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SMOOTH TYPE III DIFFEOMORPHISMS OF MANIFOLDS

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ABSTRACT. In this paper we prove that every smooth paracompact connected manifold of dimension $\geqslant 3$ admits a smooth type III_λ diffeomorphism for every $0 \leqslant \lambda \leqslant 1$. (Herman proved the result for $\lambda = 1$ in [7].) The result follows from a theorem which gives sufficient conditions for the existence of smooth ergodic real line extensions of diffeomorphisms of manifolds.

1. Introduction. Let f be a nonsingular ergodic automorphism of a Lebesgue space $(X, \mathcal{S}, \mu)$. The problem of describing the conditions under which μ is equivalent to an f-invariant measure has been the subject of much study [**2, 3, 4, 6, 11, 13**]. Herman and Katznelson have constructed examples of diffeomorphisms of the circle which do not admit any invariant measure equivalent to Lebesgue measure [**8, 11**]. In [**6**], some relationships between the rotation number of a diffeomorphism of T^1 and its measure theoretic properties are studied further. In [**8**] Herman proves the existence of smooth type III_1 diffeomorphisms of every paracompact connected manifold of dimension $m \geqslant 3$.

In this paper we study the more pathological types of non-measure-preserving diffeomorphisms of manifolds. In particular we construct examples of smooth type III_0 and III_λ, $0 < \lambda < 1$, diffeomorphisms of C^∞ manifolds. The main theorem of the paper is the following.

THEOREM 5.5. *Every smooth paracompact connected manifold of dimension* $\geqslant 3$ *admits a type* III_λ *diffeomorphism for every* $0 \leqslant \lambda \leqslant 1$.

We prove the theorem in a sequence of steps, following a general method introduced by Anosov in [**5**] and used by Herman to construct type III_1 diffeomorphisms. In order to obtain a type III_0 diffeomorphism on an m-dimensional manifold M, we construct a flow with the desired property on $T^2 \times \mathbf{R}^{m-2}$ and extend it to a globally defined flow with the same property on M.

§2 supplies some necessary definitions and notation. In §3, we prove that starting with a type III_0 diffeomorphism of the circle (which we know exists by [**11**]) we can obtain a type III_0 flow and diffeomorphisms on higher dimensional tori. We prove the following slightly more general assertion: if X is any smooth manifold which admits a type III_0 diffeomorphism f, then the diffeomorphism of $X \times T^1$ given by $(x, y) \mapsto (fx, y + t (\mathrm{mod}\ 1))$, $x \in X$, $y \in T^1$, is of type III_0 for m-a.e. $t \in (0, 1)$.

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§4 offers a proof that we can obtain real line extensions of type III_0 diffeomorphisms which preserve the ratio set (cf. §2). This result is not always true in the measure-preserving case. In fact it is well known that for most irrational numbers α, the diffeomorphism of the circle given by rotation through α admits no ergodic real line extensions of the form: $(y, z) \mapsto (y + \alpha, z + \psi y)$, where $\psi\colon T^1 \to \mathbf{R}$ is smooth. (Using Fourier coefficients the equation $\psi(y) = \varphi(y) - \varphi(y + \alpha)$ has a C^∞ solution for m-a.e. α, hence the extension by ψ is not even ergodic.) However, Theorem 4.13 states that every type III_0 diffeomorphism of a paracompact manifold (and type III_λ, $0 < \lambda < 1$, as well by a small modification) admits weakly equivalent real line extensions.

The last section, §5, then applies the methods of Anosov and Herman to extend the type III_λ flows we have constructed on $T^2 \times \mathbf{R}^{m-2}$ to arbitrary paracompact connected manifolds of dimension m.

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2. Ergodic theory preliminaries. Let $(X, \mathbb{S}, \mu)$ denote a Borel space where μ is a probability measure on $(X, \mathbb{S})$. Let f denote a nonsingular ergodic transformation of $(X, \mathbb{S}, \mu)$, i.e. every f-invariant set $B \in \mathbb{S}$ satisfies either $\mu(B) = 0$ or $\mu(B) = 1$. We define the set $\mathrm{Aut}(X, \mathbb{S}, \mu) = \{T\colon (X, \mathbb{S}) \hookleftarrow$ such that T is a nonsingular Borel automorphism of $(X, \mathbb{S})\}$, and let

$$O_f(x) = \{f^n x\colon n \in \mathbf{Z}\}.$$

The *full group* of f is defined by

$$[f] = \{V \in \mathrm{Aut}(X, \mathbb{S}, \mu)\colon Vx \in O_f(x) \text{ for } \mu\text{-a.e. } x \in X\}.$$

DEFINITION 2.1. Two transformations $f, g \in \mathrm{Aut}(X, \mathbb{S}, \mu)$ are *weakly equivalent* if there exists a measurable invertible map $\psi\colon X \to X$ with $\psi_*^{-1}\mu \sim \mu$ and $\psi(O_f(x)) = O_g(\psi x)$ for μ-a.e. $x \in X$.

We now introduce an invariant of weak equivalence.

DEFINITION 2.2. Let $f \in \mathrm{Aut}(X, \mathbb{S}, \mu)$ be an invertible, ergodic transformation. A nonnegative real number t is said to lie in the *ratio set of* f, $r^*(f)$, if for every Borel set $B \in \mathbb{S}$ with $\mu(B) > 0$, and for every $\varepsilon > 0$,

$$\mu\left(\bigcup_{n\in\mathbf{Z}}\left(B \cap f^n B \cap \left\{x \in X\colon \left|\frac{d\mu f^{-n}}{d\mu}(x) - t\right| < \varepsilon\right\}\right)\right) > 0.$$

Here $d\mu f^{-n}/d\mu$ denotes the Radon-Nikodym derivative of $f_*^n\mu$ with respect to μ. We set $r(f) = r^*(f)\setminus\{0\}$. One can show that $r(f)$ is a closed subgroup of the multiplicative group of positive real numbers $\mathbf{R}^+$, and that f admits a σ-finite invariant measure if and only if $r^*(f) = \{1\}$ [**13**]. If f has no σ-finite invariant measure equivalent to μ, there are three possibilities:

(1) $r^*(f) = \{t \in \mathbf{R}\colon t \geq 0\}$, in which case f is said to be of type III_1;

(2) $r^*(f) = \{0\} \cup \{\lambda^n\colon n \in \mathbf{Z}\}$ for $0 < \lambda < 1$; in this case f is said to be of type III_λ; or,

(3) $r^*(f) = \{0, 1\}$; then f is of type III_0.

The ratio set is actually an example of the set of essential values for a particular cocycle for f. We shall briefly introduce these more general concepts from the study of nonsingular group actions on measure spaces. For the purposes of this paper, we will give the definitions in the differentiable context; for the most general definitions, we refer the reader to [**16**].

Let $(X, \mathbb{S}, \mu)$ denote a compact Riemannian manifold with $\mathbb{S}$ the σ-algebra of Borel sets, and let μ denote a smooth probability measure. By $\mathrm{Diff}^\infty(X)$ we will denote the set of C^∞ diffeomorphisms of X. We define the metric

$$d_\infty(f, g) = \sum_{r=1}^{\infty} 2^{-r} \frac{d_r(f, g)}{1 + d_r(f, g)} \quad \text{for every } f, g \in \mathrm{Diff}^\infty(X),$$

where d_r denotes the usual C^r metric. Clearly $\mathrm{Diff}^\infty(X)$ is a Baire space.

Let $f \in \mathrm{Diff}^\infty(X)$ be μ-ergodic and let H be a locally compact second countable abelian group. The action $(n, x) \mapsto f^n(x)$ of $\mathbf{Z}$ on X is nonsingular since for every $n \in \mathbf{Z}$, $x \mapsto f^n x$ is a Borel automorphism of X which leaves μ quasi-invariant.

DEFINITION 2.3. A Borel map a: $\mathbf{Z} \times X \to H$ is called a *cocycle* for $\mathbf{Z}$ if the following condition holds:

For every $n, m \in \mathbf{Z}$ and for every $x \in X$, we have

$$a(n, f^m x) - a(n + m, x) + a(m, x) = 0.$$

A cocycle a: $\mathbf{Z} \times X \to H$ is called a *coboundary* if there exists a Borel map b: $X \to H$ with $a(n, x) = b(f^n x) - b(x)$, $n \in \mathbf{Z}$, for μ-a.e. $x \in X$. Two cocycles a_1 and a_2 are said to be *cohomologous* if their difference is a coboundary.

The following defines a cohomology invariant which generalises the concept of the ratio set.

DEFINITION 2.4. Let $(X, \mathbb{S}, \mu)$ be as above, G a countable group which acts nonsingularly and ergodically on $(X, \mathbb{S}, \mu)$ and let a: $G \times X \to H$ be a cocycle for G. An element $\alpha \in \overline{H} = H \cup \{\infty\}$ is called an *essential value* of a if, for every Borel set $B \in \mathbb{S}$ with $\mu(B) > 0$ and for every neighbourhood $N(\alpha)$ of α in $\overline{H}$,

$$\mu\big(B \cap g^{-1}B \cap \{x\colon a(g, x) \in N(\alpha)\}\big) > 0$$

for some $g \in G$. The set of essential values is denoted by $\overline{E}(a)$, and we put $E(a) = \overline{E}(a) \cap H$. We will state a few well-known properties of $\overline{E}(a)$.

(1) $\overline{E}(a)$ is a nonempty closed subset of $\overline{H}$;

(2) $E(a)$ is a closed subgroup of H;

(3) $\overline{E}(a) = \{0\}$ if and only if a is a coboundary;

(4) $\overline{E}(a_1) = \overline{E}(a_2)$ whenever a_1 and a_2 are cohomologous.

