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WEIGHTED NORM INEQUALITIES FOR THE FOURIER TRANSFORM

BY

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ABSTRACT. Given p and q satisfying $1 < p \leq q < \infty$, sufficient conditions on nonnegative pairs of functions U, V are given to imply

$$\left[\int_{R^n} |\hat{f}(x)|^q U(x)\,dx\right]^{1/q} \leq c\left[\int_{R^n} |f(x)|^p V(x)\,dx\right]^{1/p},$$

where $\hat{f}$ denotes the Fourier transform of f, and c is independent of f. For the case $q = p'$ the sufficient condition is that for all positive r,

$$\left[\int_{U(x)>Br} U(x)\,dx\right]\left[\int_{V(x)<r^{p-1}} V(x)^{-1/(p-1)}\,dx\right] \leq A,$$

where A and B are positive and independent of r. For $q \neq p'$ the condition is more complicated but also is invariant under rearrangements of U and V. In both cases the sufficient condition is shown to be necessary if the norm inequality holds for all rearrangements of U and V. Examples are given to show that the sufficient condition is not necessary for a pair U, V if the norm inequality is assumed only for that pair.

1. Introduction. Although weighted norm inequalities for classical operators have been studied extensively as described in [**4**], surprisingly little is known about the pairs U, V of nonnegative functions for which

$$\left[\int_{R^n} |\hat{f}(x)|^q U(x)\,dx\right]^{1/q} \leq c\left[\int_{R^n} |f(x)|^p V(x)\,dx\right]^{1/p}, \tag{1.1}$$

where $\hat{f}$ denotes the Fourier transform of f, and c is independent of f. The best known result is Pitt's theorem [**6**, p. 489], which for $n = 1$ asserts (1.1) if $q \geq p$, $U(x) = |x|^{-\gamma q}$, $V(x) = |x|^{\alpha p}$, $0 \leq \alpha < 1 - 1/p$, $0 \leq \gamma < 1/q$ and $\gamma = \alpha + 1/p + 1/q - 1$. Other more recent results include Knopf and Rudnick [**3**], Sagher [**5**, p. 119] and Aguilera and Harboure [**1**].

The conditions given here are less restrictive than those in the cited results. They depend only on the distribution functions of U and V and not on monotonicity or continuity conditions. The sufficiency theorems include Pitt's theorem. Surprisingly, the proofs are simpler than the proof of Pitt's theorem in [**6**] where several interpolations are needed. Here the case $q = p'$ is proved directly from the Hausdorff-Young theorem. The proof for $q < p'$ is similar but uses a lemma obtained by interpolation in place of the Hausdorff-Young theorem. The case $q > p'$ is obtained by duality.

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The theorems will be stated for integrable f to avoid difficulties with various definitions of the Fourier transform though they could easily be extended. Because of this restriction, we can use the definition

$$\hat{f}(x) = \int_{R^n} e^{-ix\cdot t} f(t)\, dt.$$

The sufficiency theorem for $q = p' = p/(p-1)$ is as follows.

THEOREM 1. *If $1 < p \leq 2$, $U(x)$ and $V(x)$ are nonnegative functions on R^n and there are positive constants A and B, independent of r, such that*

$$\left[\int_{U(x)>Br} U(x)\, dx\right]\left[\int_{V(x)<r^{p-1}} V(x)^{-1/(p-1)}\, dx\right] \leq A \tag{1.2}$$

for $r > 0$, then for every integrable f,

$$\left[\int_{R^n} |\hat{f}(x)|^{p'} U(x)\, dx\right]^{1/p'} \leq c\left[\int_{R^n} |f(x)|^p V(x)\, dx\right]^{1/p}, \tag{1.3}$$

where c depends only on A, B and p.

Note that the constant B is needed in (1.2) if the case with $U(x)$ and $V(x)$ not necessarily equal constants is to be included. Condition (1.2) is not a necessary condition for (1.3); this is shown in §6. Condition (1.2) is necessary, however, if (1.3) holds for all rearrangements of U and V as shown by the following theorem.

THEOREM 2. *If $1 < p < \infty$, $U(x)$ and $V(x)$ are nonnegative radial functions on R^n such that, as functions of $|x|$, U is nonincreasing and V is nondecreasing and* (1.3) *holds for all integrable f with c independent of f, then there exist constants A and B such that* (1.2) *holds for all $r > 0$.*

The theorems for $q \neq p'$ will be stated using rearranged functions; for a measurable function g on R^n the nonincreasing rearrangement g^* is defined on $[0, \infty)$ by

$$g^*(x) = \inf\{s: |\{t: |g(t)| > s\}| < x\}.$$

The sufficiency result for $q \neq p'$ will be done in two cases, $q < p'$ and $q > p'$. Since we will also assume $q \geq p$, the first case can be written as $p \leq q < p'$; note that this set of inequalities also implies $p < 2$. Similarly, the second case can be written $q' \leq p' < q$, which implies $q > 2$. The sufficiency theorems are as follows.

THEOREM 3. *If $1 < p < 2$, $p \leq q < p'$, $U(x)$ and $V(x)$ are nonnegative functions on R^n and there are positive constants A and B, independent of r, such that*

$$\left[\int_{[xU^*(x)]^{p'/q} > Brx} U^*(x)\, dx\right]\left[\int_{V(x)<r^{p-1}} V(x)^{-1/(p-1)}\, dx\right]^{q/p'} \leq A \tag{1.4}$$

for all $r > 0$, then for every integrable f, (1.1) *holds with c depending only on A, B, p and q.*

THEOREM 4. *If* $2 < q < \infty$, $q' < p \leqslant q$, $U(x)$ *and* $V(x)$ *are nonnegative functions on* R^n, $W(x) = [V(x)^{-1/(p-1)}]^*$ *and there are positive constants A and B, independent of r, such that*

$$\left[\int_{[xW(x)]^{q/p'} > Brx} W(x)\,dx\right]\left[\int_{rU(x)>1} U(x)\,dx\right]^{p'/q} \leqslant A \tag{1.5}$$

for all $r > 0$, *then for every integrable f*, (1.1) *holds with c depending only on A, B, p and q.*

The procedure used to prove Theorem 2 can also be applied to the case $q \neq p'$ as follows.

