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SOME CONJECTURES ON ELLIPTIC CURVES OVER CYCLOTOMIC FIELDS

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ABSTRACT. We give conjectures for the mean values of Hasse-Weil type L-functions over cyclotomic fields. In view of the Birch-Swinnerton-Dyer conjectures, this translates to interesting arithmetic information.

1. In 1967 A. Weil [**6**] showed that if the Dirichlet series

$$L_1(s) = \sum_{n=1}^{\infty} a(n)n^{-s}, \qquad L_1(s, \chi) = \sum_{n=1}^{\infty} a(n)\chi(n)n^{-s} \quad (\chi \bmod q)$$

satisfy the functional equations

$$(1) \qquad (\sqrt{N}/2\pi)^s\, \Gamma(s)L_1(s) = w_1(\sqrt{N}/2\pi)^{k-s}\, \Gamma(k-s)L_1(k-s), \qquad |w_1| = 1,$$

$$(2) \qquad \begin{gathered} (\sqrt{N}/2\pi q)^s\, \Gamma(s)\, L_1(s, \chi) = w_\chi(\sqrt{N}/2\pi q)^{k-s}\, \Gamma(k-s)\, L_1(k-s, \bar{\chi}), \\ w_\chi = w_1\varepsilon(q)\frac{\tau_\chi}{\tau_{\bar{\chi}}}\,\chi(-N), \qquad \tau_\chi = \sum_{a=1}^{q} \chi(a)e^{2\pi i a/q}, \end{gathered}$$

for "sufficiently many" q such that $(q, N) = 1$ and all primitive characters χ mod q, where N and k are positive integers and ε is a primitive character mod N, then $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$ is a modular form with multiplier ε of weight k for the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbf{Z},\ ad - bc = 1,\ c \equiv 0 \pmod{N} \right\}.$$

Our aim is to propose some conjectures on the asymptotic behaviour of the mean value

$$S(X; h) = \sum_{\substack{p \le X \\ p \equiv 1(h)}} \prod_{\substack{\chi \bmod p \\ \chi^h = \chi_0 \\ \chi^r \ne \chi_0 \text{ for } r<h}} L_1(k/2, \chi) \qquad (h \geqq 2 \text{ a fixed integer})$$

where the sum is restricted to primes p, and χ_0 is the principal character mod p.

Let $L_h(s, \chi) = \sum_{n=1}^{\infty} a(n^h)\chi(n)n^{-s}$. If $L_1(s)$ satisfies (1) and (2), then we propose the following

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MAIN CONJECTURE. *For* $h = 2$ ($\chi^2 = \chi_0$, $\chi \neq \chi_0$, $\chi \bmod p$):

$$S(X; 2) = \sum_{p \le X} L_1(k/2, \chi)$$

$$\sim \sum_{p \le X} \frac{1 + w_1 \varepsilon(p) \chi(-N)}{\Gamma(k/2)} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(k/2+z) \left(\frac{\sqrt{N}}{2\pi} p \right)^z L_2(k+2z, \chi_0) \frac{dz}{z}.$$

For $h > 2$:

$$S(X; h) \sim \sum_{\substack{p \le X \\ p \equiv 1 (h)}} \frac{1 + (w_1 \varepsilon(p))^{\phi(h)}}{\Gamma(k/2)^{\phi(h)}}$$

$$\cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(k/2+z)^{\phi(h)} \left(\frac{\sqrt{N}}{2\pi} p \right)^{\phi(h) z} H(z)^{\phi(h)/2} \frac{dz}{z} \quad (c > 0),$$

where

$$H(z) \equiv \sum_{n=1}^{\infty} \frac{a(n)^2 \chi_0(n)}{n^{k+2z}} + L_h(h(k/2 + z), \chi_0)^2.$$

The integrals in the above conjecture can be easily evaluated asymptotically by shifting the line of integration and computing the residues at $z = 0$. If we assume that $L_1(s)$ has an Euler product of the form

$$(3) \qquad L_1(s) = \prod_{p \mid N} \left(1 - \frac{a(p)}{p^s}\right)^{-1} \cdot \prod_{p \nmid N} \left(1 - \frac{\gamma_p}{p^s}\right)^{-1} \left(1 - \frac{\bar{\gamma}_p}{p^s}\right)^{-1},$$

with $|\gamma_p|^2 = p^{k-1}$ (see [1]), then we have the following

PROPOSITION. *Assuming the Main Conjecture and the Euler product* (3), *it follows that*

$$\text{(i)} \qquad S(X; 2) \sim \frac{48\pi}{N} \prod_{p \mid N} (1 - p^{-2})^{-1} \cdot \langle f, f \rangle \cdot \frac{X}{\log X},$$

where

$$\langle f, f \rangle = \iint_{\Gamma_0(N) \backslash H} \left| \sum_{n=1}^{\infty} a(n) e^{2\pi i n z} \right|^2 y^{k-2} \, dx \, dy$$

is the Petersson inner product of f with itself;