DEFINITION 2.5. Let $(X, \mathbb{S}, \mu)$ be as above, G a countable group acting nonsingularly and ergodically on X, and a: $G \times X \to H$ a cocycle for G. The cocycle a is *recurrent* if, for every $B \in \mathbb{S}$ with $\mu(B) > 0$, and for every neighbourhood $N(0)$ in H,

$$\mu\Big(\bigcup_{g \in G} \big(B \cap g^{-1}B \cap \{x\colon a(g, x) \in N(0)\} \cap \{x\colon gx \neq x\}\big)\Big) > 0.$$

We will now recall some definitions relating to flows, or $\mathbf{R}$-actions, on manifolds.

DEFINITION 2.6. A C^r flow on a manifold X is a C^r map $f\colon X \times \mathbf{R} \to X$ such that if we denote $f_t(x) = f(x, t)\,\forall x \in X$, $t \in \mathbf{R}$, then

(i) $f_{t+s}(x) = f_t f_s(x)\,\forall t, s \in \mathbf{R}$, $\forall x \in X$;

(ii) $f_0 = \mathrm{Id}_X$.

Then it follows that

(iii) $f_{-t} = f_t^{-1}$.

A flow on $(X, \mathbb{S}, \mu)$ is μ-ergodic if $f_t(A) = A$ $(A \in \mathbb{S})\,\forall t \in \mathbf{R}$ implies $\mu(A) = 0$ or $\mu(A^c) = 0$.

DEFINITION 2.7. A nonsingular flow f_t on $(X, \mathbb{S}, \mu)$ is of *type* III if it admits no σ-finite invariant measure equivalent to μ.

If f is any orientation-preserving diffeomorphism of a smooth manifold of dimension $n \geqslant 1$, there exists a canonical method for obtaining a flow on an $(n+1)$-dimensional manifold, from f, called the suspension flow of f. We first define a flow on $X \times \mathbf{R}$ by $G_t(x, y) = (x, y + t)\,\forall x \in X$, $y \in \mathbf{R}$, $t \in \mathbf{R}$. We next consider the equivalence relation on X given by $(x, y) \sim (f^n x, y + n)\,\forall n \in \mathbf{Z}$ and we see that G_t induces a flow, F_t, on $X \times \mathbf{R}/\!\sim\, \cong X \times T^1$. We call F_t the suspension flow of f.

The following lemma appears in [**7** and **14**].

LEMMA 2.8. *Let $(Y, \mathfrak{B}, \nu)$ be a Lebesgue space, ν a positive σ-finite measure and G_t a flow on Y preserving ν and ν-ergodic. If G_t has no orbit of full ν-measure, then there exists a set $B \subset [0, 1]$, $m(B) = 1$, such that for all $t_0 \in B$, $G_{t_0} \in \mathrm{Aut}(Y, \nu)$ is ν-ergodic.*

Finally, we define type III_λ flows for $0 \leqslant \lambda \leqslant 1$, and give a definition for weak equivalence of flows.

DEFINITION 2.9. A smooth flow G_t on a C^∞ manifold $(X, \mathbb{S}, \mu)$ is of *type* III_λ $(0 \leqslant \lambda \leqslant 1)$ if for m-a.e. $t_0 \in \mathbf{R}$, $G_{t_0} \in \mathrm{Aut}(X, \mu)$ is of type III_λ.

Two flows g_t and G_t are *weakly equivalent* if there exists a measurable invertible map $\psi\colon X \to X$ with $\psi_*^{-1}\mu \sim \mu$ and $\psi(g_t(x)) = G_s(\psi x)$ for μ-a.e. $x \in X$, and some $s, t \in \mathbf{R}$.

3. Type III_0 flows and diffeomorphisms of T^n. This section contains a crucial step in the process of obtaining a type III_0 diffeomorphism on an arbitrary manifold; we prove the existence of a flow with the desired property. In order to obtain a type III_0 flow on T^2, we start with a type III_0 diffeomorphism, f, of T^1 and then take the suspension flow. We need to check that for m-a.e. $t \in (0, 1)$, the diffeomorphism given by $(x, w) \mapsto (fx, w + t)$ is also of type III_0. This involves proving that, usually, the nontrivial ergodic decomposition of the skew product given by

$$(x, z) \mapsto \left(fx, z + \log \frac{d\mu f^{-1}}{d\mu}(x) \right) \text{ on } X \times \mathbf{R}$$

is preserved when we pass to one higher dimension.

In Theorem 3.2 we prove that a type III_0 diffeomorphism of an arbitrary smooth manifold X can be extended to a type III_0 flow on $X \times T^1$. As a corollary we obtain type III_0 diffeomorphisms of T^n, for $n \geqslant 2$.

Proposition 3.1. *Let $(X, \mathcal{S}, \mu)$ be a smooth manifold with μ a C^∞ σ-finite measure on X. Suppose $f \in \mathrm{Aut}(X, \mu)$ is a μ-ergodic diffeomorphism of type* III_λ $(0 \leq \lambda \leq 1)$, *and there exists a $\beta \in [0, 1]$ satisfying*:

(1) *F_β: $X \times T^1 \to X \times T^1$ defined by $(x, w) \mapsto (fx, w + \beta)$ is $\mu \otimes m$-ergodic*;

(2) *$(Y, \mathcal{T}, \rho)$ and $\{q_y: y \in Y\}$ are the Borel space and family of ergodic measures, respectively, corresponding to the ergodic decomposition of S_f: $X \times \mathbf{R} \to X \times \mathbf{R}$ given by*

$$(x, z) \mapsto \left(fx, z + \log \frac{d\mu f^{-1}}{d\mu}(x) \right)$$

(see **[16]**), *and the ergodic decomposition of $S_{F_\beta} = S_\beta$: $X \times T^1 \times \mathbf{R} \to X \times T^1 \times \mathbf{R}$ defined by*

$$(x, w, z) \mapsto \left(fx, w + \beta, z + \log \frac{d\mu \otimes mF_\beta^{-1}}{d\mu \otimes m}(x, w) \right)$$

has Borel space $(\tilde{Y}, \tilde{\mathcal{T}}, \tilde{\rho})$ and ergodic measures $\{q_{\tilde{y}}: \tilde{y} \in \tilde{Y}\}$ such that $q_{\tilde{y}} = q_y \otimes m$ for $\tilde{\rho}$-a.e. $\tilde{y} \in \tilde{Y}$ and ρ-a.e. $y \in Y$.

Then F_β is also of type III_λ.

Proof. We prove the theorem by showing that $r^*(F_\beta) = r^*(f)$ which is equivalent to proving that $\bar{E}(a_f) = \bar{E}(a_\beta)$ where a_f and a_β are the following cocycles:

$$a_f: \mathbf{Z} \otimes X \to \mathbf{R}, \qquad a_f(n, x) = \log \frac{d\mu f^{-n}}{d\mu}(x) \quad \text{and}$$

$$a_\beta: \mathbf{Z} \times X \times T^1 \to \mathbf{R}, \qquad a_\beta(n, x, w) = \log \frac{d\mu \otimes mF_\beta^{-n}}{d\mu \otimes m}(x, w) = \log \frac{d\mu f^{-n}}{d\mu}(x)$$

for every $n \in \mathbf{Z}$, $x \in X$, $w \in T^1$.

From **[16**, §5**]** we recall that $I(a_f) = \{\lambda \in \mathbf{R}: R_\lambda B = B (\mu \bmod 0)$ for every $B \in \mathcal{S} \times \mathcal{C} \subset X \times \mathbf{R}$ which is S_f-invariant$\}$, and $I(a_\beta)$ is defined analogously, where $R_\lambda(x, z) = (x, z + \lambda)$ for all $x \in X$, $z \in \mathbf{R}$, $\lambda \in \mathbf{R}$, and $R_\lambda^\beta(x, w, z) = (x, w, z + \lambda)$ for every $x \in X$, $w \in T^1$, $z \in \mathbf{R}$, $\lambda \in \mathbf{R}$. Now by Theorem 5.2 in **[16]** we have that $E(a) = I(a)$ for all cocycles a: $\mathbf{Z} \times X \to \mathbf{R}$ (and for all a: $\mathbf{Z} \times X \times T^1 \to \mathbf{R}$) so the proposition is true for $\lambda > 0$ if we show that $I(a_f) = I(a_\beta)$, and if $\lambda = 0$, $I(a_\beta) \subset I(a_f)$ plus the uniqueness of ergodic decomposition suffice to prove the result. (This will be discussed further later.)

Let π: $X \times T^1 \times \mathbf{R} \to X \times \mathbf{R}$ denote the projection mapping defined by

$$(x, w, z) \mapsto (x, z) \quad \text{for every } x \in X, w \in T^1, z \in \mathbf{R}.$$

Clearly, $S_f \circ \pi = \pi \circ S_\beta$ (as maps from $X \times T^1 \times \mathbf{R}$ to $X \times \mathbf{R}$), and also for every $\lambda \in \mathbf{R}$, $R_\lambda \circ \pi = \pi \circ R_\lambda^\beta$.

We now prove that π defines an isomorphism (mod sets of $\mu \otimes m$-measure zero) from S_f-invariant sets of $X \times \mathbf{R}$ to S_β-invariant sets of $X \times T^1 \times \mathbf{R}$. Suppose that $B \in \mathcal{S} \times \mathcal{C} \subset X \times \mathbf{R}$ is S_f-invariant. Then $S_f B = B$ implies that $B = S_f \circ \pi(\pi^{-1}B) = \pi \circ S_\beta(\pi^{-1}B)$, hence $\pi^{-1}B = S_\beta(\pi^{-1}B)$ so $\pi^{-1}B$ is S_β-invariant. Now let $D \in \mathcal{S} \times \mathcal{J} \times \mathcal{C} \subset X \times T^1 \times \mathbf{R}$ satisfy $S_\beta D = D$. Then $S_f \pi D = \pi \circ S_\beta D = \pi D$, so πD is S_f-invariant.

Now $\lambda \in I(a_f)$ implies for every S_β-invariant D, $R_\lambda(\pi D) = \pi D$, which implies that $\pi^{-1}R_\lambda \pi D = \pi^{-1}\pi D$. Equivalently, $\pi^{-1}\pi R_\lambda^\beta D = D$, so $\lambda \in I(a_\beta)$.