THEOREM 5. *If* $1 < p < \infty$, $1 < q < \infty$, $U(x)$ *and* $V(x)$ *are nonnegative radial functions on* R^n *such that, as functions of* $|x|$, *U is nonincreasing and V is nondecreasing and* (1.1) *holds for all integrable f with c independent of f, then if* $q \leqslant p'$ *there exist constants A and B such that* (1.4) *holds for* $r > 0$, *if* $q \geqslant p'$ *there exist constants A and B such that* (1.5) *holds for* $r > 0$.

This paper is organized as follows. Theorem 1 is proved in §2. Two lemmas needed to prove Theorem 3 are given in §3. One of these lemmas is the substitute for the Hausdorff-Young theorem mentioned before. The other concerns rearranging the function x on $[0, \infty)$ to a function g on R^n which bears the same relation to $U(x)$ that x has to $U^*(x)$. Theorems 3 and 4 are proved in §4; the proof of Theorem 3 is like that for Theorem 1 while Theorem 4 follows from Theorem 3 by duality. The necessity results, Theorems 2 and 5, are proved in §5. As usual this consists of choosing f so that (1.1) gives information about U and V. Finally, in §6 two examples are given to show that the sufficient conditions in Theorems 1 and 3 are not necessary.

Throughout this paper c will be used to denote constants, not necessarily the same at each occurrence. The convention $0 \cdot \infty = 0$ will be used. The symbol $|E|$ will denote the Lebesgue measure of E in the appropriate number of dimensions. Expressions such as $[f(x)g(x)]\hat{}$ and $[f(x)g(x)]^*$ will be used to denote, respectively, $\hat{h}(x)$ and $h^*(x)$ where $h(x) = f(x)g(x)$. The characteristic function of a set E will be written χ_E.

2. Proof of Theorem 1. To prove Theorem 1, observe first that if $V(x) = 0$ on a set of positive measure, then (1.2) implies $U(x) = 0$ almost everywhere and (1.3) is true trivially. If the set where $f(x) \neq 0$ and $V(x) = \infty$ has positive measure, the right side of (1.3) is infinite and (1.3) is also true trivially. Therefore, assume that $V(x) > 0$ almost everywhere and that the set where $f(x) \neq 0$ and $V(x) = \infty$ has measure 0.

To prove (1.3), start with the fact that $\int_{-\infty}^{\infty} |\hat{f}(x)|^{p'} U(x)\,dx$ is bounded by $2^{p'}$ times the sum of

$$\sum_{j=-\infty}^{\infty} \int_{2^j B < U(x) \leqslant 2^{j+1}B} \left| \int_{V(t) \geqslant 2^{jp-j-1}} f(t) e^{-ix\cdot t}\,dt \right|^{p'} U(x)\,dx \tag{2.1}$$

and

$$(2.2)\qquad \sum_{j=-\infty}^{\infty}\int_{2^jB<U(x)\leqslant 2^{j+1}B}\left|\int_{V(t)<2^{jp-j-1}} f(t)e^{-ix\cdot t}\,dt\right|^{p'} U(x)\,dx.$$

It is sufficient to show that each of these is bounded by

$$(2.3)\qquad c\left[\int_{-\infty}^{\infty}|f(x)|^pV(x)\,dx\right]^{p'/p},$$

where c depends only on A, B and p.

To estimate (2.1), observe that it is bounded by

$$\sum_{j=-\infty}^{\infty}\int_{R^n}|[f(x)\chi_{V\geqslant 2^{jp-j-1}}(x)]^{\wedge}|^{p'}2^{j+1}B\,dx.$$

By the Hausdorff-Young theorem, this is bounded by

$$2B\sum_{j=-\infty}^{\infty}2^j\left[\int_{R^n}|f(x)\chi_{V\geqslant 2^{jp-j-1}}(x)|^p\,dx\right]^{p'/p}.$$

Since $p'/p\geqslant 1$ this has the bound

$$(2.4)\qquad 2B\left[\int_{R^n}|f(x)|^ph(x)\,dx\right]^{p'/p},$$

where

$$(2.5)\qquad h(x)=\sum_{j=-\infty}^{\infty}2^{jp/p'}\chi_{V\geqslant 2^{jp-j-1}}(x).$$

Now for a given x, $h(x)=\sum_{j=-\infty}^{J}2^{j(p-1)}$ where J is the largest integer satisfying $2^{Jp-J}\leqslant 2V(x)$. Since the geometric series defining h has the ratio $2^{p-1}>1$, $h(x)$ is bounded by a constant times the last term. Therefore, $h(x)\leqslant cV(x)$ and (2.4) is bounded by (2.3). This completes the proof for (2.1).

