrow \phi & & \downarrow \rho & & \downarrow \theta \\
N^n & \xrightarrow{h} & MSO_2 & \xrightarrow{u} & MO_2 \\
\end{array}
\]

One now has \( \tilde{u}^*(v_{1,k}) = U_k^2 \) where \( U_k \) is the Thom class in the \( k \)th wedge summand of \( \bigvee_k MSO_2 \), and \( \tilde{h}^*(U_k^2) = X(v_k)h^*(U_k) \) where \( X(v_k) = c_1(v_k) \in H^2(\tilde{B}_k; Z) \) is the Euler class or first Chern class of the normal bundle \( v_k \) of \( \tilde{B}_k \) in \( M \). One then has
\[
\tilde{B}_k \to M^n \xrightarrow{\phi} N^n \xrightarrow{h} MSO_2
\]

with \((h \circ \phi \circ i)^*(U) = X(v_k) = c_1(v_k) = kc_1(v_k)\), where \( v_k \) is the \( k \)th tensor power of the complex line bundle \( \tilde{v}_k \), and \( U \) is the Thom class. Since \( H^2(N, Z) = 0 \) one has
Proposition 7. If \( \phi: M^n \to N^n \) is a branched covering of closed oriented manifolds with \( H^4(N; \mathbb{Q}) = 0 \), then \([M^n] - (\deg \phi)[N^n] \in \text{Tor}(\Omega_*).\)

Proof. Two proofs will be given, with the first being a bit sophisticated.

Proof Number 1. Let \( N \to B_{\deg \phi} \) be a classifying map for the branched covering \( \phi \). According to Brand [4], \( B_{\deg \phi} \otimes \mathbb{Q} \) is a wedge of Eilenberg-Mac Lane spaces \( K(\mathbb{Q}, 4) \), and since \( H^4(N, \mathbb{Q}) = 0 \), the map \( f \) in \( \Omega_*(B_{\deg \phi} \otimes \mathbb{Q}) \) lies in the image of \( \Omega_*(\text{point}) \otimes \mathbb{Q} \). Thus some multiple of \( \phi \) is cobordant to a trivial unbranched covering. \( \square \)

Proof Number 2. Essentially duplicating the argument of Proposition 6, one has \( k^2 \nu_{1,k} = (g \circ \phi \circ i)(\nu_{1,k}) = 0 \) in \( H^*(M; \mathbb{Q}) \) and so \( \nu_{1,k} = \nu_{1,k} = 0 \) in \( H^*(M, \mathbb{Q}) \). Thus Brand's formula becomes

\[
p(r(M) - r(N)) = 1 + 2(1 - k^2) P_{1,k} \in H^*(M; \mathbb{Q}).
\]

For any partition \( \omega \), one then has

\[
s_\omega(p)(r(M)) = s_\omega(p)((r(M) - \phi^*r(N)) \oplus \phi^*r(N))
\]

\[
= \sum_{\omega' \cup \omega'' = \omega} s_{\omega'}(p)(r(M) - \phi^*r(N)) \cup s_{\omega''}(p)(\phi^*r(N))
\]

\[
= \begin{cases} 
\phi^*(s_\omega(p)(r(N))) & \text{if } \omega \neq (\omega', 1), \\
\phi^*(s_\omega(p)(r(N))) + \phi^*(s_\omega(p)(r(N))) \cdot \left( \sum_k (1 - k^2) \nu_{1,k} \right) & \text{if } \omega = (\omega', 1),
\end{cases}
\]

since \( s_\omega(p)(r(M) - \phi^*r(N)) \) is nonzero only for \( \omega = (0) \) or \( (1) \), in rational cohomology. Thus \( s_\omega(p)[M] = (\deg \phi)s_\omega(p)[N] \) except for \( \omega = (\omega', 1) \), and

\[
s_{(\omega', 1)}(p)[M] - (\deg \phi)s_{(\omega', 1)}(p)[N] = \sum_k (1 - k^2) \phi^*(s_\omega(p)(r(N)))\nu_{1,k}[M]
\]

\[
= \sum_k (1 - k^2) i^*\phi^*(s_\omega(p)(r(N)))[\tilde{B}_k \cap \tilde{B}_k]
\]

\[
= \sum (1 - k^2) s_\omega(p)(r(N)), \phi \circ i)_* [\tilde{B}_k \cap \tilde{B}_k].
\]

However, \( (\phi \circ i)_*[\tilde{B}_k \cap \tilde{B}_k] \in H_{n-4}(N^n; \mathbb{Q}) \equiv H^4(N^n; \mathbb{Q}) = 0 \) and so this number is zero. Thus \([M^n] - (\deg \phi)[N^n] \) has all Pontrjagin numbers zero. \( \square \)