If $\lambda \in I(a_\beta)$, then for every S_f-invariant B, $R_\lambda^\beta \pi^{-1}B = \pi^{-1}B$. Then $R_\lambda B = R_\lambda \pi(\pi^{-1}B) = \pi R_\lambda^\beta \pi^{-1}B = \pi\pi^{-1}B = B$, so $\lambda \in I(a_f)$.

In the case $\lambda = 0$, we have $\bar{E}(a_f) = \{0, \infty\}$. Since $\tilde{Y} \cong Y$ by hypothesis, and by the one-to-one correspondence between S_f and S_β invariant sets, F_β cannot be of type II (i.e. have $r^*(F_\beta) = \{1\}$ which would imply that S_β is conjugate to (F_β, Id)), so it must be of type III_0.

Using Proposition 3.1, it is now an easy consequence to prove the following theorem.

THEOREM 3.2. *If X is any smooth manifold which admits a type III_0 diffeomorphism, then the suspension flow of that diffeomorphism is of type III_0 on $X \times T^1$ (i.e., for m-a.e. $t \in (0, 1)$, the diffeomorphism obtained by fixing the flow at t is of type III_0).*

PROOF. Assume that f is a type III_0 diffeomorphism of X. We consider the ergodic decomposition of the skew product defined by

$$F: X \times \mathbf{R} \to X \times \mathbf{R},$$

$$(x, z) \mapsto \left(fx, z + \log \frac{d\mu f^{-1}}{d\mu}(x) \right), \qquad x \in X, z \in \mathbf{R},$$

which preserves the measure $\nu = e^{-z}\, d\mu \otimes dz$. By **[16]** there exists a Borel probability space $(Y, \mathfrak{T}, \rho)$ and σ-finite measures q_y on $(T^k \times \mathbf{R}, \mathfrak{S}_k, \nu)$ such that:

(i) $y \mapsto q_y(B)$ is Borel for every $B \in \mathfrak{S}_k$.

(ii) $\nu(B) \sim \int_{T^k \times \mathbf{R}} q_y(B)\, d\rho(y)$.

(iii) Every q_y, $y \in Y$, is invariant and ergodic under F, and q_y and $q_{y'}$ are mutually singular when $y \neq y'$.

(iv) Let $\mathfrak{X} = \{B \in \mathfrak{S}_k: F(B) = B\}$. For every $B \in \mathfrak{X}$, put $B_y = \{y \in Y: q_y(B) > 0\}$. Then $\mathfrak{X}_Y = \{B_Y: B \in \mathfrak{X}\}$ is equal to $\mathfrak{T}$ mod sets of ρ-measure zero.

For each $y \in Y$, the map F_t: $X \times \mathbf{R} \times T^1 \to X \times \mathbf{R} \times T^1$ defined by

$$(x, z, w) \to \left(fx, z + \log \frac{d\mu f^{-1}}{d\mu}(x), w + t \right)$$

is $q_y \otimes m$-ergodic for m-a.e. $t \in [0, 1]$. This can be shown by taking the suspension flow of F and applying Lemma 2.8 for each $y \in Y$. If we can prove that the set $Q = \{(y, t) \in (Y \times I, \mathfrak{T} \times \mathfrak{I}, \rho \times m)$: F_t is $q_y \otimes m$-ergodic$\}$ is $\rho \otimes m$-measurable, then by Fubini's Theorem, since $\rho \otimes m(Q) = 1$ there exists a set $C \subset [0, 1]$, $m(C) = 1$ such that if $t \in C$ for ρ-a.e. $y \in Y$, F_t is $q_y \otimes m$-ergodic. By the uniqueness of ergodic decomposition, this implies that F_t is of type III_0.

We conclude the proof with the following lemma.

LEMMA 3.3. *The set*

$$Q = \left\{ (y, t) \in (Y \times I, \mathfrak{T} \times \mathfrak{I}, \rho \otimes m): F_t \textit{ is } q_y \otimes m\textit{-ergodic} \right\}$$

is measurable.

PROOF. Let $\tilde{X} = X \times \mathbf{R}$, and denote by X_n the manifold defined by $X_n = X \times (-n, n)$. The skew product F defined in 3.1 is always conservative, so we can induce on X_n. Let F_n denote the induced transformation on X_n, and we write F_{nt} for the map

$$F_{nt}: X_n \times T^1 \to X_n \times T^1,$$

$$(x, z, w) \mapsto (F_n(x, z), w + t) \quad \text{for every } x \in X, z \in (-n, n), w \in T^1.$$

We define, for every $y \in Y$ and $n \geqslant 1$, a normalised measure on X_n equivalent to the induced measure obtained from q_y restricted to X_n. Call these measures $p_y^{(n)}$; then $p_y^{(n)}(X_n) = 1$ for $n \geqslant 1$. Clearly we have F_t is $q_y \otimes m$-ergodic if and only if F_{nt} is $p_y^{(n)} \otimes m$-ergodic for every $n \geqslant 1$. To show Q is measurable, we show that $Q_n = \{(y, t) \in Y \times I \mid F_{nt} \text{ is } p_y^{(n)} \otimes m\text{-ergodic}\}$ is $\rho \otimes m$-measurable for each $n \in N$.

To show Q_n is measurable, we use the following claim, a different version of which is proved in [7].

Claim. With the above notation, F_{nt} is $p_y^{(n)} \otimes m$-ergodic if and only if

$$\inf_{m \geqslant 1} \left| \frac{1}{m} \sum_{j=0}^{m-1} h_i \circ F_{nt}^j - \int_{X_n \times I} h_i \, dp_y^{(n)} \otimes m \right|_{L^2(X_n \times I, p_y^{(n)} \otimes m)} = 0,$$

where $|\ |_{L^2(X_n \times I, p_y^{(n)} \otimes m)}$ denotes the relevant L^2 norm and $h_i \in \{h_k\}_{k \in \mathbf{N}}$ and $\{h_k\}_{k \in \mathbf{N}}$ is a countable dense sequence of Borel (L^2) functions on $X_n \times I$ (hence measurable for $p_y^{(n)} \otimes m$, for every $y \in Y$).

Since the infimum of measurable functions is measurable, and since the countable intersection of measurable sets is measurable, it suffices to show that for each fixed m, i, and $n \in \mathbf{N}$, the map

$$(x, z, w, y, t) \to \left| \frac{1}{m} \sum_{j=0}^{m-1} h_i \circ F_{nt}^y(x, z, w) - \int_{X_n \times I} h_i \, dp_y^{(n)} \otimes m \right|_{L^2(X_n \times I, p_y^{(n)} \otimes m)}$$

is measurable, for every $x \in X_n$, $z \in (-n, n)$, $w \in T^1$, $y \in Y$, and $t \in [0, 1]$.

Using the definition of Lebesgue integral and elementary facts about measurable functions, it is not difficult to see that the map $\Phi_{(m,i,n)}: X \times (-n, n) \times T^1 \times Y \times I \to \mathbf{R}$ given by

$$(x, z, w, y, t) \mapsto \left| \frac{1}{m} \sum_{j=0}^{m-1} h_i \circ F_{nt}^j(x, z, w) - \int_{X_n \times T^1} h_i \, dp_y^{(n)} \otimes m \right|_{L^2(X_n \times I)}$$

is Borel for each fixed triplet (m, i, n) and hence the infimum map, denoted $\Phi_{i,n}$, is Borel also.

Thus the set $\overline{Q} = \bigcap_{n \in \mathbf{N}} \bigcap_{i \in \mathbf{N}} \Phi_{i,n}^{-1}(0)$ is a measurable set in $X \times \mathbf{R} \times T^1 \times Y \times I$, and by Fubini's Theorem we have that the projection on $Y \times I$ of $\overline{Q}$ is measurable in $Y \times I$. Now we conclude by observing that

$$\pi_{Y \times I}(\overline{Q}) = \{(y, t) \in Y \times I \mid F_t \text{ is } \rho \otimes m\text{-ergodic}\} = Q,$$

so the lemma is proved.

COROLLARY 3.4. *For every $n \geqslant 1$, there exist type* III_0 *diffeomorphisms of* T^n.

PROOF. We use induction. For $n = 1$, the theorem is true by [11]. At the kth step we take the suspension flow and apply Theorem 3.2.

4. Type III_0 diffeomorphisms of $T^1 \times \mathbf{R}$. In this section we prove the existence of smooth, ergodic real line extensions for the $\mathbf{Z}$-action of a type III_λ diffeomorphsim on a smooth, paracompact manifold X, for any $0 \leqslant \lambda \leqslant 1$. It is not difficult to show that in the type III_0 case, most of these ergodic extensions give type III_0 diffeomorphisms of $X \times \mathbf{R}$. Our goal is to obtain type III_0 diffeomorphisms of $T^1 \times \mathbf{R}$, so we can complete our construction on arbitrary manifolds in §5. Special thanks are due to Klaus Schmidt and Ralf Spatzier for helpful discussions on the construction given in Theorem 4.8.

We begin with a theorem proved in [16].

THEOREM 4.1. *Let $(X, \mathcal{S}, \mu)$ denote a C^∞ manifold with smooth probability measure μ, and let $f \in \mathrm{Diff}^\infty(X)$ be μ-ergodic. If H denotes a locally compact second countable abelian group with Haar measure λ_H, and if a: $\mathbf{Z} \times X \to H$ is a cocycle for the $\mathbf{Z}$-action of f on X, then the map S_a: $X \times H \to X \times H$ given by $(x, h) \mapsto (fx, h \cdot a(1, x))$ is $\mu \otimes \lambda_H$-ergodic if and only if $E(a) = H$.*

Any real-valued function ϕ on a smooth manifold $(X, \mathcal{S}, \mu)$ gives rise to a cocycle for the $\mathbf{Z}$-action of any ergodic diffeomorphism, f, of X in the following way.