To estimate (2.2), observe first that it is bounded by

$$(2.6)\qquad \int_{R^n}\left[\int_{2V(t)<[U(x)/B]^{p-1}}|f(t)|\,dt\right]^{p'}U(x)\,dx.$$

Now let J be the least integer such that $2^J\geqslant \|f\|_1$, and let $r_J=\infty$. Let $W(x)$ be a function such that $V(x)\leqslant 2W(x)\leqslant 2V(x)$ for all x and for all $r>0$, $|\{x\colon W(x)=r\}|=0$. For $j<J$ let r_j be chosen so that

$$(2.7)\qquad \int_{2W(t)<r_j^{p-1}}|f(t)|\,dt=2^j.$$

Then (2.6) is bounded by

$$(2.8)\qquad \sum_{j=-\infty}^{J}\int_{Br_{j-1}<U(x)\leqslant Br_j}\left|\int_{2W(t)<r_j^{p-1}}|f(t)|\,dt\right|^{p'}U(x)\,dx.$$

By (2.7) and the definition of r_J, we have for $j \leqslant J$,

$$\int_{2W(t)<r_j^{p-1}} |f(t)|\, dt \leqslant 2^j = 4\int_{D_j} |f(t)|\, dt, \tag{2.9}$$

where D_j is the set of all t for which $r_{j-2}^{p-1} \leqslant 2W(t) < r_{j-1}^{p-1}$. Furthermore, since the set where $V(t) = 0$ has measure 0 and $f(t) = 0$ almost everywhere on the set where $V(t) = \infty$, we have

$$|f(t)| = \left[|f(t)|\, V(t)^{1/p}\right] V(t)^{-1/p}$$

for almost every t with the convention $0 \cdot \infty = 0$. This and (2.9) then show that (2.8) is bounded by

$$c \sum_{j=-\infty}^{\infty} \left[\int_{Br_{j-1}<U(x)\leqslant Br_j} U(x)\, dx\right]\left[\int_{D_j}\left[|f(t)|\, V(t)^{1/p}\right] V(t)^{-1/p}\, dt\right]^{p'}. \tag{2.10}$$

By Hölder's inequality the jth term in (2.10) is bounded by the product of

$$\left[\int_{Br_{j-1}<U(x)} U(x)\, dx\right]\left[\int_{2W(t)<r_{j-1}^{p-1}} V(t)^{-p'/p}\, dt\right] \tag{2.11}$$

and

$$\left[\int_{D_j} |f(t)|^p V(t)\, dt\right]^{p'/p}.$$

Now since the set where $2W(t) < r_{j-1}^{p-1}$ is a subset of the set where $V(t) < r_{j-1}^{p-1}$, the hypothesis (1.2) implies that (2.11) is bounded by A. Then since $p'/p \geqslant 1$, we obtain the bound

$$c\left[\sum_{j=-\infty}^{J} \int_{D_j} |f(t)|^p V(t)\, dt\right]^{p'/p}$$

for (2.10). Since the D_j's are disjoint, this shows that (2.2) is bounded by (2.3) and completes the proof of Theorem 1.

3. Two basic lemmas. For the proof of Theorem 3 we need an inequality that plays the same role that the Hausdorff-Young theorem did in the proof of Theorem 1. This is given in Lemma 1. We also need a function that bears the same relation to U that x bears to the nonincreasing rearrangement U^*; this is done in Lemma 2.

LEMMA 1. *If $U(x)$ is a nonnegative function on R^n, $1 < p \leqslant q < p'$ and $U^*(x) \leqslant x^{-1+q/p'}$, then for every integrable f on R^n we have*

$$\left[\int_{R^n} |\hat{f}(x)|^q U(x)\, dx\right]^{1/q} \leqslant c\left[\int_{R^n} |f(x)|^p\, dx\right]^{1/p}, \tag{3.1}$$

where c is independent of f.

To prove this, observe first that we may assume that $U^*(x) = x^{-1+q/p'}$, since increasing U increases only the left side of (3.1). Define $r = 1 + q'/p'$. Then

$r' = 1 + p'/q'$, $r \leq 2$, and by the Hausdorff-Young theorem we have

$$\left[\int_{R^n} |\hat{f}(x)U_1(x)|^{r'} U_1(x)^{-r'}\,dx\right]^{1/r'} \leq c\left[\int_{R^n} |f(x)|^r\,dx\right]^{1/r}, \tag{3.2}$$

where $U_1(x) = U(x)^{-q'/(p'-q)}$. Next, since $|\hat{f}(x)| \leq \|f\|_1$, we have for $s > 0$,

$$\int_{|\hat{f}(x)|U_1(x)>s} U_1(x)^{-r'}\,dx \leq \int_{U_1(x)>s/\|f\|_1} U_1(x)^{-r'}\,dx. \tag{3.3}$$

The second integral in (3.3) equals

$$\int_{U^*(x)^{-q'/(p'-q)}>s/\|f\|_1} U^*(x)^{(p'+q')/(p'-q)}\,dx.$$

By the assumption $U^*(x) = x^{-1+q/p'}$, this equals

$$\int_{x^{q'/p'}>s/\|f\|_1} x^{-1-q'/p'}\,dx = \frac{p'}{q's}\|f\|_1.$$

Combining these results gives

$$\int_{|\hat{f}(x)|U_1(x)>s} U_1(x)^{-r'}\,dx \leq \frac{p'}{q's}\int_{R^n} |f(x)|\,dx. \tag{3.4}$$

Now apply the Marcinkiewicz interpolation theorem [**8**, p. 183] to the strong type result (3.2) and the weak type inequality (3.4) with $t = 1/q - 1/p'$. Since $(1-t)/r' + t = 1/q$, $(1-t)/r + t = 1/p$ and $U_1(x)^{q-r'} = U(x)$, this gives (3.1).

LEMMA 2. *If $U(x)$ is a nonnegative function on R^n, $|\{t\colon U(t) = x\}| = 0$ for all $x > 0$, $S = |\{t\colon U(t) > 0\}| < \infty$, $L(x) = |\{t\colon U(t) \geq x\}|$ and $g(x) = L[U(x)]$, then $|\{x\colon g(x) < a\}| = \min(a, S)$ for $a > 0$.*

To prove this, observe first that since $L[U(x)]$ and $L[U^*(x)]$ are equimeasurable, it is sufficient to show that

$$|\{x\colon L[U^*(x)] < a\}| = \min(a, S). \tag{3.5}$$

If $x < S$, $U^*(x) > 0$ and the fact $|\{t\colon U(t) = U^*(x)\}| = 0$ shows that $L[U^*(x)] = |\{t\colon U(t) > U^*(x)\}| \leq x$ by Lemma 3.4, p. 189 of [**8**]. By the definition of $U^*(x)$, we also have $L[U^*(x)] \geq x$ for all x. Therefore $L[U^*(x)] = x$ for $0 \leq x < S$. By the definition, if $x > S$, then $U^*(x) = 0$ and $L[U^*(x)] = \infty$. Combining these facts proves (3.5) and completes the proof of Lemma 2.