(ii) *For* $h > 2$,

$$S(X; h) \sim 2 \left(\frac{24\pi}{N} \prod_{p \mid N} \left(1 + p^{-1}\right)^{-1} \cdot \langle f, f \rangle \right)^{\phi(h)/2} \frac{X(\phi(h) \log X)^{\phi(h)/2-1}}{(\frac{1}{2}\phi(h))!}.$$

In the case that the weight k is 2, that (1), (2) and (3) are satisfied and the $a(n)$ are rational, it is known (see [4, Theorems 7.14, 7.15]) that $L_1(s)$ is the Hasse-Weil L-function of some elliptic curve E, and $\langle f, f \rangle$ can always be expressed as the product of an algebraic number, a power of π, and the two periods of E. For example (see [5]), when $k = 2$, $N = 11$ there is a unique cusp form of weight 2:

$$f(z) = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^2 \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^2 = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z},$$

and $L_1(s) = L_E(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ is the Hasse-Weil L-function of the elliptic curve E: $y^2 + y = x^3 - x^2$. For this curve, our conjectures take the following form:

$$S(X; 2) \sim \frac{11}{5\pi}\Omega^+\Omega^- \frac{X}{\log X},$$

$$S(X; h) \sim 2\left(\frac{\Omega^+\Omega^-}{\pi}\right)^{\phi(h)/2} \frac{X(\phi(h)\log X)^{\phi(h)/2-1}}{(\frac{1}{2}\phi(h))!} \quad (h > 2).$$

Here $\Omega^+ = 0.6346047 \cdots$ and $\Omega^- = 1.4588166 \cdots$ are the real period and the absolute value of the imaginary period of E, respectively.

Now let p be an odd prime. The Hasse-Weil L-function of E over the cyclotomic field $\mathbf{Q}(\sqrt[p]{1})$ is given by

$$L_{\mathbf{Q}(\sqrt[p]{1})\backslash E}(s) = \prod_{\chi \bmod p} L_E(s, \chi).$$

It is reasonable to expect that the average value (as p varies) is given by

$$\text{average value of } L_{\mathbf{Q}(\sqrt[p]{1})\backslash E}(1) \sim L_E(1)\cdot\frac{11}{5\pi}\Omega^+\Omega^-\cdot \prod_{\substack{h|(p-1)\\ h>2}} \frac{2\left(\frac{\phi(h)}{\pi}\Omega^+\Omega^- \log p\right)^{\phi(h)/2}}{\phi(h)(\frac{1}{2}\phi(h))!}.$$

Sharper forms of part (ii) of the Proposition can be derived on assuming the analytic continuation of $L_h(s)$ for $h > 2$. Then there will be extra terms involving lower powers of $\log X$, whose coefficients are expressible in terms of the special values of $L_h(s)$, some of which can be given by the conjectures of Deligne **[2]**.

2. In order to lend credence to our Main Conjecture, we give the following arguments. Firstly, the conjecture for $S(X; 2)$ has already been dealt with in [3]. We therefore consider $S(X; h)$ for $h > 2$. For any prime $p \equiv 1 \pmod h$ there are $\phi(h)$ characters mod p of exact order h, and moreover these characters are all primitive. It follows by (2) that, for $p \nmid N$,

$$\left(\left(\frac{\sqrt{N}}{2\pi}p\right)^s \Gamma(s)\right)^{\phi(h)} \prod_{\chi \bmod p}{}' L_1(s, \chi) = W\left(\left(\frac{\sqrt{N}}{2\pi}p\right)^{k-s} \Gamma(k-s)\right)^{\phi(h)} \prod_{\chi \bmod p}{}' L_1(k-s, \chi),$$

$$W = W(p, h) = \prod_{\chi \bmod p}{}' w_\chi = (w_1\varepsilon(p))^{\phi(h)},$$

where $\prod'$ means that the product is taken over all characters of exact order h, which we denote by $\chi_1, \ldots, \chi_{\phi(h)}$. By Lavrik's method (see [3]), we have

$$\prod{}' L_1(k/2, \chi) = \frac{1 + W}{\Gamma(k/2)^{\phi(h)}} \sum_{n_1} \cdots \sum_{n_{\phi(h)}} a(n_1) \cdots a(n_{\phi(h)})\, \chi_1(n_1) \cdots \chi_{\phi(h)}(n_{\phi(h)})(n_1 \cdots n_{\phi(h)})^{-k/2}$$