Using additive notation we have

$$a_\phi(n, x) = \begin{cases} \sum_{k=0}^{n-1} \phi(f^k x) & \text{if } n \geqslant 1, \\ 0 & \text{if } n = 0, \\ -a(-n, f^n x) & \text{if } n \leqslant -1 \end{cases}$$

for μ-a.e. $x \in X$.

From the above definition one can check that a_ϕ is a coboundary if and only if ϕ can be written as $\phi(x) = \eta - \eta \circ f(x)$ for μ-a.e. $x \in X$ and for some Borel map η: $X \to \mathbf{R}$. We say that ϕ is a coboundary if a_ϕ is a coboundary and we will write $\bar{E}(a_\phi) = \bar{E}(\phi)$ unless confusion arises.

If $V \in [f]$, then recall that $V(x) = f^{n(x)}(x)$ for μ-a.e. $x \in X$. Using the notation of [16] we let $a_\phi(V, x) = a_\phi(n(x), x)$ if $V(x) = f^{n(x)}(x)$.

We prove a result which is motivated by a result of Jones and Parry [10] for continuous and compact extensions.

PROPOSITION 4.2. *Let $(X, \mathcal{S}, \mu)$ be a smooth connected compact manifold with μ a C^∞ probability measure on X. Let $f \in \mathrm{Diff}^\infty(X)$ be an ergodic diffeomorphism. Suppose there exists an element which is not a coboundary in the set*

$$\mathcal{C} = \overline{\{ \phi \in C^\infty(X, R) \mid \phi = \eta - \eta \circ f \text{ for some Borel map } \eta \colon X \to \mathbf{R} \}}$$

(where the closure is taken with respect to the C^∞ topology). Then the set of coboundaries is meagre in $\mathcal{C}$.

PROOF. The proof is similar to that of [10]. We consider $\mathcal{C}$ as a complete topological group under pointwise addition and with respect to the C^∞ topology. If we let

$$E = \{\phi\colon X \to R \mid \phi \text{ is a Borel map and } \phi - \phi \circ f = h \text{ a.e. for some } h \in \mathcal{C}\},$$

and identify two functions in E if and only if they are equal μ-a.e., then we see that E is a group under pointwise addition.

We define the map $L\colon E \to \mathcal{C}$ by setting $L(\phi) = \phi - \phi \circ f$. We see that L is a group homomorphism and $\ker L =$ constant maps $\cong R$. We now define a metric on E by

$$\delta_{\mathbf{R}}(\phi_1, \phi_2) = \int_X \frac{|\phi_1 - \phi_2|}{1 + |\phi_1 - \phi_2|} d\mu + \|L\phi_1 - L\phi_2\|_\infty \quad \text{for all } \phi_1, \phi_2 \in E;$$

then we see that E is complete and separable with respect to $\delta_{\mathbf{R}}$, and that L is a continuous group homomorphism.

Using the Open Mapping Theorem, and the assumption that there exists $\psi \in \mathcal{C}$ such that $\psi \notin$ image L, the proposition is proved.

REMARK 4.3. Under the assumption that there exists at least one element which is not a coboundary in $\mathcal{C}$, we will prove that there is in fact a dense G_δ of elements in $\mathcal{C}$ which contain all of $\mathbf{R}$ in their essential range; i.e. there is a dense G_δ in $\mathcal{C}$, call it $\mathcal{E}_{\mathcal{C}}$, such that if $\phi \in \mathcal{E}_{\mathcal{C}}$, then the skew product given by $F\colon X \times \mathbf{R} \to X \times \mathbf{R}$, where $(x, y) \mapsto (fx, y + \phi x)$ is $\mu \otimes m$-ergodic. We begin with an easy lemma, whose proof is similar to [**16**, 9.6].

LEMMA 4.4. *Let* $(X, \mathcal{S}, \mu)$ *and* f *be as in* 4.2. *The set* $\mathcal{E}_{\mathcal{C}} = \{\phi \in C^\infty(X, R) \mid F\colon X \to \mathbf{R}$ *defined in Remark* 4.3 *is* $\mu \otimes m$*-ergodic*$\}$ *is a* G_δ.

PROOF. By Theorem 4.1, $\mathcal{E}_{\mathcal{C}} = \{\phi \in C^\infty(X, R) \mid E(\phi) = \mathbf{R}\}$. Let $\mathcal{S}_0$ denote a countable, dense subalgebra for X. Let $\{\lambda_i\}_{i \in \mathbf{N}}$ be a dense sequence in $\mathbf{R}$. Fix any element $B \in \mathcal{S}_0$ and any number $\beta \in \{\lambda_i\}_{i \in \mathbf{N}}$. We claim that for every fixed $\delta > 0$ and $\varepsilon > 0$ the set

$$U(B, \beta, \varepsilon, \delta) = \left\{\phi \in C^\infty(X, \mathbf{R}) \mid \sup_{V \in [f]} \mu\big(B \cap V^{-1}B \cap \{x\colon a_\phi(V, x) \in (\beta - \varepsilon, \beta + \varepsilon)\}\big) > \delta\right\}$$

is open in $C^\infty(X, \mathbf{R})$ with the C^∞ topology.

(PROOF OF CLAIM. Clearly, for fixed n, the map $\phi \mapsto \sum_{i=0}^{n-1} \phi \circ f^i$ is continuous in $C^\infty(X, \mathbf{R})$ with respect to the C^∞ topology. We recall that $Vx = f^{n(x)}x$ for some $n \in \mathbf{Z}$ and for μ-a.e. $x \in X$. By [**16**, 2.6], $a_\phi(V, x) = a_\phi(n(x), x) = \sum_{i=0}^{n(x)-1} \phi \circ f^i(x)$. Therefore by continuity, we can find $\delta > 0$ small enough s.t. $\|\phi - \tilde{\phi}\|_\infty < \delta$ implies that $\mu\{x\colon |a_\phi(V, x)| < \varepsilon\} = \mu\{x\colon |a_{\tilde{\phi}}(V, x)| < \varepsilon\}$.) Then

$$\bigcap_{B \in \mathcal{S}_0} \bigcap_m U\left(B, \beta, \frac{1}{m}, \frac{\mu(B)}{4}\right) = \{\phi \in C^\infty(X, R) \mid \beta \in E(\phi)\},$$

and finally we have

$$= \bigcap_i \bigcap_B \bigcap_m U\left(B, \lambda_i, \frac{1}{m}, \frac{\mu(B)}{4}\right) = \{\phi \in C^\infty(X, R) \mid E(\phi) = \mathbf{R}\},$$

which proves the proposition.

REMARK 4.5. Since $\mathcal{C} \subset C^\infty(X, \mathbf{R})$ is a closed subgroup, the set $\mathcal{E}_\mathcal{C} = \{\phi \in C^\infty(X, R) \mid E(\phi) = \mathbf{R}\}$ is also a G_δ. It is easy to see that the set $\mathcal{E}_\mathcal{C}$ is dense in $\mathcal{C}$ if it is not empty. For if there exists an element $\psi \in \mathcal{E}_\mathcal{C}$, then by adding a suitable coboundary to ψ we can find another element of $\mathcal{E}_\mathcal{C}$ arbitrarily close in the C^∞ topology to any $\phi \in \mathcal{C}$.

In the next lemma we will prove that if there exists an element $\psi \in \mathcal{C}$ such that $E(\psi) = \{n\lambda\}_{n\in\mathbf{Z}}$ for some $\lambda > 0$, then there exists an element $\tilde{\psi} \in \mathcal{E}_\mathcal{C}$.

LEMMA 4.6. *Let $(X, \mathcal{S}, \mu)$ and f be as in* 4.2. *Suppose there exists an element $\psi \in \mathcal{C}$ such that $E(\psi) = \{n\lambda\}_{n\in\mathbf{Z}}$ for some $\lambda > 0$. Then $\mathcal{E}_\mathcal{C}$ is a dense G_δ in $\mathcal{C}$.*

PROOF. Since $\lambda \in E(\psi)$, we have for all $B \in \mathcal{S}$, $\mu(B) > 0$ and all $\varepsilon > 0$,

$$\sup_{V\in[f]} \mu\Big(B \cap V^{-1}B \cap \big\{x: |a_\psi(V, x) - \lambda| < \varepsilon\big\}\Big) > \frac{\mu(B)}{2},$$

as proved in **[16]**.

We can choose an irrational scalar $c \in \mathbf{R}$, $0 < c < 1$, such that $\beta = c\lambda$, and λ and β are rationally independent. Then for all $x \in \{x: | a_\psi(V, x) - \lambda | < \varepsilon\}$, we have

$$a_{c\psi}(V, x) = \sum_{i=0}^{n(x)-1} c\psi \circ f^i(x) = c \sum_{i=0}^{n(x)-1} \psi \circ f^i(x),$$

for some $n(x) \in \mathbf{Z}$, so $| a_{c\psi}(V, x) - \beta | < \varepsilon$ as well. Let $\tilde{\psi} = c\psi$. Thus $\beta \in E(\tilde{\psi})$. By adding suitable coboundaries to $\tilde{\psi}$, we see that the set $\{\phi \in \mathcal{C} \mid E(\phi) \ni \{n\beta\}_{n\in\mathbf{Z}}\}$ is dense in $\mathcal{C}$, and the proof of Lemma 4.4 shows us that it is in fact a G_δ. Similarly, we have that $U_\lambda = \{\phi \in \mathcal{C} \mid E(\phi) \ni \{n\lambda\}_{n\in\mathbf{Z}}\}$ is a dense G_δ. Then $U_\lambda \cap U_\beta$ is also a dense G_δ in $\mathcal{C}$, and since the set of essential values for $\phi \in \mathcal{C}$ forms a closed additive subgroup of $\mathbf{R}$, $\lambda \in E(\phi)$ and $\beta \in E(\phi)$ imply that $E(\phi) = \mathbf{R}$ since λ and β are rationally independent. Therefore the proposition is proved.