4. Proof of Theorems 3 and 4. To prove Theorem 3, observe first that as in the proof of Theorem 1 we may assume that $f(x) = 0$ almost everywhere on the set where $V(x) = \infty$ and $V(x) > 0$ almost everywhere. We may also assume that the set where $V(x) < \infty$ has positive measure; otherwise $f(x) = 0$ almost everywhere and (1.1) is trivial. Then the second integral in (1.4) is positive for some $r > 0$. If the set where $U(x) = \infty$ had positive measure, the first integral in (1.4) would be infinite for all $r > 0$. Therefore, (1.4) implies that $U(x)$ is finite almost everywhere.

We will first consider the case that the set where $U(x) > 0$ has finite measure and for every $r > 0$ the set where $U(x) = r$ has measure 0. Then define $g(x)$ as in

Lemma 2 and for every integer j let E_j be the set where $2^jBg(x) < [g(x)U(x)]^{p'/q} \leqslant 2^{j+1}Bg(x)$. Then $\int_{-\infty}^{\infty} |\hat{f}(x)|^q U(x)\,dx$ is bounded by 2^q times the sum of

$$\sum_{j=-\infty}^{\infty} \int_{E_j} \left| \int_{V(t) \geqslant 2^{jp-j-1}} f(t) e^{-ix\cdot t}\,dt \right|^q U(x)\,dx \tag{4.1}$$

and

$$\sum_{j=-\infty}^{\infty} \int_{E_j} \left| \int_{V(t) < 2^{jp-j-1}} f(t) e^{-ix\cdot t}\,dt \right|^q U(x)\,dx. \tag{4.2}$$

It is sufficient to show that each of these is bounded by

$$c\left[\int_{-\infty}^{\infty} |f(x)|^p V(x)\,dx\right]^{q/p}, \tag{4.3}$$

where c depends only on A, B, p and q.

To estimate (4.1), observe that it is bounded by

$$c \sum_{j=-\infty}^{\infty} \int_{R^n} |[f(x)\chi_{V \geqslant 2^{jp-j-1}}(x)]\hat{}\,|^q 2^{jq/p'} g(x)^{-1+q/p'}\,dx. \tag{4.4}$$

From Lemma 2, it follows that $[1/g(x)]^* = 1/x$ for $0 < x \leqslant S$ and $[1/g(x)]^* = 0$ for $x > S$. Therefore $[g(x)^{-1+q/p'}]^* \leqslant x^{-1+q/p'}$. Then Lemma 1 can be applied to show that (4.4) is bounded by

$$c \sum_{j=-\infty}^{\infty} 2^{jq/p'} \left[\int_{R^n} |f(x)\chi_{V \geqslant 2^{jp-j-1}}(x)|^p\,dx\right]^{q/p}.$$

Since $q/p \geqslant 1$, this has the bound

$$c\left[\int_{R^n} |f(x)|^p h(x)\,dx\right]^{q/p}, \tag{4.5}$$

where h is the function defined in (2.5). As shown in §2, $h(x) \leqslant cV(x)$; this completes the proof for (4.1).

To estimate (4.2), observe first that since $2^j < T(x) = [g(x)U(x)]^{p'/q}/Bg(x)$ on E_j, if follows that (4.2) is bounded by

$$\int_{R^n} \left[\int_{2V(t) < T(x)^{p-1}} |f(t)|\,dt\right]^q U(x)\,dx. \tag{4.6}$$

As in the estimation of (2.6) choose $W(x)$, J and r_j satisfying the conditions stated there. Then (4.6) is bounded by

$$\sum_{j=-\infty}^{J} \int_{r_{j-1} < T(x) \leqslant r_j} \left| \int_{2W(t) < r_j^{p-1}} |f(t)|\,dt \right|^q U(x)\,dx. \tag{4.7}$$

By (2.9), the fact that $V(t) > 0$ almost everywhere and the fact that $f(t) = 0$ at almost every point where $V(t) = \infty$, it follows that the jth term in (4.7) is bounded by

$$c\left[\int_{r_{j-1} < T(x)} U(x)\,dx\right]\left[\int_{D_j} [|f(t)|\,V(t)^{1/p}]\,V(t)^{-1/p}\,dt\right]^q,$$

where D_j is the set of all t for which $r_{j-2}^{p-1} \leqslant 2W(t) < r_{j-1}^{p-1}$. By Hölder's inequality this is bounded by the product of

$$(4.8) \qquad \left[\int_{r_{j-1}<T(x)} U(x)\,dx\right]\left[\int_{2W(t)<r_{j-1}^{p-1}} V(t)^{-1/(p-1)}\,dt\right]^{q/p'}$$

and

$$(4.9) \qquad c\left[\int_{D_j} |f(t)|^p V(t)\,dt\right]^{q/p}.$$

Now the set where $T(x) > r_{j-1}$ is the set where $U(x)^{p'/q}L[U(x)]^{-1+p'/q} > Br_{j-1}$. Consequently if $E = \{y: y^{p'/q}L(y)^{-1+p'/q} > Br_{j-1}\}$, the set where $T(x) > r_{j-1}$ equals the set where $U(x) \in E$. Therefore,

$$(4.10) \qquad \int_{r_{j-1}<T(x)} U(x)\,dx = \int_{U(x)\in E} U(x)\,dx = \int_{U^*(x)\in E} U^*(x)\,dx.$$