Collecting together everything one knows, one then has

Proposition 8. Let \( \phi: M^n \to S^n \) be a branched covering with \( M^n \) closed and if \( n = 4 \) assume \( B_\phi \) orientable. Then \( M^n \) is orientable and \([M^n] \in \text{Tor} \Omega_*\). If \( M^n \) is a Spin manifold or if \( B_\phi \) is orientable, then \([M^n] = 0 \) in \( \Omega_*\).
Proof. \( w_1(M) = \phi^* w_1(S^n) = 0 \), so \( M \) is orientable. Let \( \phi\colon M_i \to S^n \) be the restriction to the component \( M_i \) of \( M \) of \( \phi \). Then \( M_i \) with the orientation induced by the covering satisfies the conditions of Proposition 6 for \( n = 4 \) and otherwise Proposition 7. Thus \([M_i] = [M_i] - (\deg \phi_i)[S^n] \in \text{Tor } \Omega_*\). Reversing the orientation of \( M \) does not change that, and hence the class of \( M \) belongs to \( \text{Tor } \Omega_* \) no matter how orientations are chosen.

If \( M^n \) is a Spin manifold, one applies Corollary 3.5 of Bernstein and Edmonds [2], remarking as in §3 that the hypothesis that \( M \) have even Euler characteristic is unnecessary since \( w_m(M^n) \) is the square of the Wu class \( v_{n/2}(M^n) \). This gives \([M^n] = 0 \) in \( \Omega_* \). If \( B_\phi \) is orientable, one applies the corollary from §3 to obtain \([M^n] = 0 \) in \( \mathcal{R} \). Finally, one recalls that the Tor \( \Omega_* \) injects into \( \mathcal{R}_* \), and hence \( M^n \) is an oriented boundary in both of these cases. \( \square \)

Note. For \( n = 4, \pi\colon CP^2 \to CP^2/\text{conjugation} = S^4 \) has nonorientable branch set. One also has the \( k\)-dric \( \rho\colon Q^k \to CP^2 \) branched along the \( k\)-dric \( Q^k = \{ z \in CP^2 | z_0^k + z_1^k + z_2^k \} \) with \( Q^k \) a Spin manifold for \( k \) even and \( Q^k \) does not meet \( RP^2 \). Thus the composite \( \rho \circ \pi\colon Q^k \to S^4 \) is a branched cover with \( Q^k \) being Spin, provided \( k \) is even and \( v_1(Q^k) = (4 - k^2)k \neq 0 \) for \( k = 4 \).

To see that there is a branched covering \( \phi\colon P^5 \to S^5 \) with \([P^5] \neq 0 \), one may proceed as follows. One has a cofibration \( RP^\infty \to B_2 \to M(\mu_2(\gamma_2)) \) where \( B_2 \) is Brand's classifying space for 2-fold branched covers, and \( \gamma_2 \) is the 2-plane bundle over \( BO_2 \), and a cofibration \( M(\mu_2(\gamma_2)) \to M(\mu_2(\gamma_2)) \to M(\mu_2(\gamma_2) \oplus \gamma_2) \). Now \( M(\mu_2(\gamma_2)) \to M(\mu_2(\gamma_2) \oplus 1) \approx M(\mu_2 \oplus 1) \approx \Sigma RP^\infty \). The composite \( \beta \circ \alpha\colon B_2 \to M(\mu_2(\gamma_2) \oplus \gamma_2) \) is then a homotopy equivalence. It is readily seen to induce an isomorphism on unoriented bordism, hence on \( Z_2 \) homology, while \( \alpha \) and \( \beta \) are isomorphisms in \( Z_2 \) homology for odd \( p \). Finally, both spaces are simply connected. One then observes that \( \mu(\gamma_2) \oplus \gamma_2 \) is orientable to produce a map \( \theta\colon M(\mu_2(\gamma_2) \oplus \gamma_2) \to K(Z; 4) \times K(Z_2; 5) \) via the classes \( \phi(1) \) and \( \phi(w_1) \), where \( \phi \) is the Thom isomorphism. \( \theta \) induces an isomorphism in mod 2 cohomology through dimension 7. Thus \( \pi_3(B_2) \approx Z_2 \) (plus possible odd torsion) with nontrivial image in \( H_5(B_2; Z_2) \). The branch set for this map is the nonzero class in \( \mathcal{R}_*(BO_2) \) with \( w_1 \) being the nonzero number.

6. Coverings of spheres. Since all classes of manifolds branched over \( S^n \) with \( n > 5 \) belong to \( \text{Tor } \Omega_n \subset \mathcal{R}_n \), one should analyze the possible Stiefel-Whitney numbers for manifolds branched over \( S^n \). This section will do so.

Observation. The set \( B(S^n) \) of classes \([M^n] \in \Omega_n\), with \( \phi\colon M^n \to S^n \) a branched covering, is a subgroup of \( \Omega_n \).