We have verified the existence of a nonempty set $\mathcal{E}_\mathcal{C}$ if there exists $\psi \in \mathcal{C}$ such that $\overline{E}(\psi) = \mathbf{R}$ or $\{n\lambda\}_{n\in\mathbf{Z}}$. The only remaining possibility is that every element $\phi \in \mathcal{C}$ which is not a coboundary satisfies $\overline{E}(\phi) = \{0, \infty\}$. In Theorem 4.8 we will show that even under this assumption we can construct elements in $\mathcal{E}_\mathcal{C}$. This theorem is sufficient to ensure that type III_0 diffeomorphisms of compact manifolds have ergodic real line extensions, which is what is needed to extend type III_0 diffeomorphisms to arbitrary manifolds.

We first prove a proposition which is necessary for the construction in Theorem 4.8.

PROPOSITION 4.7. *Let $(X, \mathcal{S}, \mu)$ be a smooth paracompact connected manifold with μ a smooth σ-finite measure on X. Let $f \in \mathrm{Diff}^\infty(X)$ be a μ-ergodic diffeomorphism. We denote by $\mathcal{S}_0$ a countable dense subalgebra of $\mathcal{S}$, and $\mathcal{C}$ is as in* 4.2. *Suppose there exists $\phi \in \mathcal{C}$ such that ϕ is recurrent and $\overline{E}(\phi) = \{0, \infty\}$. Then there exists a set $\Psi \subset \mathcal{C}$, such that Ψ is a dense G_δ in $\mathcal{C}$ with respect to the C^∞ topology and every element $\psi \in \Psi$ satisfies the following condition*:

For every $\varepsilon > 0$, for every $M \in \mathbf{R}^+$, and for every $B \in \mathcal{S}_0$, there exists $\phi \in \mathcal{C}$ with $\|\phi\|_\infty \leq 1$ and $\overline{E}(\phi) = \{0, \infty\}$, and $V \in [f]$ such that

$$\mu\Big(B \cap V^{-1}B \cap \big\{x: | a_\psi(V, x) | < \varepsilon\big\} \cap \big\{x: | a_\phi(V, x) | > M\big\}\Big) \geq \mu(B)/2.$$

PROOF. We choose a countable dense set $\{\phi_i\}_{i\in\mathbf{N}}$ in the unit ball of $\mathcal{C}$, where for every $i \in \mathbf{N}$, $\bar{E}(\phi_i) = \{0, \infty\}$ and ϕ_i is recurrent. We choose a countable dense set in the full group of f, denoted $\{V_k\}_{k\in\mathbf{N}}$. Let $M \in \mathbf{N}$ denote a positive integer. We define the set

$$\Lambda(B, M, \varepsilon, l, j, k, i) = \left\{\psi \in \mathcal{C} \mid \mu\left(B \cap V_k^{-1}B \cap \left\{x: |a_\psi(V_k, x)| < \varepsilon\right\} \cap \left\{x \middle| a_{\phi_i/j}(V_k, x)\right| > M\right\}\right) > \left(1 - \frac{1}{l}\right)\mu(B)\cdot 2^{-1}.$$

By the continuity of ψ, and using techniques from 4.4 we can show that for each fixed $(B, M, \varepsilon, l, j, k, i)$ the set $\Lambda(B, M, \varepsilon, l, j, k, i)$ is open in $\mathcal{C}$ with respect to the C^∞ topology.

We also claim that $\Gamma(B, M, \varepsilon, l, j) = \bigcup_k \bigcup_i \Lambda(B, M, \varepsilon, l, j, k, i)$ is open and dense in $\mathcal{C}$. Clearly it is open, and it is dense because each $\Gamma(B, M, \varepsilon, l, j)$ contains the coboundaries. To show this, fix ε_0, M_0, B_0, j_0 and l_0. Suppose $\psi \in \mathcal{C}$ is a coboundary. Then choose any $\phi_0 \in \mathcal{C}$ which satisfies $\bar{E}(\phi_0) = \{0, \infty\}$ and $\|\phi_0\|_\infty/j_0 \leqslant 1$. Since ψ is a coboundary we write $\psi = \eta - \eta \circ f$ where η is a Borel function on X, and we find a set $D_0 \subset B_0$ such that $|\eta(x) - \eta(y)| < \varepsilon_0/4$ for all $x, y \in D_0$. Since $\infty \in \bar{E}(\phi_0)$, we can find an integer p such that

$$\mu\left(D_0 \cap f^{-p}D_0 \cap \left\{x: |a_{\phi_0/j_0}(p, x)| > M_0\right\}\right) > 0.$$

Using the exhaustion argument method of [15, 9.4], we can find an element $V_k \in [f]$ such that

$$\mu\left(B \cap V_k^{-1}B \cap \left\{x: |a_\psi(V_k, x)| < \varepsilon_0\right\} \cap \left\{x: |a_{\phi_0/j_0}(V_k, x)| > M_0\right\}\right) > \left(1 - \frac{1}{l_0}\right)\mu(B_0)2^{-1}.$$

This proves that $\psi \in \Gamma(B_0, M_0, \varepsilon_0, l_0, j_0)$.

We now define $\Psi = \bigcap_{B\in\mathbb{S}_0} \bigcap_M \bigcap_\varepsilon \bigcap_l \bigcap_j \Gamma(B, M, \varepsilon, l, j)$ (where $\varepsilon \in \{\varepsilon_r\}_{r\in\mathbf{N}}$ is a countable set such that $\varepsilon_r \leqslant 1/r$).

Clearly Ψ is a dense G_δ, and it remains to show that $\psi \in \Psi$ satisfies the hypotheses of the proposition. If $\psi \in \Psi$, then for every $\varepsilon_r > 0$, for every $M \in \mathbf{N}$, for every $B \in \mathbb{S}_0$, and for every $j, l \in \mathbf{N}$ there exists $\phi_i \in \mathbf{C}$, $\bar{E}(\phi_i) = \{0, \infty\}$, ϕ_i is recurrent, and there exists $V_k \in [f]$ satisfying

$$\mu\left(B \cap V_k^{-1}B \cap \left\{x: |a_\psi(V_k, x)| < \varepsilon\right\} \cap \left\{x: |a_{\phi_i/j}(V_k, x)| > M\right\}\right) > \left(1 - \frac{1}{l}\right)\mu(B)\cdot 2^{-1}.$$

This concludes the proof.

We make Ψ into a complete metric space by defining a metric on Ψ given by

$$D_\infty(\phi_1, \phi_2) = \|\phi_1 - \phi_2\|_\infty + d_\infty(\phi_1, \phi_2),$$

where $d_\infty(\phi_1, \phi_2)$ is defined in the following way.

Let $d(\phi, A) = \inf_{\psi \in A} \|\phi - \psi\|_\infty$ for any set $A \subset \mathcal{C}$. We index the countable set of sets $\Gamma(B, \varepsilon, M, j, k)$ by $s \in \mathbf{N}$, say. Then we define

$$d_s(\phi_1, \phi_2) = \left| \frac{1}{d(\phi_1, \Gamma_s^c)} - \frac{1}{d(\phi_2, \Gamma_s^c)} \right|,$$

where Γ_s^c denotes the complement of the open set $\Gamma_s = \Gamma_s(B, M, \varepsilon, l, j)$.

Finally we let $d_\infty(\phi_1, \phi_2) = \sum_{s=1}^\infty 2^{-s} d_s(\phi_1, \phi_2)/(1 + d_s(\phi_1, \phi_2))$. An easy calculation shows that D_∞ is a metric on Ψ, and that Ψ is complete with respect to D_∞.

We are now ready for the main theorem of this section.

THEOREM 4.8. *Let $(X, \mathcal{S}, \mu)$ and $f \in \mathrm{Diff}^\infty(X)$ be as in 4.7. Suppose there exists an element $\phi \in \mathcal{C}$ which is recurrent and not a coboundary. Then $\mathcal{E}_\mathcal{C}$ is a dense G_δ in $\mathcal{C}$ with the C^∞ topology.*

PROOF. By 4.4–4.6 it suffices to assume that every element of $\mathcal{C}$ which is not a coboundary satisfies $\bar{E}(\phi) = \{0, \infty\}$.

Let $\mathcal{S}_0$ denote a countable, dense subalgebra of X. We fix an element $B \in \mathcal{S}_0$, $\mu(B) > 0$, and we choose and fix any $\varepsilon > 0$.

We will construct $\psi \in \mathcal{C}$ and $V \in [f]$ such that

$$\mu\left(B \cap V^{-1}B \cap \left\{x: \left|a_\psi(V, x) - 1\right| < \varepsilon\right\}\right) \geq \mu(B)/2.$$

Then, using the notation and methods of Lemmas 4.4–4.6 we see that $U(B, 1, \varepsilon, \mu(B)/2)$ is open, dense, and nonempty in $\mathcal{C}$ (in the C^∞ topology) for each $B \in \mathcal{S}_0$ and $\varepsilon > 0$, and therefore the theorem is proved.

A. *Setting up the construction.* Let Ψ be defined as in 4.7. We start the induction process by defining $\phi_0 = 0$, $\tilde{B}_0 = B$, $M_0 = 1$, and $\varepsilon_0 = \varepsilon/2$. Since $\phi_0 \in \Psi$ we apply 4.7 to obtain p_1 and ϕ_1 satisfying

$$(4.1)\quad \mu\left(\tilde{B}_0 \cap f^{-p_1}\tilde{B}_0 \cap \left\{x: \left|a_{\phi_0}(p_1, x)\right| < \varepsilon_0\right\} \cap \left\{x: \left|a_{\phi_1}(p_1, x)\right| > M_0\right\}\right) > 0.$$

Since the set Ψ is dense in $\mathcal{C}$, we can perturb ϕ_1 slightly if necessary so that $\phi_1 \in \Psi$, and (4.1) still holds. We choose

$$B_1 \subseteq \tilde{B}_0 \cap f^{-p_1}\tilde{B}_0 \cap \left\{x: \left|a_{\phi_0}(p_1, x)\right| < \varepsilon_0\right\} \cap \left\{x: \left|a_{\phi_1}(p_1, x)\right| > M_0\right\}$$

such that $B_1 \cap f^{p_1}B_1 = \varnothing$. Choose $c_1 \leq 1$ satisfying

$$\mu\left(\tilde{B}_0 \cap f^{-p_1}\tilde{B}_0 \cap \left\{x: \left|a_{\phi_0}(p_1, x)\right| < \varepsilon_0\right\} \cap \left\{x: \left|a_{c_1\phi_1}(p_1, x) - 1\right| < \varepsilon_0\right\}\right) > 0.$$

We define $V_1 \in [f]$ by

$$V_1(x) = \begin{cases} f^{p_1}x & \text{if } x \in B_1, \\ f^{-p_1}x & \text{if } x \in f^{p_1}B_1 \\ x & \text{otherwise,} \end{cases}$$

and let $\tilde{B}_1 = \tilde{B}_0 \backslash (B_1 \cap f^{p_1}B_1)$. We define $\zeta_1 = c_1\phi_1$.

