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A CORRECTION AND SOME ADDITIONS TO "FUNDAMENTAL SOLUTIONS FOR DIFFERENTIAL EQUATIONS ASSOCIATED WITH THE NUMBER OPERATOR"

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ABSTRACT. Let (H, B) be an abstract Wiener pair and $\mathcal{N}$ the operator defined by $\mathcal{N}u(x) = -\mathrm{trace}_H D^2u(x) + (x, Du(x))$, where $x \in B$ and$(\cdot, \cdot)$ denotes the B-B^* pairing. In this paper, we point out a mistake in the previous paper concerning the existence of fundamental solutions of $\mathcal{N}^k$ and intend to make a correction. For this purpose, we study the fundamental solution of the operator $(\mathcal{N} + \lambda I)^k$ $(\lambda > 0)$ and investigate its behavior as $\lambda \to 0$. We show that there exists a family $\{Q_\lambda(x, dy)\}$ of measures which serves as the fundamental solution of $(\mathcal{N} + \lambda I)^k$ and, for a suitable function f, we prove that the solution of $\mathcal{N}^k u = f$ can be represented by $u(x) = \lim_{\lambda \to 0} \int_B f(y) Q_\lambda(x, dy) + C$, where C is a constant.

In our previous paper [**2**, §3], we have shown that the solution of the equation $\mathcal{N}^k u(x) f(x)$ $(f \in \mathcal{L}_0)$ is of the form $G^k f(x) +$ a constant, where $Gf(x) = \int_0^\infty [\int_B f(y) o_t(x, dy)]\, dt$ and $G^k f = G(G^{k-1} f)$ with $G^0 f = f$. Viewing the representation of $G^k f$, we then intuitively claimed that the family $\{Q(x, dy)\}$ of k-fold convolution of $G(x, dy) = \int_0^\infty o_t(x, dy)\, dt$ forms rigorously the "fundamental solution" of $\mathcal{N}^k$. Unfortunately, the "fundamental solution" is only formal. The mistake is caused by the fact that $G^k f(x)$ may not equal $\int_B f(y) Q(x, dy)$ when $f \in \mathcal{L}_0$ (though $G^k f(x) = \int f(y) Q(x, dy)$ for all $f \geqslant 0$). In order to obtain a correct representation of $G^k f(x)$ by an integral with respect to certain measure, we study the fundamental solution of the differential operator $(\mathcal{N} + \lambda I)^k$, where $\lambda > 0$, and then investigate its behavior as λ goes to zero. We show that the fundamental solution of $(\mathcal{N} + \lambda I)^k$ exists in the sense of measure, which means that there exists a family of measures, say $\{Q_\lambda(x, dy)\}$, so that, for any member f of a certain reasonable large class of functions, the integral $Q_\lambda f(x) = \int_B f(y) Q_\lambda(x, dy)$ exists and $(\mathcal{N} + \lambda I)^k (Q_\lambda f)(x) = f(x)$. As λ goes to zero, we show that $\lim_{\lambda \to 0} \int_B f(y) Q_\lambda(x, dy) = G^k f(x)$ for any f in $\mathcal{L}_0$.

DEFINITIONS AND NOTATION. We give in the following some new definitions and notations which did not appear in the previous paper. For the others, we refer the reader to [**2**].

For each x in B and for each Borel set A in B, we define

$$G_\lambda(x, A) = \int_0^\infty e^{-\lambda t} o_t(x, A)\, dt \qquad (\lambda > 0),$$

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$$R_\lambda(x, A) = \int_0^\infty e^{-\lambda t}[o_t(x, A) - p_1(A)]\, dt;$$

and let

$$G_\lambda f(x) = \int_0^\infty \int_B e^{-\lambda t} f(y) o_t(x, dy)\, dt \quad \text{(if it exists)},$$

$$R_\lambda f(x) = \int_0^\infty \int_B e^{-\lambda t} f(y)[o_t(x, dy) - p_1(dy)]\, dt.$$

Evidently, $G_\lambda f$ and $R_\lambda f$ exist when f is bounded and continuous. Furthermore, we have

LEMMA 1. (a) *$G_\lambda(x, \cdot)$ and $R_\lambda(x, \cdot)$ are Borel measures with total variation λ^{-1} and $2\lambda^{-1}$, respectively.*

(b) *If $f \in \mathfrak{L}$, then $R_\lambda f \in \mathfrak{L}$ and $G_\lambda f \in \mathfrak{L}$; and, if $f \in \mathfrak{L}_0$, then $R_\lambda f(x) = G_\lambda f(x)$ and $G_\lambda f \in \mathfrak{L}_0$.*

(c) *If $f \in \mathfrak{L}$, f is integrable with respect to $R_\lambda(x, \cdot)$ and $G_\lambda(x, \cdot)$. Moreover, we have*:

$$R_\lambda f(x) = \int_B f(y) R_\lambda(x, dy), \tag{1}$$

$$G_\lambda f(x) = \int_B f(y) G_\lambda(x, dy). \tag{2}$$

PROOF. (a) follows from the fact that $o_t(x, \cdot)$ and $p_1(\cdot)$ are mutually singular probability measures.

(b) follows by arguments similar to [**2**, Proposition 3.1].

It remains to prove (c). First of all, we observe that $R_\lambda f(x) = G_\lambda f(x) - \lambda^{-1}\int_B f(y)p_1(dy)$ and $R_\lambda(x, \cdot) = G_\lambda(x, \cdot) - \lambda^{-1}p_1(\cdot)$, so it suffices to verify (2).