Proof. If \( \phi\colon M^n \to S^n \) is a branched covering, \( B_\phi \subset S^n \) is a proper closed subset of \( S^n \) and hence one may find a closed disc contained in \( S^n - B_\phi \). By reparametrizing \( S^n \), one may assume that disc is the "southern" hemisphere \( D^n_- \) and hence that \( B_\phi \subset \text{interior}(D^n_-) \). If \( \psi\colon N^n \to S^n \) is a second branched covering one may similarly suppose \( B_\psi \subset \text{interior}(D^n_-) \). The union \( \phi \cup \psi\colon M^n \cup N^n \to S^n \) is then a branched cover and gives the sum of the classes in \( B(S^n) \). \( \square \)

Note. If \( \phi\colon M^n \to S^n \) and \( \psi\colon N^n \to S^n \) both have degree \( d \), one may realize the sum by a branching of degree \( d \). One simply joins \( \phi^{-1}(D^n_-) \) and \( \psi^{-1}(D^n_-) \) along their
common boundaries which are copies of $S^{n-1} \times \{1,2,\ldots,d\}$. The resulting manifold is obtained by surgery on $d$ copies of $D^n \times S^0$ in $M \cup N$. This phenomenon is much more general since one could sew together two $d$-fold coverings over $M^n$ and $N^n$ to obtain a $d$-fold covering of $M^n \# N^n$.

If one now considers a branched covering $\phi: M^n \to S^n$ with $n \geq 5$, one has by Brand's formula
\[
\omega(M^n) = \phi^*\omega(S^n) \left\{ 1 + U_{ev} + w_1U_{ev} + \cdots + w_{d-2}U_{ev} + \cdots \right\}
\]
where $U_{ev} = \sum_k \text{ev}_k U_k$ and induced homomorphisms are ignored. Letting $B_{ev}$ be the points of $\phi^{-1}(B_{ev})$ of even local branching degree, $B_{ev}$ is the submanifold of $M^n$ dual to the class $U_{ev}$. One then has
\[
\omega(B_{ev}) = \frac{1}{1 + w_1} = 1 + w_1 + w_1^2 + w_1^3 + \cdots + w_1^{n-2}
\]
and
\[
\omega(B_{ev}) = 1 + w_1 + w_2
\]
where $w_2$ is the restriction of $U_{ev}$ to $B_{ev}$.

**Lemma 3.** The Stiefel-Whitney numbers of $M^n$ are given by
\[
\omega_i \cdots \omega_{i_0}(M^n) = \begin{cases} 0 & \text{if any } i_a = 1, \\ w_{n-r}w_{r-2}^{-1}[B_{ev}-2] & \text{if each } i_a > 1. \end{cases}
\]

**Proof.** Since $w_1(M) = 0$, the first formula is obvious. For the second, one has from the proof of Proposition 2' that $w_a[M^n] = \hat{w}_a[RP(\tilde{r}_{ev} \oplus 1)]$, where
\[
\hat{w}(RP(\tilde{r}_{ev} \oplus 1)) = 1 + U + w_1U + \cdots + w_1U
\]
and $U = c^2 + w_1c + w_2$ with $cU = 0$, and here the classes $w_1U$ are actual products. Thus
\[
\hat{w}_i \cdots \hat{w}_{i_0}(RP(\tilde{r}_{ev} \oplus 1)) = w_i^{i_1} \cdots w_{i_0}^{i_0-2} U^{i_0-2}[RP(\tilde{r}_{ev} \oplus 1)] = w_i^{n-r}w_{r-2}^{-1}[B_{ev}].
\]

**Note.** One may obtain the second formula directly by considering the map $g: M \to M(\tilde{r}_{ev})$ by collapsing. Then $w_i \cdots w_j[M] = g^* \Phi(w_i^{i_1} \cdots \Phi(w_j^{j_0-2}))[M] = g^* \Phi(w_i^{n-r}w_{r-2}^{-1})[M]$, where $\Phi$ is the Thom isomorphism. This is $(\Phi(w_i^{n-r}w_{r-2}^{-1}), g_{*}[M]) = (w_i^{n-r}w_{r-2}^{-1}, \phi g_{*}[M])$ where $\phi$ is the homology Thom isomorphism, and $\phi g_{*}[M] = [B_{ev}]$ gives the result.

**Note.** This argument does not depend on the use of $S^n$, and shows that $x(w_i^{i_1}U_{ev})[M] = xw_i^{i_1}[B_{ev}]$ for any branching and class $x$, i.e. $w_1^{i_1}U$ acts like a product.