(4)

$$\cdot \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\Gamma\left(\frac{k}{2} + z\right)\left(\frac{\sqrt{N}}{2\pi}p\right)^z\right)^{\phi(h)} (n_1 \cdots n_{\phi(h)})^{-z}\, \frac{dz}{z}.$$

Now, summing over p should give a lot of cancellation except for those $\phi(h)$-tuples

$(n_1, \ldots, n_{\phi(h)})$ for which

$$\prod_{i=1}^{\phi(h)} \chi_i(n_i) = 1 \tag{5}$$

for all $p \nmid n_i$ $(i = 1, \ldots, \phi(h))$. We can arrange the characters so that $\chi_i = \bar{\chi}_{\phi(h)-i+1}$. It follows that

$$\chi_i(n_i)\chi_{\phi(h)-i+1}(n_{\phi(h)-i+1}) = 1$$

whenever $n_i = n_{\phi(h)-i+1}$ and $p \nmid n_i$. Also $\chi_i(n_i) = 1$ whenever $n_i = m_i^h$ is an hth-power and $p \nmid m_i$. Combinations of these two cases are the only ways in which the aforementioned tuples can be constructed. Hence, every tuple $(n_1, \ldots, n_{\phi(h)})$ satisfying (5) is given as follows. Let $0 \leqq r \leqq \frac{1}{2}\phi(h)$; choose an r-tuple $(i_1, \ldots, i_r)$ with $1 \leqq i_1 < \cdots < i_r \leqq \frac{1}{2}\phi(h)$. Put $n_{i_j} = n_{\phi(h)-i_j+1}$ $(j = 1, \ldots, r)$. Also let $n_i = m_i^h$ be a perfect hth-power for $i \neq i_j$, $i \neq \phi(h) - i_j + 1$ $(j = 1, \ldots, r)$. Since there are exactly $\binom{\phi(h)/2}{r}$ such r-tuples, it is reasonable to expect that, after summing (4) over primes $p \equiv 1 \pmod h$, $S(X; h)$ should be given asymptotically as

$$\begin{aligned} S(X;h) \sim \sum_{\substack{p \leq X \\ p \equiv 1(h)}} \frac{1+W}{\Gamma(k/2)^{\phi(h)}} \sum_{r=0}^{\phi(h)/2} \binom{\frac{1}{2}(\phi h)}{r} \\ \times \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\Gamma\left(\frac{k}{2}+z\right)\left(\frac{\sqrt{N}}{2\pi}p\right)^z\right)^{\phi(h)} \\ \cdot \left(\sum_{n=1}^{\infty} \frac{a(n)^2\chi_0(n)}{n^{k+2z}}\right)^r \left(\sum_{m=1}^{\infty} \frac{a(m^h)\chi_0(m)}{m^{h(k/2+z)}}\right)^{\phi(h)-2r} \frac{dz}{z} \\ = \sum_{\substack{p \leq X \\ p \equiv 1(h)}} \frac{1+W}{\Gamma(k/2)^{\phi(h)}} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\Gamma\left(\frac{k}{2}+z\right)\left(\frac{\sqrt{N}}{2\pi}p\right)^z\right)^{\phi(h)} H(z)^{\phi(h)/2} \frac{dz}{z}. \end{aligned}$$

In order to derive the Proposition from the Main Conjecture, note that the Rankin-Selberg L-function $\sum_{n=1}^{\infty} |a(n)|^2 n^{-s}$ has a simple pole at $s = k$ with residue

$$\frac{48\pi}{N} \prod_{p \mid N} (1 + p^{-1})^{-1} \cdot \langle f, f \rangle.$$

If $L_1(s)$ satisfies (3), then the coefficients $a(n)$ are real, and $w_1\varepsilon(p) = \pm 1$. Also, $L_h(s)$ is regular for $\operatorname{Re}(s) > 1 + h(k-1)/2$. Hence, for $h > 2$, $H(z)$ is regular for $\operatorname{Re}(z) > \frac{1}{h} - \frac{1}{2}$ except for a simple pole at $z = 0$, with residue

$$\frac{24\pi}{N} \prod_{p \mid N} (1 + p^{-1})^{-1} \cdot \langle f, f \rangle.$$

Shifting the line of integration to $c = \frac{1}{h} - \frac{1}{2} + \varepsilon$ $(0 < \varepsilon < \frac{1}{2} - \frac{1}{h})$ and using standard estimates for the growth of the Rankin-Selberg L-function, the proposition follows on computing the main term of the residue and applying the prime number theorem for arithmetic progressions.

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