Next, noting that if f is in $\mathfrak{L}$ then f^+, f^- and $|f|$ are also in $\mathfrak{L}$; it suffices to prove that any nonnegative member f in $\mathfrak{L}$ is integrable with respect to $G_\lambda(x, \cdot)$ and (2) holds. But, by the definition of $G_\lambda(x, \cdot)$, it is easy to see that (2) holds when f is a simple function and so, by the monotone convergence theorem, (2) holds if f is a nonnegative function. Now the integrability of a nonnegative member in $\mathfrak{L}$ follows immediately from (b). □

PROPOSITION 1. *For each x in B and each Borel set E in B, define*

$$Q_\lambda(x, E) = \underbrace{\int_B \cdots \int_B}_{(k-1 \text{ times})} G_\lambda(y_{k-1}, E) G_\lambda(y_{k-2}, dy_{k-1}) \cdots G_\lambda(y_1, dy_2) G_\lambda(x, dy_1). \tag{3}$$

We have:

(a) *The total variation of $Q_\lambda(x, \cdot)$ is λ^{-k}.*

(b) *$\mathfrak{L} \subset L^1(Q_\lambda(x, \ ))$ for each x in B and $\lambda > 0$ and*

$$G_\lambda^k f(x) = \int_B f(y) Q_\lambda(x, dy). \tag{4}$$

(c) *If f is a function in $\mathfrak{L}$, then $u(x) = G_\lambda^k f(x)$ satisfies the equation $(\mathfrak{N} + \lambda I)^k u = f$ (cf. [**1**]).*

PROOF. (a) follows from Lemma 1(a).

(b) Using the same idea as in the proof of Lemma 1(c), we see that f^+ and f^- are integrable with respect to $Q_\lambda(x, dy)$ and $G_\lambda^k f^+(x) = \int_B f^+(y)Q_\lambda(x, dy)$, $G_\lambda^k f^-(x) = \int_B f^-(y)Q_\lambda(x, dy)$, which yield the identity (4).

Finally, imitating the proof of [**2**, Theorem 3.5], (c) follows immediately. □

REMARK. Proposition 1 shows that the fundamental solution of $(\mathfrak{N} + \lambda I)^k$ exists in the sense of measure which is given by the family $\{Q_\lambda(x, \cdot)\}$. □

PROPOSITION 2. *Let* $\{f_\lambda : \lambda \in R^+\}$ *be a net of functions in* $\mathfrak{L}$ *satisfying the following conditions*:

(C-1) *There exist constants* c, c' *such that*

$$|f_\lambda(x) - f_\lambda(y)| \leq c \cdot e^{c'\|x\|} e^{c'\|y\|} \|x - y\|$$

for all $x, y \in B$ *and* $\lambda \in R^+$.

(C-2) $\lim_{\lambda \to 0} f_\lambda(x) = f(x)$.

Then we have

$$\lim_{\lambda \to 0} R_\lambda f_\lambda(x) = \int_0^\infty [o_t f(x) - p_1 f(0)]\, dt. \tag{5}$$

In particular, if $f \in \mathfrak{L}$, *then* $\lim_{\lambda \to 0} R_\lambda f(x) = Rf(x)$, *where* $Rf(x)$ *is defined by the limit function of* (5).

PROOF. Write out the expression of $R_\lambda f_\lambda(x)$ and use Lebesgue's dominated convergence theorem. □

COROLLARY 1. *Assume* $f \in \mathfrak{L}_0$. *Then*

$$G^k f(x) = \lim_{\lambda \to 0} \int_B f(y) Q_\lambda(x, dy).$$

PROOF. Noting that the net $\{G_\lambda f\}$ satisfies (C-1) and (C-2) of Proposition 2, the Corollary follows immediately. □

REMARK. To correct the previous paper, we should change properly all the statements concerning the fundamental solution of $\mathfrak{N}^k$ according to the above results. In view of Corollary 1. Theorem 3.5(b) of [**2**] should read:

Assume f is a function in $\mathfrak{L}_0$ and $Q_\lambda(x, \cdot)$ is defined as in (3). Then $G^k f(x) = \lim_{\lambda \to 0} \int_B f(y) Q_\lambda(x, dy)$ exists, $G^k f \in \mathfrak{L}(k)_0$ and $\mathfrak{N}^k(G^k f)(x) = f(x)$. □

REMARK. It is not known so far if the fundamental solution of $\mathfrak{N}^k$ exists in the sense of measure. When $k = 1$ and $f \in \mathfrak{L}_0$, we see that $p_1 f(0) = 0$ and

$$Gf(x) = Rf(x) = \int_0^\infty (o_t f(x) - 0)\, dt = \int_0^\infty \int_B f(y)[o_t(x, dy) - p_1(dy)]\, dt.$$

Since the last integral exists for all $f \in \mathfrak{L}$, one might conjecture that the the set function $R(x, A) = \int_0^\infty [o_t(x, A) - p_1(A)]\, dt$ could define a measure and the family $\{R(x, A)\}$ might form the fundamental solution of $\mathfrak{N}$. Unfortunately, if one takes $A =$ the concentrated set of p_1, then $R(x, A) = -\infty$ and $R(x, A^c) = +\infty$, thus

$R(x, \cdot)$ fails to be a measure. From this observation, we conjecture that the fundamental solution of $\mathfrak{N}$ does not exist in the sense of measure and neither does that of $\mathfrak{N}^k$. However, a proof is lacking. □

REFERENCES

1. H.-H. Kuo, *Potential theory associated with Uhlenbeck-Ornstein process*, J. Funct. Anal. **21** (1976), 63–75.

2. Y.-J. Lee, *Fundamental solutions for differential equations associated with the number operator*, Trans. Amer. Math. Soc. **268** (1981), 467–476.

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