**Note.** This gives an alternative proof that $\phi: M^n \to S^n$ with $M$ Spin implies $M$ bounds without using a category argument. Use $w_n = \epsilon_n^2/2$ to get an equivalent number $w_n[M] = \sum w_i \cdots w_{i_0}[M]$ with $r > 0$. Then $w_i \cdots w_{i_0}[M] = w_{n-2r+r}^{n-r}w_{r-1}^{-1}[M] = 0$ whenever $r > 0$, so all numbers of $M$ are zero.
LEMMA 4. The Wu class of $B_{ev}$ is given by

$$v = 1 + w_1 + w_1^3 + \cdots + w_1^{2^i-1} + \cdots,$$

and

$$0 = \bar{w}_{n-2}(\tilde{v}_{ev})[B_{ev}] = \sum_{k=0}^{[(n-2)/2]} \binom{n-2-k}{k} w_1^{n-2-k} w_2^k [B_{ev}].$$

PROOF. The first formula is obtained by $v = Sq^1 w$ with $w = 1 + w_1 + w_1^2 + \cdots + w_1^t + \cdots$. One calculates the dual Stiefel-Whitney class $\bar{w} = 1/w$ of $M$ by calculation in $MO_2$, and to do that calculation one may calculate in $BO_2$, following Brand. One has

$$\frac{1 + w_1}{1 + w_1 + w_2} = \frac{1 + w_1 + w_2 + w_2}{1 + w_1 + w_2} = 1 + \frac{w_2}{1 + w_1 + w_2},$$

and so

$$\bar{w}(M) = 1 + U_{ev}/(1 + w_1 + U_{ev})$$

expanded in the usual formal way. Thus

$$0 = \bar{w}_n[M^n] = \left( \frac{1}{1 + w_1 + w_2} \right)[B_{ev}] = \bar{w}_{n-2}(\tilde{v}_{ev})[B_{ev}].$$

Finally, the degree $i$ component of $1/(1 + w_1 + w_2)$ is $\sum_{j=0}^{i/2} \binom{i-2j+j}{j} w_1^{i-2j} w_2^j$, which is easily seen by induction on $i$. □

Note. The condition $\bar{w}_{n-2}(\tilde{v}_{ev})[B_{ev}] = 0$ is equivalent to the assertion that $RP(\tilde{v}_{ev}) \to S^n \times RP^\infty$ bounds. One has $w(RP(\tilde{v}_{ev})) = 1$ and the only relation is $0 = c^{n-1}(RP(\tilde{v}_{ev}))$ which is this relation on $B_{ev}$.

LEMMA 5. If $n$ is even, $n > 4$, then $M^n$ bounds.

PROOF. One considers the numbers $w_1^a w_2^b[B_{ev}]$ with $a + 2b = n - 2$. Thus $a$ must be even.

For $b$ odd and $a > 0$, one has

$$w_1^{2^p+2} w_2^{2^q+1}[B_{ev}] = w_1 \left( (w_1 w_2) \cdot (w_1^{p} w_2^q)^2 \right)[B_{ev}] = Sq^1 \left( (Sq^1 w_2) \cdot x^2 \right)[B_{ev}] = 0$$

for $Sq^1 = 0$ and $Sq^1(x^2) = 0$.

For $b > 0$ and even, one has

$$w_1^{2^p} w_2^{2^q}[B_{ev}] = Sq^{(n-2)/2}(w_1^p w_2^q)[B_{ev}]$$

$$= \begin{cases} 0 & \text{if } (n-2)/2 \neq 2^t - 1, \\ w_1^{2^t-1+p} w_2^q[B_{ev}] & \text{if } (n-2)/2 = 2^t - 1, \end{cases}$$

by the formula for $v$. One notes that $2^t - 1 + p > 0$ and that the power of 2 dividing $b$ has been reduced in the second case. Inductively, on this power of 2, these numbers are zero.
The only possible nonzero numbers are then those with $a = 0$ and $b$ odd or with $b = 0$, with the latter only for $(n - 2)/2 = 2^r - 1$ by the same Wu class argument.

Now

$$w_2^{2q+1}[B_{ev}] = w_2^{2q+2}[M^n] = (w_2^2)^{q+1}[M^n] \equiv (p_1)^{q+1}[M^n] \mod 2$$

and since $[M^n] \in \text{Tor}(\Omega_n)$, this is zero. (Note. This does not hold for $n = 4$, and is the crucial number for the case.)

If $(n - 2)/2 = 2^r - 1$, $n = 2^{r+1}$, and the coefficient of $w_1^{2^{r+1}-2}$ in $\bar{w}_n - 2(\bar{r}_{ev})$ is 1. Since all other $w_1^a w_2^b[B_{ev}]$ are zero, one must have $w_1^{2^{r+1}-2}[B_{ev}] = 0$. □

The situation for $n$ odd is much harder. Since $\beta_1 = \beta_3 = \beta_7 = 0$ and since $\beta_5 \in Z_2$ has a nonzero class known to branch over $S_5$, one may suppose $n > 9$. One may then divide up into the cases $2^k + 1 > n > 2^k$, where with no loss $k \geq 3$. (Everything could be checked for smaller $k$.) Further, it is convenient to consider

$$2^k + 2^{r+1} - 3 \geq n > 2^k + 2^r - 3,$$

with $1 \leq r \leq k$. (Note. For $r = 1$, $n = 2^k + 1$ only, and for $r = k$, $n = 2^{k+1} - 1$ only.)