Next, observe that since $L[U^*(x)] = x$ on the set where $U^*(x) > 0$, we have

$$(4.11) \qquad \int_{U^*(x)\in E} U^*(x)\,dx = \int_{[xU^*(x)]^{p'/q}>Br_{j-1}x} U^*(x)\,dx.$$

Combining (4.10) and (4.11) and using the fact that $2W(t) \geqslant V(t)$ shows that (4.8) is bounded by

$$\left[\int_{[xU^*(x)]^{p'/q}>Br_{j-1}x} U^*(x)\,dx\right]\left[\int_{V(t)<r_{j-1}^{p-1}} V(t)^{-1/(p-1)}\,dt\right]^{q/p'}.$$

By (1.4) this is bounded by A. Therefore (4.2) is bounded by

$$Ac\sum_{j=-\infty}^{J}\left[\int_{D_j} |f(t)|^p V(t)\,dt\right]^{q/p}.$$

Since the sets D_j are disjoint and $q/p \geqslant 1$, this is bounded by (4.3). This completes the proof of Theorem 3 for the case that the set where $U(x) > 0$ has finite measure and $|\{x: U(x) = r\}| = 0$ for $r > 0$.

Next we consider the case in which the set where $U(x) > 0$ has infinite measure and $|\{x: U(x) = r\}| = 0$ for $r > 0$. First note that if $s > 0$ and $|\{x: U(x) > s\}| = \infty$, then since $p'/q > 1$, the set where $[xU^*(x)]^{p'/q} > Brx$ and $U^*(x) > s$ has infinite measure. Therefore, the first integral in (1.4) is infinite for all r, $V(x) = \infty$ almost everywhere and the theorem is true trivially. Therefore, assume that $|\{x: U(x) > s\}| < \infty$ for all $s > 0$ and define $U_n(x) = U(x)$ if $U(x) > 1/n$ and $U_n(x) = 0$ if $U(x) \leqslant 1/n$. Then since $U_n^*(x)$ is a truncation of U^*, (1.4) is true with U^* replaced by U_n^* with the same constants A and B. By the previous case (1.1) follows with U replaced by U_n with c independent of n. The proof is completed by applying the monotone convergence theorem.

Finally, if $|\{x: U(x) = r\}| > 0$ for some $r > 0$, choose $U_1(x)$ so that $U(x) \leqslant 2U_1(x) \leqslant 2U(x)$ for all x and $|\{x: U_1(x) = r\}| = 0$ for all $r > 0$. Inequality (1.4) is true with $U(x)$ replaced by $U_1(x)$ and the same A and B. Inequality (1.1) follows with U replaced by U_1. Since $U(x) \leqslant 2U_1(x)$ this implies (1.1) with the constant multiplied by $2^{1/q}$. This completes the proof of Theorem 3.

To prove Theorem 4, observe that the sufficient condition in Theorem 3 for

$$\left[\int_{R^n} |\hat{f}(x)|^{p'} V(x)^{-1/(p-1)}\,dx\right]^{1/p'} \leqslant c\left[\int_{R^n} |f(x)|^{q'} U(x)^{-1/(q-1)}\,dx\right]^{1/q'} \tag{4.12}$$

reduces to (1.5). Therefore, the hypothesis of Theorem 4 implies that (4.12) holds for all integrable f, and (1.1) follows by duality.

5. Necessity proofs. This section consists of the proof of Theorem 5; Theorem 2 then follows as a special case. The basic fact needed for the proof is the following lemma.

LEMMA 3. *If* $1 < p < \infty$, $1 \leqslant q < \infty$, $a > 0$, $U(x)$ *and* $V(x)$ *are nonnegative radial functions on* R^n *such that, as functions of* $|x|$, U *is nonincreasing and* V *is nondecreasing and* (1.1) *holds for all integrable* f *with* c *independent of* f, *then there is a constant* B, *depending only on* a, n, p, q *and the* c *in* (1.1), *such that for* $r > 0$,

$$\left[\sup\{x\colon [xU^*(x)]^{p'/q} > Brx\}\right] |\{x\colon V(x) < r^{p-1}\}| \leqslant a. \tag{5.1}$$

Given positive numbers B and r, define

$$M = M(B, r) = \left[\sup\{x\colon [xU^*(x)]^{p'/q} > Bxr\}\right]^{1/n}. \tag{5.2}$$

The proof will consist of showing that if B and r are positive numbers such that

$$[M(B, r)]^n |\{x\colon V(x) < r^{p-1}\}| > a, \tag{5.3}$$

then B is less than a positive constant that depends only on a, n, p, q and the c in (1.1). The conclusion follows since any B larger than this constant cannot satisfy (5.3) for any $r > 0$ and, therefore, must satisfy (5.1) for all $r > 0$. We may assume that $a < (2\sqrt{n})^n$ since (5.1) for one value of a implies (5.1) for all larger values of a.

To show that (5.3) implies that B is less than a positive constant for $0 < a < (2/\sqrt{n})^n$, fix such an a and B and r that satisfy (5.3). Define $f(x)$ by $f(x) = 1$ for $|x| \leqslant a^{1/n}/2M$ and $f(x) = 0$ for $|x| > a^{1/n}/2M$. Since the real part of $e^{ix\cdot t}$ is greater than $\frac{1}{2}$ if $|x||t| < 1$, we have $|\hat{f}(x)| \geqslant \frac{1}{2}\int_{R^n} f(t)\,dt$ if $|x| \leqslant M/a^{1/n}$. Since $a < (2/\sqrt{n})^n$, this implies that for $|x| \leqslant M\sqrt{n}/2$ we have $|\hat{f}(x)| \geqslant DM^{-n}$, where D depends only on n and a. Since all x in a cube with side M centered at the origin satisfy $|x| \leqslant M\sqrt{n}/2$, we have $|\{x\colon |x| \leqslant M\sqrt{n}/2\}| \geqslant M^n$. Combining this with the estimate of $\hat{f}$ then gives

$$DM^{-n}\left[\int_0^{M^n} U^*(x)\,dx\right]^{1/q} \leqslant \left[\int_{|x| \leqslant M\sqrt{n}/2} |\hat{f}(x)|^q U(x)\,dx\right]^{1/q}. \tag{5.4}$$