B. *The jth stage.* We will define inductively, $\phi_j \in \mathcal{C}$, $\|\phi_j\|_\infty \leq 1$, $c_j \in \mathbf{R}^+$, $s_j \in \mathbf{N}$, $M_j \in \mathbf{R}^+$, $\varepsilon_j > 0$, $B_j \subset B$, $\tilde{B}_j \subset B$, $p_j \in \mathbf{N}$, $\zeta_j \in \Psi$ and $V_j \in [f]$ satisfying:

$(1)_j$ $\zeta_j = \sum_{l=1}^j c_l\phi_l \in \Psi$,

$(2)_j$ $\mu(\tilde{B}_{j-1} \cap f^{-p_j}\tilde{B}_{j-1} \cap \{x: |a_{c_j\phi_j}(p_j, x) - 1 < \varepsilon_j\} \cap \{x: |a_{\zeta_{j-1}}(p_j, x)| < \varepsilon_j\}) > 0$,

$(3)_j$ $D_\infty(\zeta_{j-1}, \zeta_j) < \varepsilon_j$,

$(4)_j$ $c_j \|\Sigma_{l=0}^{p_l-1} \phi_l\|_\infty < \varepsilon(2^{j+2l})^{-1}$ for $0 \leqslant l \leqslant j-1$,

$(5)_j$ $p_j > p_{j-1}$, $\varepsilon_j = \varepsilon(2^{j+1})^{-1}$, $M_j \geqslant M_{j-1}$,

$(6)_j$ $B_j \subset \tilde{B}_{j-1} \cap f^{-p_j}\tilde{B}_{j-1} \cap \{x: |a_{c_j\phi_j}(p_j, x) - 1| < \varepsilon_j\}$, $\mu(B_j) > 0$, and $B_j \cap f^{p_j}B_j = \varnothing$. We define $\tilde{B}_j = \tilde{B}_{j-1} \backslash (B_j \cup f^{p_j}B_j)$.

$(7)_j$

$$V_j(x) = \begin{cases} V_k(x) & \text{if } x \in B_k, k \leqslant j-1, \\ f^{p_j}(x) & \text{if } x \in B_j, \\ f^{-p_j}(x) & \text{if } x \in f^{p_j}B_j, \\ x & \text{otherwise.} \end{cases}$$

C. *The induction step*. Assume we are at the jth stage. First we choose $s_{j+1} \in \mathbf{N}$ large enough so that $\Sigma_{s=s_{j+1}}^\infty 2^{-s} < \varepsilon(2^{j+3})^{-1}$. Then we define $\gamma_{s_{j+1}} = \gamma_{j+1} = \min_{s < s_{j+1}} d(\zeta_j, \Gamma_s^c)$. Since $\zeta_j \in \Psi$, $\gamma_{j+1} > 0$.

Now we choose $M_{j+1} \geqslant \max(M_j, \varepsilon_j^{-1}\gamma_{j+1}^{-2}s_{j+1}2^{4j}p_j)$. Using $(1)_j$ and Proposition 4.7 we can find $p_{j+1} > p_j$ and $\phi \in \mathcal{C}$ such that $\|\phi\|_\infty \leqslant 1$, and for $\varepsilon_{j+1} = \varepsilon_j \cdot 2^{-1}$ we have (4.2)

$$\mu\Big(\tilde{B}_j \cap f^{-p_{j+1}}\tilde{B}_j \cap \Big\{x: \big|a_{\zeta_j}(p_{j+1}, x)\big| < \varepsilon_{j+1}\Big\} \cap \Big\{x: \big|a_\phi(p_{j+1}, x)\big| > M_{j+1}\Big\}\Big) > 0.$$

Let $\phi_{j+1} = \phi$. We choose $c_{j+1} < 1/M_{j+1}$ such that

$$(4.3) \quad \mu\Big(\tilde{B}_j \cap f^{-p_{j+1}}\tilde{B}_j \cap \Big\{x: \big|a_{\zeta_j}(p_{j+1}, x)\big| < \varepsilon_{j+1}\Big\} \cap \Big\{x: \big|a_{c_{j+1}\phi_{j+1}}(p_{j+1}, x) - 1\big| < \varepsilon_{j+1}\Big\}\Big) > 0.$$

We define the set

$$B_{j+1} \subseteq \tilde{B}_j \cap f^{-p_{j+1}}\tilde{B}_j \cap \Big\{x: \big|a_{\zeta_j}(p_{j+1}, x)\big| < \varepsilon_{j+1}\Big\} \cap \Big\{x: \big|a_{c_{j+1}\phi_{j+1}}(p_{j+1}, x) - 1\big| < \varepsilon_{j+1}\Big\}$$

such that $\mu(B_{j+1}) > 0$ and $B_{j+1} \cap f^{p_{j+1}}B_{j+1} = \varnothing$.

We can assume that $\zeta_j + c_{j+1}\phi_{j+1} \in \Psi$, because if not we could have chosen $\tilde{c}_{j+1}$ or $\tilde{\phi}_{j+1}$ arbitrarily close by such that (4.2) still holds and $\zeta_j + \tilde{c}_{j+1}\tilde{\phi}_{j+1} \in \Psi$ (since Ψ is dense in $\mathcal{C}$). Define $\zeta_{j+1} = \zeta_j + c_{j+1}\phi_{j+1} = \Sigma_{i=1}^{j+1} c_i\phi_i$.

We must now check to see if $(1)_{j+1}$ through $(7)_{j+1}$ hold.

For $(7)_{j+1}$, we just define

$$V_{j+1}(x) = \begin{cases} V_k(x) & \text{if } x \in B_k, k \leqslant j, \\ f^{p_{j+1}}(x) & \text{if } x \in B_{j+1}, \\ f^{-p_{j+1}}(x) & \text{if } x \in f^{p_{j+1}}B_{j+1}, \\ x & \text{otherwise.} \end{cases}$$

By our construction, $(1)_{j+1}$, $(2)_{j+1}$, $(5)_{j+1}$, and $(6)_{j+1}$ obviously are satisfied.

To check $(3)_{j+1}$ we proceed as follows:

$$\begin{aligned}
D_\infty(\zeta_j, \zeta_{j+1}) &= \|\zeta_j - \zeta_{j+1}\|_\infty + \sum_{s=1}^{\infty} 2^{-s} \frac{d_s(\zeta_j, \zeta_{j+1})}{1 + d_s(\zeta_j, \zeta_{j+1})} \\
&= \|\zeta_j - \zeta_{j+1}\|_\infty + \sum_{s=1}^{s_{j+1}} 2^{-s} \frac{d_s(\zeta_j, \zeta_{j+1})}{1 + d_s(\zeta_j, \zeta_{j+1})} + \sum_{s=s_{j+1}+1}^{\infty} 2^{-s} \frac{d_s(\zeta_j, \zeta_{j+1})}{1 + d_s(\zeta_j, \zeta_{j+1})} \\
&\leqslant c_{j+1}\|\phi_{j+1}\|_\infty + \sum_{s=1}^{s_{j+1}} 2^{-s} \frac{d_s(\zeta_j, \zeta_{j+1})}{1 + d_s(\zeta_j, \zeta_{j+1})} + \sum_{s=s_{j+1}+1}^{\infty} 2^{-s} \\
&\leqslant c_{j+1}\|\phi_{j+1}\|_\infty + \sum_{s=1}^{s_{j+1}} 2^{-s} \frac{d_s(\zeta_j, \zeta_{j+1})}{1 + d_s(\zeta_j, \zeta_{j+1})} + \varepsilon(2^{j+3})^{-1},
\end{aligned}$$

by our choice of s_{j+1}.