**Lemma 6.** For $n$ odd, $w_1^{n-2p} w_2^{p-1}[B_{ev}] = 0$ if $p$ is odd.

**Proof.** $w_1^{n-2p} w_2^{p-1}[B_{ev}] = \text{Sq}(w_1^{n-2p-1/2} w_2^{p-1/2})^2[B_{ev}] = 0$ for $\text{Sq}^1 x^2 = 0$. □

**Lemma 7.** For $n$ odd, $2k + x > n > 2k$, $w_1^{n-2p} w_2^{p-1}[B_{ev}] = 0$ except for $((n - l)/4) - (2^{k-2} - 1) \leq p/2 \leq ((n - l)/4)$.

**Proof.** To have $w_1^{n-2p} w_2^{p-1}[B_{ev}] \neq 0$ one must have $2(p - 1) \leq n - 2$ and since $n$ is odd, $2(p - 1) \leq n - 3$ or $p/2 \leq ((n - l)/4)$. From $2^{k+1} > n > 2^k$, one has $2^{k-1} > (n - 2)/2 > 2^{k-1} - 1$. Since $v = 1 + w_1 + w_2^3 + \cdots + w_1^{2^{k-1}} + \cdots$ and $v_i = 0$ if $i > [(n - 2)/2]$, one has $v = 1 + w_1 + w_2 + \cdots + w_1^{2^{k-1}} - 1$ and $w_1^{2^{k-1}} = 0$. To have $w_1^{n-2p} w_2^{p-1}[B_{ev}] \neq 0$, one must then have $n - 2p < 2^{k-1} - 1$ or $(n + 1 - 2^k)/2 < p$ and $(n + 1 - 2^k)/2 + 1 > p$. Thus $(n - l)/2 - (2^{k-1} - 2) < p$ and dividing by 2 gives the result. □

**Lemma 8.** For $n$ odd, the numbers $w_1^{n-2p} w_2^{p-1}[B_{ev}]$ depend only on the numbers $w_1^{n-2p} w_2^{p-1}[B_{ev}]$ with $2^r \leq p$.

**Proof.** If $p \neq 2^r$, there are integers $a, b$ with $a + b = p - 1$ and $(a) \equiv 1 \mod 2$, for example, if $p = 2(2s + 1)$, $r, s > 0$, one may let $b = 2^r$. One then has

$$0 = \text{Sq}^{2b}(w_1^{n-2p} w_2^a)[B_{ev}],$$

and

$$\text{Sq}(w_1^{n-2p} w_2^a) = w_1^{n-2p} w_2^a(1 + w_1)^{n-2p}(1 + w_1 + w_2)^a.$$ One wishes to examine terms of $(1 + w_1)^{n-2p}(1 + w_1 + w_2)^a$ of dimension $2b$, but

$$(1 + w_1)^{n-2p}(1 + w_1 + w_2)^a = \sum_{i=0}^{a} \binom{a}{i} w_2^i (1 + w_1)^{n-2p+a-i},$$
so that the coefficient of $w_2^k$ is \( \binom{a}{b} \). Thus one has an equation

$$w_1^{n-2p} w_2^{p-1} [B_{ev}] = w_1^{n-2p} w_2^{q} \cdot w_2^{b} [B_{ev}] = \sum_{q < p} \alpha_q w_1^{n-2q} w_2^{q-1} [B_{ev}]$$

where $\alpha_q$ depends only on $n$, $p$, and $a$, and of course $q$. Inductively, the result holds for $q < p$, giving the result. \( \square \)

**Corollary.** For $n$ odd, $2^k + 2^{r+1} - 3 \geq n > 2^k + 2^r - 3$, $2^{k+1} > n$, $\dim_{Z_s} (B(S^n)) \leq k - r$.

**Proof.** From Lemmas 6 and 7, the class of $M^n$ is determined by the numbers $w_1^{n-2p} w_2^{p-1} [B_{ev}]$ with $p = 2^{r+1}$ and

$$((n-1)/4) - (2^{k-2} - 1) \leq 2^r \leq ((n-1)/4).$$

Now $((n-1)/4) - (2^{k-2} - 1) > (2^k + 2^r - 4)/4 - (2^{k-2} - 1) = 2^{r-2}$, so $s \geq r - 1$. Also $n - 1 < 2^{k+1}$, and so $s < k - 1$. Thus $r - 1 \leq s \leq k - 2$. Thus one has $k - r$ choices for $s$. \( \square \)

**Corollary.** If $n = 2^{k+1} - 1$, $M^n$ bounds.

**Proof.** This is the case $r = k$, and $B(S^n) = 0$. \( \square \)

**Lemma 8'.** If $n$ is odd and $w_1^{n-2p} w_2^{p-1} [B_{ev}] = 0$ for $p < 2^{r+1}$, then $w_1^{n-2p} w_2^{p-1} [B_{ev}] = 0$ for $2^{r+1} < p < 2^{r+1} + 2^r$.