Next, since $U^*(x)$ is nonincreasing, we obtain from the definition of M that

$$U^*(x) \geqslant (BrM^n)^{q/p'}/M^n \tag{5.5}$$

for $0 \leqslant x \leqslant M^n$. Combining this with (5.4) gives

$$DM^{-n}[BrM^n]^{1/p'} \leqslant \left[\int_{R^n} |\hat{f}(x)|^q U(x)\,dx\right]^{1/q}. \tag{5.6}$$

Since $V(x)$ is radial and nondecreasing and $V(x) < r^{p-1}$ on a set of measure greater than aM^{-n} by (5.3), we have $V(x) < r^{p-1}$ for $|x| < a^{1/n}/2M$. Therefore

$$\left[\int_{R^n} |f(x)|^p V(x)\,dx\right]^{1/p} \leqslant r^{(p-1)/p}[aM^{-n}]^{1/p}.$$

Combining this with (5.6) and (1.1) then gives

$$DM^{-n}[BrM^n]^{1/p'} \leqslant cr^{(p-1)/p}[aM^{-n}]^{1/p}$$

which reduces to $B^{1/p'} \leqslant ca^{1/p}/D$. This completes the proof of Lemma 3.

To prove Theorem 5 for $q \leqslant p'$, let $a = (2/n)^n$ and B be as in Lemma 3. Fix $r > 0$ and for each positive integer j define

$$\begin{aligned} f_j(x) &= V(x)^{-1/(p-1)}, \qquad 1/j < V(x) < r^{p-1}, \\ &= 0, \qquad \text{elsewhere.} \end{aligned}$$

If $\int_{R^n} f_j(x)\,dx = \infty$ for some j, then $|\{x\colon V(x) < r^{p-1}\}| = \infty$. Lemma 3 then implies that $|\{x\colon [xU^*(x)]^{p'/q} > Brx\}| = 0$ and (1.4) follows for this r with $A = 0$.

Therefore, assume that $\int_{R^n} f_j(x) < \infty$ for all j. The proof will be completed by showing that

$$\left[\int_0^{M^n} U^*(x)\,dx\right]\left[\int_{R^n} f_j(x)\,dx\right]^{q/p'} \leqslant A, \tag{5.7}$$

where M is as defined in (5.2) and A depends only on p, q, B, n and the constant in (1.1). This is, of course, sufficient by the monotone convergence theorem. If $\int_{R^n} f_j(x)\,dx = 0$, then (5.7) is immediate with $A = 0$. Therefore assume that $0 < \int_{R^n} f_j(x)\,dx < \infty$. By Lemma 3 $|\{x\colon V(x) < r^{p-1}\}| \leqslant [2/Mn]^n$, and since V is radial and nondecreasing, this implies that $V(x) \geqslant r^{p-1}$ if $|x| \geqslant 1/M\sqrt{n}$. Therefore, f_j is supported on $|x| < 1/M\sqrt{n}$ and since the real part of $e^{ix\cdot t}$ is greater than $\frac{1}{2}$ if $|x||t| < \frac{1}{2}$, we have

$$|\hat{f}_j(x)| \geqslant \frac{1}{2}\int_{R_n} f_j(t)\,dt \tag{5.8}$$

for $|x| < M\sqrt{n}/2$. Since the set where $|x| < M\sqrt{n}/2$ has measure greater than M^n,

$$\int_0^{M^n} U^*(x)\,dx \leqslant \int_{|x| < M\sqrt{n}/2} U(x)\,dx. \tag{5.9}$$

From (1.1), (5.8), and (5.9) we then conclude that

$$\left[\frac{1}{2}\int_{R_n} f_j(t)\,dt\right]\left[\int_0^{M^n} U^*(x)\,dx\right]^{1/q} \leqslant c\left[\int_{R^n} |f_j(t)|^p V(t)\,dt\right]^{1/p}. \tag{5.10}$$

From the definition of f_j we have

$$\int_{R^n} |f_j(t)|^p V(t)\,dt = \int_{R_n} f_j(t)\,dt.$$

Using this in (5.10), dividing by $[\int_{R_n} f_j(t)\,dt]^{1/p}$ and taking the q power then proves (5.7). This completes the proof of Theorem 5 for $q \leqslant p'$. For $q > p'$ use the fact that (1.1) implies (4.12) and apply the previous case to (4.12).

6. Necessity counterexamples. The two examples given here are simple consequences of the theorems of this section. To simplify the presentation, the theorems and examples are given in one dimension though they are easy to generalize. The example in Theorem 6 is a modified version of an example by Aguilera and Harboure [**1**]. It shows that (1.4) is not a necessary condition for (1.1) for $1 < p \leqslant q < 2$. Unfortunately, it does not seem easy to modify this example to include the case $2 \leqslant q \leqslant p'$. Therefore, a more complicated example is given which shows that (1.4) is not necessary for $1 < p \leqslant q \leqslant p'$ and (1.2) is not necessary for $1 < p \leqslant q = p'$. This example was suggested by a result of Dahlberg [**2**] for fractional integrals; the proof here, however, does not use Dahlberg's theorem. Counterexamples for $1 < q' \leqslant p \leqslant q$ can be generated from those given here by duality.

That Theorems 6 and 7 do provide examples of pairs that satisfy (1.1) but not (1.4) or (1.2) is shown after the statements of the theorems. The proofs of the theorems are given after the description of the counterexamples.