We have that

$$\begin{aligned}
d_s(\zeta_j, \zeta_{j+1}) &= \left| \frac{1}{d(\zeta_j, \Gamma_s^c)} - \frac{1}{d(\zeta_{j+1}, \Gamma_s^c)} \right| \\
&\leqslant \left| \frac{1}{d(\zeta_j, \Gamma_s^c)} - \frac{1}{d(\zeta_j, \Gamma_s^c) + \|\zeta_j - \zeta_{j+1}\|_\infty} \right|,
\end{aligned}$$

since $|d(\zeta_j, \Gamma_s^c) - d(\zeta_{j+1}, \Gamma_s^c)| \leqslant \|\zeta_j - \zeta_{j+1}\|_\infty$ for each s, so the denominators can vary by at most $\|\zeta_j - \zeta_{j+1}\|_\infty$, and now we have

$$\begin{aligned}
\|\zeta_j - \zeta_{j+1}\|_\infty &\leqslant c_{j+1}\|\phi_{j+1}\|_\infty \leqslant c_{j+1} \cdot 1 \\
&\leqslant \varepsilon_j \cdot \gamma_{j+1}^2 \cdot 2^{-4j} \cdot p_j^{-1} \cdot s_{j+1}^{-1} \leqslant \varepsilon \cdot \gamma_{j+1}^2 \cdot 2^{-5j} \cdot p_j^{-1} \cdot s_{j+1}^{-1}
\end{aligned}$$

and recalling our choice of $\gamma_{j+1} > 0$, we have

$$\begin{aligned}
D_\infty(\zeta_j, \zeta_{j+1}) &\leqslant \varepsilon \cdot 2^{-5j} + \sum_{s=1}^{s_{j+1}} \frac{\gamma_{j+1}^2 \cdot \varepsilon \cdot 2^{-(j+6)} s_{j+1}^{-1}}{\gamma_{j+1}^2} \Big/ 1 + \frac{\gamma_{j+1}^2 \cdot \varepsilon \cdot 2^{-(j+6)} s_{j+1}^{-1}}{\gamma_{j+1}^2} \\
&\quad + \varepsilon \cdot 2^{-(j+3)} \\
&\leqslant \varepsilon \cdot 2^{-5j} + s_{j+1}\left(\varepsilon \cdot 2^{-(j+6)} s_{j+1}^{-1}\right) + \varepsilon \cdot 2^{-(j+3)} \\
&\leqslant \varepsilon \cdot 2^{-5j} + \varepsilon \cdot 2^{-(j+6)} + \varepsilon \cdot 2^{-(j+3)} \\
&< \varepsilon \cdot 2^{-(j+3)} = \varepsilon_{j+1}.
\end{aligned}$$

We check $(4)_{j+1}$ as follows. Clearly

$$\begin{aligned}
c_{j+1}\left\| \sum_{i=0}^{p_l - 1} \phi_{j+1} \circ f^i \right\|_\infty &\leqslant c_{j+1} \cdot p_j \cdot \|\phi_{j+1}\|_\infty \quad \text{since } 0 \leqslant l \leqslant j, \\
&\leqslant \varepsilon_j \cdot \gamma_{j+1}^2 \cdot s_{j+1}^{-1} \cdot p_j^{-1} \cdot 2^{-4j} \cdot p_j \cdot \|\phi_{j+1}\|_\infty \\
&\leqslant \varepsilon : 2^{-(j+1)} \cdot 2^{-(4j)} \leqslant \varepsilon \cdot 2^{-(j+1+2l)}.
\end{aligned}$$

D. *Taking the limit.* We let $\psi = \sum_{i=1}^{\infty} c_i \phi_i$. Since $(3)_j$ holds for all $j \geqslant 1$, ψ is the limit of a Cauchy sequence in Ψ. More precisely, given any $\delta > 0$ (we might as well assume $0 < \delta < \varepsilon$), we choose $j_0 \in \mathbf{N}$ such that $\sum_{j=j_0}^{\infty} 2^{-j} < \delta/\varepsilon$. Then for any $m > n \geqslant j_0$ we have

$$D_\infty(\zeta_n, \zeta_m) \leqslant \sum_{k=0}^{m-n-1} D(\zeta_{n+k}, \zeta_{n+k+1}) \leqslant \sum_{k=0}^{m-n-1} \varepsilon_{n+k+1}, \quad \text{by } (3)_j.$$

Now

$$\sum_{k=0}^{m-n-1} \varepsilon_{n+k+1} \leqslant \varepsilon \sum_{k=0}^{m-n-1} 2^{-(n+k+1)} \leqslant \varepsilon \sum_{k=j_0}^{\infty} 2^{-k} < \delta,$$

so $D_\infty(\zeta_n, \zeta_m) < \delta$.

Therefore $\psi \in \Psi$, since Ψ is complete. We can apply 4.7 to ψ and continue the induction process. Then, using an exhaustion argument, we obtain a sequence of sets B_i such that $B_i \cap B_j = B_i \cap f^{p_j}B_j = B_j \cap f^{p_i}B_i = B_i \cap f^{p_i}B_i = B_j \cap f^{p_j}B_j = \varnothing$, for all $i \neq j$, and such that $\mu(\bigcup_{i \in \mathbf{N}}(B_i \cup f^{p_i}B_i)) > \mu(B)/2$. We also obtain $V \in [f]$ such that

$$V(x) = \begin{cases} V_k(x) & \text{if } x \in B_k \cup f^{p_k}B_k, \\ x & \text{otherwise.} \end{cases} \tag{4.4}$$

For all $x \in B_j$, we have $V(x) = f^{p_j}(x)$, and this implies for $x \in B_j$,

$$\begin{aligned} |a_\psi(V, x)| &= |a_\psi(p_j, x)| \\ &\leqslant |a_{\zeta_{j-1}}(p_j, x)| + |a_{c_j\phi_j}(p_j, x)| + \left| \sum_{k=0}^{p_j-1} \left(\sum_{i=j+1}^{\infty} c_i\phi_i \right) \circ f^k(x) \right| \\ &\leqslant \varepsilon_j + 1 + \varepsilon_j + \left| \sum_{i=j+1}^{\infty} c_i \left(\sum_{k=0}^{p_j-1} \phi_i \circ f^k(x) \right) \right| \end{aligned}$$

by $(2)_j$, and now by $(4)_j$,

$$\begin{aligned} &\leqslant \varepsilon_j + 1 + \varepsilon_j + \sum_{i=j+1}^{\infty} \varepsilon(2^{i+2(i-1)})^{-1} \\ &\leqslant \varepsilon_j + 1 + \varepsilon_j + \varepsilon/8 \leqslant 1 + (5/8)\varepsilon. \end{aligned}$$

From the above and (4.4), an easy calculation shows that

$$\mu\big(B \cap V^{-1}B \cap \{x : |a_\psi(V, x) - 1| < \varepsilon\}\big) \geqslant \mu(B)/2,$$

and we are done.

We should point out that 4.4 is true for noncompact X, and the hypotheses on $(X, \mathbb{S}, \mu)$ in 4.7 are sufficient for 4.8 to be true. We have proved the existence of a dense G_δ of $\mathcal{C}$ whose elements give ergodic extensions for f; we now need to see which of these skew products have the same ratio sets as f. We will give a necessary and sufficient condition, but first we will recall some easily proved facts.

LEMMA 4.9. *Let* $(X, \mathcal{S}, \mu)$ *be as in* 4.7 *and let* $f \in \mathrm{Diff}^\infty(X)$ *be any ergodic diffeomorphism of* X. *If we consider the skew product* F_ϕ *defined by*

$$F_\phi\colon X \times \mathbf{R} \to X \times \mathbf{R}, \qquad (x, y) \mapsto (fx, y + \phi x) \quad \textit{where } \phi \in \mathcal{C},$$

then $r^*(F_\phi) \subset r^*(f)$.

PROOF. Let $F = F_\phi$. By definition, $r^*(F) = \bar{E}(\log(d\mu \otimes mF^{-1}/d\mu \otimes m))$ where $(d\mu \otimes mF^{-1}/d\mu \otimes m)(x, y)$ denotes the Radon-Nikodym derivative of the measure $\mu \otimes m(F^{-1})$ with respect to $\mu \otimes m$ at the point $(x, y) \in X \times \mathbf{R}$. This implies

$$\frac{d\mu \otimes mF^{-1}}{d\mu \otimes m}(x, y) = \det DF(x, y) = \det\begin{vmatrix} df(x) & 0 \\ d\phi(x) & 1 \end{vmatrix}$$

$$= df(x) = \frac{d\mu f^{-1}}{d\mu}(x).$$

From this we see that $\lambda \in r^*(F) \Rightarrow \lambda \in r^*(f)$.

REMARK 4.10. Given two cocycles on X, $\phi_1, \phi_2 \in C^\infty(\mathbf{Z} \times X, \mathbf{R})$ we can define a cocycle $(\phi_1, \phi_2)\colon \mathbf{Z} \times X \to \mathbf{R}^2$ by $(\phi_1, \phi_2)(n, x) = (\phi_1(n, x), \phi_2(n, x))\, \forall n \in \mathbf{Z}$, $x \in X$. We compactify $\mathbf{R}^2$ by adding lines of the form (α, ∞), $(\alpha, -\infty)$, (∞, α), $(-\infty, \alpha)$ for all $\alpha \in \mathbf{R}$, plus four points at $(-\infty, \infty)$, (∞, ∞), $(\infty, -\infty)$, $(-\infty, -\infty)$. Then $(\lambda, \beta) \in \bar{E}(\phi_1, \phi_2)$ means that for every $B \in \mathcal{S}$, $\mu(B) > 0$ and for every $\varepsilon > 0$, there exists $n \in \mathbf{Z}$ such that $\mu(B \cap f^{-n}B \cap \{x\colon |(\phi_1(n, x), \phi_2(n, x)) - (\lambda, \beta)| < \varepsilon\}) > 0$, or equivalently, $\mu(B \cap f^{-n}B \cap \{x\colon |\phi_1(n, x) - \lambda| < \varepsilon\} \cap \{x\colon |\phi_2(n, x) - \beta| < \varepsilon\}) > 0$.

It is clear that $(\lambda, \beta) \in \bar{E}(\phi_1, \phi_2)$ implies that $\lambda \in \bar{E}(\phi_1)$ and $\beta \in \bar{E}(\phi_2)$, but the converse is not necessarily true. We give an example of the usefulness of considering two cocycles together in the next proposition.

PROPOSITION 4.11. *With* $(X, \mathcal{S}, \mu)$ *an m-dimensional manifold and f as in* 4.7, *we assume further that f is of type* III_0 *and that the map* F_ϕ *defined in* 4.9 *is* $\mu \otimes m$*-ergodic. Then* $(0, \infty) \in \bar{E}(\phi, \log(d\mu f^{-1}/d\mu))$ *if and only if* F_ϕ *is of type* III_0.