**Proof.** This requires more precision in the proof of Lemma 8. Assume $w_1^{n-2p} w_2^{p-1} [B_{ev}] = 0$ for $2^{r+1} < p' < p$, which is true for $p' = 2^{r+1} + 1$ since $p'$ is then odd. One then has the formula

$$w_1^{n-2p} w_2^{p-1} [B_{ev}] = \alpha_{2^{r+1}} w_1^{2^{r+2} - 1} w_2^{2^{r+1} - 1} [B_{ev}]$$

since all other terms are zero. The coefficient of $w_2^{2^{r+1} - 1 - a}$ in

$$(1 + w_1)^{n-2p} (1 + w_1 + w_2)^a$$

is

$$\binom{a}{2^{r+1} - 1 - a} (1 + w_1)^{n-2p+a-(2^{r+1} - 1 - a)}$$

and the binomial coefficient can be nonzero only when $2^{r+1} - 1 - a = 0$. If one can choose $a \neq 2^{r+1} - 1$ one then has $\alpha_{2^{r+1}} = 0$ and so $w_1^{n-2p} w_2^{p-1} [B_{ev}] = 0$. For $p = 2^{r+1} + t$, $t < 2^r$, one may let $b = 2^r$, $a = 2^r + t - 1$ to obtain $a < 2^{r+1} - 1$. \( \square \)

**Lemma 9.** If $n \equiv (2^q - 1)\mod{2^q}$ and $w_1^{n-2p} w_2^{p-1} [B_{ev}]$ is zero for $p < 2^q$, nonzero for $p = 2^q$ and $s > q$, then $n < 2^{s+1} + 2^q$.

**Proof.** Suppose $n - 2^{s+1} - 2^q > 0$, and consider

$$0 = v_{2^q} \left( w_1^{n-2^{s+1} - 2^q} w_2^{2^{s'} - 1} \right) [B_{ev}] = S q_{2^q} \left( w_1^{n-2^{s+1} - 2^q} w_2^{2^{s'} - 1} \right) [B_{ev}].$$
One has

\[
    \text{Sq}\left(w_1^{n-2^{s+1}}w_2^{2^{s}-1}\right) = w_1^{n-2^{s+1}}w_2^{2^{s}-1}\left(\sum_{j=0}^{2^{s}-1} \binom{2^{s}-1}{j}w_2^j(1+w_1)^{n-2^{s+1}-2^{s}+2^{s}-1-j}\right)
\]

and in the terms for \(\text{Sq}^{2^s}\) the powers \(w_2^j\) occur for \(0 \leq j \leq 2^{q-1} < 2^{q-1}\). By Lemma 8 only the term with \(j = 0\) can be nonzero, giving

\[
    0 = \left(\frac{n-2^{s+1}-2^q+2^s-1}{2^q}\right)w_1^{n-2^{s+1}}w_2^{2^{s}-1}[B_{e^q}].
\]

Now \(n-2^{s+1}-2^q+2^s-1 = n-2^q-2^q-1 \geq 2^s-1 > 0\) is congruent to \(-2\) mod \(2^{q+1}\), and the binomial coefficient is \(1\) mod \(2\). Thus one has a contradiction, and so \(n < 2^{s+1} + 2q\).

Lemma 10. If \(n \equiv (2^q-1) \text{mod} 2^{q+1}\), then \(B(S^n) = 0\) except for \(n = 2^k + (2^q - 1)\), \(k > q\), and for \(n = 2^k + (2^q - 1)\) with \(k > q\),

\[
    \dim_{Z_2} B(S^n) \leq \begin{cases} 1, & k = q + 1, \\ 2, & k > q + 1. \end{cases}
\]

Proof. If \(n = 2^q - 1\), \(B(S^n) = 0\) by the second corollary to Lemma 8. Thus, one may suppose \(n = 2^{q+t} + (2^q - 1)\) with \(t > 0\).

If \(t = 1\), one has \(2^{q+2} > n > 2^{q+1}\) and \(2^{q+1} + 2^{q+1} - 3 > n > 2^{q+1} + 2^q - 3\), i.e. \(k = q + 1, r = q\). By the first corollary to Lemma 8, \(\dim_{Z_2} B(S^n) \leq k - r = (q + 1) - q \leq 1\).

Now suppose \(t > 1\), and choose \(k, r\) with \(2^{k+1} > n > 2^k\), \(2^k + 2^{r+1} - 3 \geq n > 2^k + 2^{r+1} - 3\). Because \(t > 1, k \geq q + 2\), and \(n \neq 2^k + 2^{r+1} - 1\) so \(k > r\). For \(r < q\), the interval \((2^k + 2^r - 3, 2^k + 2^{r+1} - 3)\) contains no integer congruent to \(2^q - 1\) mod \(2^{q+1}\), and hence \(r \geq q\).