THEOREM 6. *If $U(x)$ is the characteristic function of $\bigcup_{n=1}^{\infty}[2^n, 2^n + 1/n]$ and $1 < p \leqslant q < 2$, then for integrable f,*

$$\left[\int_{-\infty}^{\infty} |\hat{f}(x)|^q U(x)\,dx\right]^{1/q} \leqslant c\left[\int_{-\infty}^{\infty} |f(x)|^p\,dx\right]^{1/p}, \tag{6.1}$$

where c is independent of f.

THEOREM 7. *If $1 < p \leqslant q \leqslant p'$, $0 < a < 1/p'$, $1/s = -a + 1/p'$, $b = (1 - q/p')/(1 - q/s)$, $D = (1 - b)p/q(p - 1 - ap)$, $\{J_k\}_{k=1}^{n}$ is a finite set of intervals of length n^{-b}, the intervals with the same centers as the J_k's with length $2n^D + 1$ are disjoint, and $U(x)$ is the characteristic function of $\bigcup_{k=1}^{n} J_k$, then for integrable f,*

$$\left[\int_{-\infty}^{\infty} |\hat{f}(x)|^q U(x)\,dx\right]^{1/q} \leqslant c\left[\int_{-\infty}^{\infty} |f(x)|^p |x|^{ap}\,dx\right]^{1/p}, \tag{6.2}$$

where c is independent of f, n and the choice of the J_k's.

It is easy to see that Theorem 6 provides an example of functions U and V that satisfy (1.1) but not (1.4) for $1 < p \leqslant q < 2$. With $V(x) \equiv 1$ and $r = 2$, (1.4) requires that $[xU^*(x)]^{p'/q} \leqslant 2Bx$ for almost every x. Since the set where the U of Theorem 6 is equal to 1 has infinite measure, $U^*(x) = 1$ for all $x \geqslant 0$, and this condition reduces to $x^{p'/q} \leqslant 2Bx$ for almost every x. This is impossible for positive B, however, since $p'/q > 1$.

For the examples based on Theorem 7, note first that since (1.4) reduces to (1.2) if $q = p'$, we do not have to consider (1.2) separately. Assuming the truth of Theorem 7, we first show that the functions $U(x)$ and $V(x) = |x|^{ap}$ cannot satisfy (1.4) with A and B depending only on p, q and the c of (6.2). This is easy to do since $U^*(x) = 1$ on $[0, n^{1-b}]$ and this implies that $[xU^*(x)]^{p'/q} > x/2$ on $[1, n^{1-b}]$. Therefore, with $r = 1/2B$, the first term in (1.4) exceeds $-1 + n^{1-b}$ and the second is a positive constant. Since $b < 1$, this shows that (1.4) cannot hold with an A independent of n.

It is also easy to use Theorem 6 to generate a $U(x)$ that satisfies (6.2) but which violates (1.4) with $V(x) = |x|^{ap}$ for any positive A and B. To do this, choose E

satisfying $E > [ap'(1-b)]^{-1}$, let j be a positive integer and define $U_j(x)$ to be 2^{-j} times the characteristic function of the union of 2^{Ej} intervals J_k of length 2^{-bEj} with the intervals of length $1 + 2^{DEj+1}$ and the same centers disjoint. Then by Theorem 7

$$\int_{-\infty}^{\infty} |f(x)|^q U_j(x)\,dx \leqslant c2^{-j}\left[\int_{-\infty}^{\infty} |f(x)|^p |x|^{ap}\,dx\right]^{q/p}$$

with c independent of j. Adding these shows that (6.2) holds with $U(x) = \sum_{j=1}^{\infty} U_j(x)$. With $V(x) = |x|^{ap}$, (1.4) reduces to

$$\left[\int_{[xU^*(x)]^{p'/q} > Brx} U^*(x)\,dx\right] \leqslant A\left[\frac{(1-ap')r^{1-1/ap'}}{2}\right]^{q/p'}. \tag{6.3}$$

Now since $U_j(x) = 2^{-j}$ on a set of measure $2^{Ej(1-b)}$, we have $U^*(x) \geqslant 2^{-j}$ on $[1, 2^{Ej(1-b)}]$. Therefore, with $r = 2^{-jp'/q}/2B$, the inequality $[xU^*(x)]^{p'/q} > Brx$ is satisfied on $[1, 2^{Ej(1-b)}]$. With this r, then

$$2\int_{[xU^*(x)]^{p'/q} > Brx} U^*(x)\,dx \geqslant 2^{j(E-Eb-1)} = (2Br)^{q(Eb+1-E)/p'}.$$

Since r can be arbitrarily close to 0 and $Eb + 1 - E < 1 - 1/ap'$, this last inequality shows that (6.3) cannot hold for all $r > 0$. This pair, therefore, satisfies (1.1) but not (1.4).

To prove Theorem 6, start with the fact that

$$\int_{-\infty}^{\infty} |\hat{f}(x)|^q U(x)\,dx = \sum_{n=1}^{\infty} \int_{2^n}^{2^{n+1/n}} |\hat{f}(x)|^q\,dx.$$

By Hölder's inequality, the right side is bounded by

$$\sum_{n=1}^{\infty} \left[\int_{2^n}^{2^{n+1/n}} |\hat{f}(x)|^{p'}\,dx\right]^{q/p'} n^{-1+q/p'}. \tag{6.4}$$

Now define the operator S_n by $[S_n f]\hat{} = \hat{f}\chi_{[2^n, 2^{n+1})}$. Then by the Hausdorff-Young theorem, (6.4) has the bound

$$\sum_{n=1}^{\infty} \left[\int_{-\infty}^{\infty} |S_n f(x)|^p\,dx\right]^{q/p} n^{-1+q/p'}.$$

Since $q/p \geqslant 1$, this is bounded by

$$\left[\int_{-\infty}^{\infty}\left[\sum_{n=1}^{\infty} n^{p/p'-p/q} |S_n f(x)|^p\right] dx\right]^{q/p}.$$

Now apply Hölder's inequality to the sum to get

$$\left[\int_{-\infty}^{\infty}\left[\sum_{n=1}^{\infty} |S_n f(x)|^2\right]^{p/2} dx\right]^{q/p}\left[\sum_{n=1}^{\infty} n^{(2p/p'-2p/q)/(2-p)}\right]^{q/p-q/2}.$$

It is easy to verify that $q < 2$ implies that the exponent of the n in the second sum is less than -1. Therefore, the second sum is finite and Theorem 5 [7, p. 104] gives the bound $c\,|\int_{-\infty}^{\infty} |f(x)|^p\,dx\,|^{q/p}$. This completes the proof of Theorem 6.