PROOF. ($\Rightarrow$) Assume that $(0, \infty) \in \bar{E}(\phi, \log(d\mu f^{-1}/d\mu))$. By 4.9 it suffices to show that $\infty \in r^*(F_\phi)$. Let $C \in \mathcal{S} \times \mathcal{G} \subset X \times \mathbf{R}$ be such that $\mu \otimes m(C) > 0$. Choose t_0 to be a point of density of C. Then there exists an $m + 1$-dimensional cube $R \subset X \times I$ of volume $\delta > 0$, centered at $t_0 = (t_1, t_2)$ such that $\mu \otimes m(R \cap C) > .99\delta$. By setting $B = \Pi_X(R \cap C)$, we see that $\mu(B) > 0$. Since $(0, \infty) \in \bar{E}(\phi, \log(d\mu f^{-1}/d\mu))$, there exists $n \in \mathbf{Z}$ such that

$$\mu\left(B \cap f^{-n}B \cap \left\{x\colon \left|\sum_{i=0}^{n-1} \phi \circ f^i(x)\right| < \delta^{1/(m+1)}\right\} \cap \left\{x\colon \left|\log \frac{d\mu f^n}{d\mu}(x)\right| > M\right\}\right) > 0.$$

This implies that

$$\mu \otimes m\left((R \cap C) \cap \left\{(x, y) \in R \cap C\colon \left(f^{-n}x, y - \sum_{i=0}^{n-1} \phi \circ f^i(x)\right) \in R \cap C\right\}\right.$$
$$\left. \cap \left\{(x, y) \left|\log \frac{d\mu f^n}{d\mu}(x)\right| > M\right\}\right) > 0.$$

Therefore $\infty \in r^*(F)$.

($\Leftarrow$) Suppose that F is of type III_0, i.e. $r^*(F) = \{0, \infty\}$. Then for every set $C \subset \mathfrak{S} \times \mathfrak{I}$, $\mu \otimes m(C) > 0$, and for every $M \in \mathbf{R}^+$, we can find an integer n such that

$$\mu \otimes m\left(C \cap F^{-n}C \cap \left\{(x, y): \left|\log \frac{d\mu \otimes mF^n}{d\mu \otimes m}(x, y)\right| > M\right\}\right) > 0.$$

Since $\log(d\mu \otimes mF^n/d\mu \otimes m)(x, y) = \log(d\mu f^n/d\mu)(x)$, and since $F^{-n}C = \{(f^{-n}x, y - \sum_{i=0}^{n-1} \phi \circ f^i(x)), (x, y) \in C\}$, then for any $B \in \mathfrak{S}$, $\mu(B) > 0$, we just choose $C = B \times (-\varepsilon/2, \varepsilon/2)$.

Then clearly there exists $n \in \mathbf{Z}$ such that

$$\mu\left(B \cap f^{-n}B \cap \left\{x: \left|\sum_{i=0}^{n-1} \phi \circ f^i(x)\right| < \varepsilon\right\} \cap \left\{x: \left|\log \frac{d\mu f^{-n}}{d\mu}(x)\right| > M\right\}\right) > 0.$$

This implies that $(0, \infty) \in \bar{E}(\phi, \log(d\mu f^{-1}/d\mu))$.

Finally we prove the existence of a dense G_δ of elements in $\mathcal{C}$ which satisfy the hypotheses of 4.11. We assume X and f are as in 4.7.

Proposition 4.12. *Let $\psi: X \to \mathbf{R}$ be a fixed (C^∞) cocycle for f with $\bar{E}(\phi) = \{0, \infty\}$. Then the set $\mathfrak{A} = \{\phi \in \mathcal{C} \mid (0, \infty) \in \bar{E}(\phi, \psi)\}$ is a dense G_δ in $\mathcal{C}$.*

Proof. Let $\mathfrak{S}_0$ be a countable dense subalgebra for X. Choose any $B \in \mathfrak{S}_0$, and fix $M \in \mathbf{R}^+$. Since $\infty \in \bar{E}(\phi)$, there exists $V \in [f]$ such that

$$\mu\left(B \cap V^{-1}B \cap \{x: |a_\psi(V, x)| > M\}\right) > \mu(B)/2.$$

If we define the set

$$\mathcal{Q}(B, M, \varepsilon) = \Big\{\phi \in \mathcal{C} \mid \sup_{V \in [f]} \mu\big(B \cap V^{-1}B \cap \{x: |a_\psi(V, x)| > M\} \cap \{x: |a_\phi(V, x)| < \varepsilon\}\big) > \mu(B)/4\Big\},$$

using the same argument as in Lemma 4.4 we see that it is open for fixed B, M, and ε. Now

$$\mathfrak{A} = \bigcap_{B \in \mathfrak{S}_0} \bigcap_{m \in \mathbf{N}} \bigcap_{M \in \mathbf{N}} \mathcal{Q}\left(B, M, \frac{1}{m}\right) = \{\phi \in \mathcal{C} \mid (0, \infty) \in \bar{E}(\phi, \psi)\}.$$

Clearly this set is a G_δ. To show that it is dense, we observe that the coboundaries are dense in $\mathcal{C}$ and obviously lie in $\mathfrak{A}$.

Theorem 4.13. *With X and f as in* 4.7, *suppose further that f is of type* III_0. *Then the set $\mathcal{C}_0 = \{\phi \in \mathcal{C} \mid (x, y) \mapsto (fx, y + \phi x)$ is of type $\mathrm{III}_0\}$ is a dense G_δ in $\mathcal{C}$.*

Proof. By 4.8, we have that $\mathfrak{G}_{\mathcal{C}}$ is a dense G_δ. By 4.11 and 4.12, we have $\mathfrak{A}$ is a dense G_δ. Then $\mathfrak{G}_{\mathcal{C}} \cap \mathfrak{A}$ is a dense G_δ of $\mathcal{C}$ and $\mathcal{C}_0 = \mathfrak{G}_{\mathcal{C}} \cap \mathfrak{A}$.

Corollary 4.14. *There are uncountably many C^∞ type III_0 diffeomorphisms on $T^n \times \mathbf{R}^p$, for every $n \geqslant 1$, $p \geqslant 0$.*

PROOF. For $n = 1$, $p = 0$, we use Katznelson's construction. By 4.2, the result is true for all $n \geqslant 1$ when $p = 0$. By repeated applications of 4.4–4.13 and an induction argument, the corollary is proved.

REMARK 4.15. All of the results from this section hold true for diffeomorphisms of type III_λ, with $0 < \lambda \leqslant 1$. In some sense, type III_λ transformations are better behaved than type III_0; there is only one type III_λ ergodic transformation, up to weak equivalence, for each $0 < \lambda \leqslant 1$ which is not true in the III_0 case. In particular, using obvious modifications of the preceding theorems, we can prove the following.

THEOREM 4.16. *For every λ, $0 < \lambda \leqslant 1$, there exists a C^∞ type III_λ diffeomorphism of $T^n \times \mathbf{R}^p$, for every $n \geqslant 1$, $p \geqslant 0$.*

5. Type III_λ, $0 \leqslant \lambda \leqslant 1$ diffeomorphisms of arbitrary manifolds. Herman proved in [7] that every connected paracompact manifold of dimension $\geqslant 3$ has a C^∞ type III_1 diffeomorphism on it. He used a nice method for exending maps on $T^2 \times \mathbf{R}^{m-2}$ to any connected paracompact m-dimensional manifold for $m \geqslant 3$; this procedure was introduced by Anosov in 1974 [5]. We will outline the method here without proof, also including some modifications for our particular circumstances.

LEMMA 5.1. *Let X be an m-dimensional C^∞ paracompact connected manifold and μ a C^∞ measure on X. Then there exists an open set $V \subset X$, diffeomorphic to $\mathbf{R}^m$ and satisfying $\mu(X - V) = 0$.*

LEMMA 5.2. *If $m \geqslant 3$, there exists an open set U of $\mathbf{R}^m$ diffeomorphic to $T^2 \times \mathbf{R}^{m-2}$ such that $\mu(\mathbf{R}^m - U) = 0$.*

LEMMA 5.3. *There exists a C^∞ type III_0 (III_λ, $0 < \lambda \leqslant 1$) flow on $T^2 \times \mathbf{R}^p$ for every $p \in \mathbf{N}$.*

PROOF. We apply Corollary 4.14 (Theorem 4.16) to obtain a C^∞ type III_0 (III_λ, $0 < \lambda \leqslant 1$) diffeomorphism of $T^1 \times \mathbf{R}^p$, then take the suspension flow.

LEMMA 5.4. *Let U be an open set of $\mathbf{R}^m$, and let f_t be a C^∞ flow of type III_λ, $0 \leqslant \lambda \leqslant 1$, on U. Let χ be the infinitesimal generator of f_t, i.e., χ is defined by*

$$\left.\frac{\partial f_t}{\partial t}(x)\right|_{t=0} = \chi \circ f_t(x).$$

Let $\phi \in C^\infty(U, \mathbf{R})$, $\phi > 0$, be defined such that the vector field $\phi\chi$ is globally integrable and defines a flow g_t. Then the flow g_t is weakly equivalent to f_t.

LEMMA 5.5. *There exists a C^∞ type III_0 (III_λ, $0 < \lambda \leqslant 1$) flow on every paracompact, connected manifold X of dimension $m \geqslant 3$.*

PROOF. By 5.1 and 5.2 we have an open set $U \subset X$ of full μ-measure and such that U is diffeomorphic to $T^2 \times \mathbf{R}^{m-2}$. Let f_t be a type III_0 (III_λ, $0 < \lambda \leqslant 1$) flow on U with infinitesimal generator χ; such a flow exists by 5.3. Let $\phi \in C^\infty(X, R)$ be such that $\phi > 0$ on U, $\phi = 0$ on $X - U$, and such that the vector field

$$Y(x) = \begin{cases} \phi(x)\chi(x), & \text{if } x \in U, \\ 0, & \text{if } x \in X - U \end{cases}$$

is C^∞ on X and globally integrable, thus defining a flow, $\tilde{f}_t$, on X. The flow $\tilde{f}_t$ is of type III_0 (III_λ) by 5.4.

COROLLARY 5.6. *There exists a C^∞ type* III_0 (III_λ, $0 < \lambda \leq 1$) *diffeomorphism on every connected paracompact manifold of dimension* ≥ 3.

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