If \(r > q\), then from the argument for the first corollary of Lemma 8, \(w_1^{n-2^{s+1}}w_2^{2^{s}-1}[B_{e^q}] \neq 0\) only for \(r \leq s \leq k - 1\), and let \(s'\) be the smallest such \(s\) giving a nonzero value, so that \(s' \geq r > q\). By Lemma 9, \(n < 2^{s'+1} + 2^q \leq 2^k + 2^q\), so \(2^k + 2^r - 1 \leq n \leq 2^k + 2^q - 1\) contradicting the assumption \(r > q\), or that \(s'\) exists. Thus \(B(S^n) = 0\).

For \(r = q\), \(2^k + 2^{q+1} - 3 \geq n > 2^k + 2^q - 3\) gives \(n = 2^k + (2^q - 1)\). Consider the subspace of \(B(S^n)\) consisting of those manifolds for which \(w_1^{n-2^{s+1}}[B_{e^q}] = 0\). On this subspace one has \(w_1^{n-2^{s+1}}w_2^{2^{s}-1}[B_{e^q}] \neq 0\) only for \(q + 1 \leq s \leq k - 1\), and letting \(s'\) be the smallest such \(s\), one has \(s' > q\). By Lemma 9, \(n < 2^{s'+1} + 2^q\) and so \(s' + 1 = k\), i.e. \(s' = k - 1\). Thus, the subspace of \(B(S^n)\) for which \(w_1^{n-2^{s+1}}w_2^{2^{s}-1}[B_{e^q}] = 0\) has dimension at most one and is detected by \(w_1^{n-2^k}w_2^{2^{k+1}-1}[B_{e^q}]\). Hence, \(\dim_{Z_2} B(S^n) \leq 2\). \(\square\)
Combining all of the pieces, one has

**Proposition 9'.** For $n$ even, $B(S^n) = \Omega_n = \mathbb{Z}$ if $n = 0$ or 4, and $B(S^n) = 0$ otherwise. For $n$ odd, $n \equiv (2^q - 1) \mod 2^{q+1}$, $B(S^n) = 0$ except possibly for $n = 2^k + (2^q - 1)$, $k > q$, and for $n = 2^k + (2^q - 1)$ one has

$$\dim \mathbb{Z}_2 B(S^n) \leq \begin{cases} 1, & \text{if } k = q + 1 \\ 2, & \text{if } k > q + 1. \end{cases}$$

**Notes.** (1) The arguments actually work for manifolds $S^n$ more general than the sphere. One could assume $w(S^n) = 1$ and take $B(S^n)$ to be the subgroup of $\Omega_n$ (or $\mathcal{R}_n$) generated by the classes $[M^n] - (\deg \phi)[S^n]$ (i.e. use $\phi: M^n \to S^n$ a union of branched coverings). In the proof of Lemma 5 one has $w_2^{2q+1}[B_{ev}] = \tilde{w}_{n-2}(\tilde{\nu}_{ev})[B_{ev}] = 0$ if $n \neq 2^{q+1}$, while $n = 2^{q+1}$ gives only $w_1^{2^{q+1} - 1}[B_{ev}] = w_1^{2^{q+1} - 1}[B_{ev}]$. Assuming $H^4(S^n; \mathbb{Q}) = 0$, one has the full result for the classes $[M^n] - (\deg \phi)[S^n]$, and if $[S^n] = 0$ in $\Omega_n$ the full result for the classes $[M^n]$. With no extra assumption one has an unoriented result with an extra case $n = 2^{q+1}$, with $\dim \mathbb{Z}_2 B(S^n) \leq 1$.

(2) For $n = 2^k + 2^q - 1$, $k > q$, consider the Dold manifold

$$P^n = P(2^q - 1, 2^k - 1) = S^{2q-1} \times CP^{2^k-1} / (-1 \times \text{conjugation})$$

and the Milnor hypersurface

$$H^n = H(2^q, 2^k) = \left\{ ([x], [y]) \in RP^{2^q} \times RP^{2^k} \mid \sum_{i=0}^{2^q} x_i y_i = 0 \right\}.$$ 

Over each of these manifolds one has a 2-plane bundle $\eta$, $S^{2q-1} \times (\text{Hopf bundle}) / (-1 \times \text{conjugation})$ or $\xi_1 \oplus \xi_2 / H^n$ respectively, and hence a composite map $M^n \to BO_2 \to MO_2$. One has

$$w(P^n) = (1 + c)^{2^{q-1}}(1 + c + d)^{2^k - 1} = (1 + d + cd + \cdots + c^{2^k-1}d)(1 + d^{2^k-1})$$

and

$$w(H^n) = \frac{(1 + \alpha)^{2^{q-1}}(1 + \beta)^{2^k - 1}}{(1 + \alpha + \beta)} = \left(1 + \frac{\alpha \beta}{1 + \alpha + \beta}\right)(1 + \alpha^{2^q} + \beta^{2^k})$$