To prove Theorem 7, observe that by use of the substitution $g(x) = [f(x)\,|x|^a]\hat{}$ that (6.2) becomes

$$\left[\int_{-\infty}^{\infty} |I_a g(x)|^q U(x)\,dx\right]^{1/q} \leq c\left[\int_{-\infty}^{\infty} |\hat{g}(x)^p\,dx\right]^{1/p},$$

where

$$I_a g(x) = \int_{-\infty}^{\infty} \frac{g(t)}{|x-t|^{1-a}}\,dt$$

is the usual fractional integral. Because of the Hausdorff-Young theorem, it is sufficient to prove that

$$\int_{-\infty}^{\infty} |I_a g(x)|^q U(x)\,dx \leq c\left[\int_{-\infty}^{\infty} |g(x)|^{p'}\,dx\right]^{q/p'}, \tag{6.5}$$

where c is independent of g, n and the choice of the J_k's. Dahlberg in [2] gives a necessary and sufficient condition on $U(x)$ so that (6.5) holds if $q = p'$; in particular, this gives a necessary and sufficient condition on $U(x)$ for (6.2) to hold if $p = q = 2$. To prove Theorem 7, however, we will prove (6.5) directly for the asserted functions $U(x)$.

To prove (6.5) for a U of the type described in Theorem 6, fix an n and a collection $\{J_k\}_{k=1}^n$ of intervals satisfying the hypotheses. The left side of (6.5) is bounded by the sum of

$$2^q \sum_{k=1}^{n} \int_{J_k} |I_a g_k(x)|^q\,dx \tag{6.6}$$

and

$$2^q \sum_{k=1}^{n} \int_{J_k} |I_a[g(x) - g_k(x)]|^q\,dx, \tag{6.7}$$

where $g_k(x) = g(x)$ on the interval of length $2n^D + 1$ with the same center as J_k and $g_k(x) = 0$ elsewhere. By Hölder's inequality and the fact that $|J_k| = n^{-b}$, it follows that (6.6) is bounded by

$$2^q n^{-b(1-q/s)} \sum_{k=1}^{n} \left[\int_{J_k} |I_a g_k(x)|^s\,dx\right]^{q/s}.$$

Now apply Theorem 1 of [7, p. 119] and the definition of b to obtain the bound

$$cn^{-1+q/p'} \sum_{k=1}^{n} \left[\int_{-\infty}^{\infty} |g_k(x)|^{p'}\,dx\right]^{q/p'}.$$

Now apply Hölder's inequality to the sum; this gives

$$cn^{-1+q/p'}\left[\sum_{k=1}^{n} \int_{-\infty}^{\infty} |g_k(x)|^{p'}\,dx\right]^{q/p'} n^{1-q/p'}.$$

Since the g_k's have disjoint support, this completes the proof that (6.6) is bounded by the right side of (6.5) with c independent of n, g and the choice of the J_k's.

To estimate (6.7), start with the fact that (6.7) is bounded by

$$2^q \sum_{k=1}^{n} \int_{J_k} \left[\int_{-\infty}^{\infty} \frac{|g(t) - g_k(t)|}{|x - t|^{1-a}} dt \right]^q dx.$$

Hölder's inequality applied to the inner integral gives the bound

$$2^q \sum_{k=1}^{n} \int_{J_k} \left[\int_{-\infty}^{\infty} |g(t) - g_k(t)|^{p'} dt \right]^{q/p'} \left[\int_{|x-t|>n^D} \frac{dt}{|x - t|^{p(1-a)}} \right]^{q/p} dx.$$

The hypothesis $a < 1/p'$ implies $p(1 - a) > 1$; therefore, the last integral is finite. Replacing that integral by its value, performing the outer integration and using the definition of D shows that (6.7) is bounded by

$$\frac{c}{n} \sum_{k=1}^{n} \left[\int_{-\infty}^{\infty} |g(t) - g_k(t)|^{p'} dt \right]^{q/p'}.$$

Since all the integrals are bounded by $\int_{-\infty}^{\infty} |g(t)|^{p'} dt$, this shows that (6.7) is bounded by the right side of (6.5). This completes the proof of Theorem 7.

References

1. N. E. Aguilera and E. O. Harboure, *On the search of weighted norm inequalities for the Fourier transform* (to appear).
2. B. Dahlberg, *Regularity properties of Riesz potentials*, Indiana Univ. Math. J. **28** (1979), 257–268.
3. Peter Knopf and Karl Rudnick, *Weighted norm inequalities for the Fourier transform* (preprint).
4. B. Muckenhoupt, *Weighted norm inequalities for classical operators*, Proc. Sympos. Pure Math., vol. 35, part 1, Amer. Math. Soc., Providence, R. I., 1979, pp. 69–83.
5. Y. Sagher, *Real interpolation with weights*, Indiana Univ. Math. J. **30** (1981), 113–121.
6. E. Stein, *Interpolation of linear operators*, Trans. Amer. Math. Soc. **83** (1956), 482–492.
7. ______, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970.
8. E. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, N. J., 1971.

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