with the classes $d^{2^k-1}$, $\alpha^{2^q}$, and $\beta^{2^k}$ making no contributions to Stiefel-Whitney numbers. Thus, the elements $(M^n, i \circ \eta)$ in $\mathcal{R}_n(MO_2)$ have the same characteristic numbers as if $w(M^n)$ were $(i \circ \eta)^*(1 + \sum w_i[U])$. Making the maps $i \circ \eta$ transverse to $BO_2 \subset MO_2$, one obtains codimension 2 submanifolds $B_{ev}(M^n)$, and this can be done explicitly to give

$$B_{ev}(P) = P(2^q - 1, 2^{k-1} - 1) \quad \text{and} \quad B_{ev}(H) = H(2^q - 1, 2^{k-1} - 1)$$

with

$$w(B_{ev}(P)) = 1 / (1 + c), \ w(\tilde{\nu}_{ev}) = 1 + c + d,$$

$$w(B_{ev}(H)) = 1 / (1 + \alpha + \beta), \ w(\tilde{\nu}_{ev}) = 1 + (\alpha + \beta) + \alpha \beta.$$

One then has

$$w_{1}^{n-2^{q+1}}w_{2}^{2^{q+1}-1}[B_{ev}(P)] = 0 \quad (k > q + 1), \quad w_{1}^{n-2^{q}}w_{2}^{2^{q+1}-1}[B_{ev}(P)] \neq 0,$$
and
\[ w_1^n-2q+1 w_2^{2q-1} [B_{ev}(H)] \neq 0, \quad w_1^{n-2k} w_2^{k-1} [B_{ev}(H)] = 0 \quad (\text{if } k > q + 1). \]

These provide examples of manifolds with the correct Stiefel-Whitney numbers, and with \(B_{ev}(M)\) actually having the correct Stiefel-Whitney class structure, to all of the exceptional cases in Proposition 9'. Thus the argument cannot be improved at the Stiefel-Whitney number level.

In the case \(n = 5\), there are maps \(f: M^5 \to MO_2\), not lifting to \(BO_2\), for which Brand's stable bundle actually pulls back to \(t(A^5)\). For \(P^5\), one has the branching \(\phi: P^5 \to S^1 \times S^4 = P(1,2)/(1 \times \text{conjugation})\), and for \(H^5\) one has the involution \(T([x], [y_0, y_1, y_2, y_3, y_4]) = ([x], [y_0, y_1, y_2, -y_3, -y_4])\) giving a branched cover \(H^5 \to N^5 = H(2,4)/T\). In order to identify \(N^5\), consider \(RP^2 \times RP^4\) as \(RP(5)\), the projective space bundle of a trivial 5-plane bundle over \(RP^2\), and observe that the defining relation \(\Sigma_6 x_i y_i = 0\) gives orthogonality, i.e. \(H(2,4) = RP(\lambda_+ + 2)\) where \(\lambda_+\) is the orthogonal complement of the line bundle \(\lambda\) inside 3. \(T\) is multiplication by \(-1\) in the fibers of 2, and so \(N^5\) is the quotient of \(S(\lambda_+ + 2)\) by the \(Z_2 \times Z_2\) given by \(-1\) in the fibers of \(\lambda_+\) and \(-1\) in the fibers of 2, the product of these being \(-1\) on the sphere. Thinking of \(S(\lambda_+ + 2)\) as the fiberwise join \(S(\lambda_+) \ast S(2), S(\lambda_+ + 2)/Z_2 \times Z_2 = S(\mu_2 \lambda_+) \ast S(\mu_2 2) = S(\mu_2 \lambda_+ + 2)\). Now \(\mu_2 \lambda_+ + 2 = \lambda + 3\), being 4-plane bundles with the same Stiefel-Whitney class and so \(N^5\) is the normal sphere bundle of \(RP^2\) imbedded in \(R^6\), hence a framed manifold.

One may generalize this construction for \(H^5\). Noting that \(4\lambda = 4\) over \(RP^2\), \(H^5 \cong RP(\lambda_+ + 2) = RP(3 + 1) = RP(\lambda + 3)\) and branches over
\[ S(\mu_2 (\lambda + 1) + \mu_2 2) = S(\lambda + 1 + 2) = S(\lambda + 3). \]
Specifically, for \(n = 2^k + 2^q - 1\), one may consider
\[ Q^n = RP((\lambda_1 + 1) + (\lambda_2 + 1)) \]
over \(RP^{2^k - 2} \times RP^{2^q - 2}\) as branching over \(U^n = S((\lambda_1 + 1) + (\lambda_2 + 1))\). One notes that \(w(U^n) = 1\), and that \(U^n\) is frameable for \(3 \geq k \geq q \geq 1\). By [17, Lemma 3.4] \(Q^n\) is indecomposable in \(\mathfrak{M}_n\) and one may check very painfully that \(w_{n+2-2q+2}^{2^k-1} [Q^n]\) \(\neq 0\) to see that \(Q^n\) is cobordant to \(H^n\). Thus, one actually has branchings over manifolds having \(w = 1\) in every exceptional dimension, and over framed manifolds when \(n = 5, 9,\) and 11.

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