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SHRINKING COUNTABLE DECOMPOSITIONS OF S^3

BY

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ABSTRACT. Conditions are given which imply that a countable, cellular usc decomposition G is shrinkable. If the embedding of each element in G has the bounded nesting property, defined in this paper, then S^3/G is homeomorphic to S^3. The bounded nesting property is a condition on the defining sequence of cells for an element of G which implies that G satisfies the Disjoint Disk criterion for shrinkability [**S1**, Theorem 3.1]. From this result, one deduces that countable, star-like equivalent usc decompositions of S^3 are shrinkable—a result proved independently by E. Woodruff [**W**]. Also, one deduces the shrinkability of countable bird-like equivalent usc decompositions (a generalization of the star-like result), and the recently proved theorem that if each element of a countable usc decomposition G of S^3 has a mapping cylinder neighborhood, then G is shrinkable [**E**; **S1**, Theorem 4.1; **S-W**, Theorem 1].

1. Introduction. Let G be an upper semicontinuous (abbreviated usc) cellular, countable decomposition of S^3 (countable means that G has only countably many nondegenerate elements). Under what circumstances is S^3/G homeomorphic to S^3, i.e., is G shrinkable? It has been shown [**S2**, Main Theorem] that no conditions on the topological nature of the nondegenerate elements of G are sufficient to guarantee that G is shrinkable. Conditions on G, then, must restrict the allowable embeddings of the elements of G.

In this paper, we describe a property of an embedding of a cellular set in S^3 called the bounded nesting property which is defined in §5. This property restricts the depth of nesting of certain curves on the boundary of each cell in a defining sequence for the cellular set. Each curve is in the intersection of a disk with a cell boundary. The main theorem in this paper is the following one.

BOUNDED NESTING THEOREM 5.3. *Let G be a countable, cellular, usc decomposition of S^3. If each element of G has the bounded nesting property, then S^3/G is homeomorphic to S^3.*

This theorem allows us to deduce as corollaries most known theorems about shrinkability of countable decompositions of S^3. Also it implies more—for example, the star-like equivalent case, stated below, which was recently proved independently by E. Woodruff [**W**].

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STAR-LIKE THEOREM 6.1. *Let G be a usc decomposition of S^3 into points and countably many nondegenerate elements $\{g_i\}_{i\in\omega}$ where for each i in ω, there is a homeomorphism h_i: $S^3 \to S^3$ so that $h_i(g_i)$ is star-like. Then S^3/G is homeomorphic to S^3.*

In addition, the Bounded Nesting Theorem encompasses the case of bird-like equivalent elements—an extension of the star-like equivalent result. Also, the Bounded Nesting Theorem implies the recent theorem, which follows from [**E**; **S1**, Theorem 4.1; **S-W**, Theorem 1], that a countable, cellular usc decomposition is shrinkable if each element has a mapping cylinder neighborhood.

The Bounded Nesting Theorem 5.3 is a result which depends on the Disjoint Disks Property for shrinkability of 0-dimensional, cell-like usc decompositions of S^3 [**S1**, Theorem 3.1]. The bounded nesting property is a condition on the sequence of defining cells for an element which allows one to prove that the Disjoint Disks Property is satisfied for a countable decomposition of elements with the bounded nesting property.

§2 contains statements of the Disjoint Disks Property results which are used in this paper. §3 gives examples illustrating those features of a defining sequence of cells which allow application of the Disjoint Disks results. §4 develops preliminaries associated with the bounded nesting property. §5 contains the proof of the Bounded Nesting Theorem. §6 contains the star-like, bird-like, and mapping cylinder applications of the Bounded Nesting Theorem.

2. The DDP in $\mathbf{S^3}$ [S1]. In this section we describe some criteria for shrinkability of certain cellular, usc decompositions of S^3. All these criteria demonstrate that the cellular, usc decomposition G in question satisfies Bing's Shrinking Criterion [**B1**] and hence is shrinkable, i.e., S^3/G is homeomorphic to S^3.

Consider a 0-dimensional, cellular (or cell-like) usc decomposition G of S^3, and a triangulation T of S^3. By [**S1**, Theorem 2.1], there is a homeomorphism of S^3, restricted by any given saturated open cover of S^3, which moves the nondegenerate elements of G off the 1-skeleton of T. If mesh T is small, and one wishes to shrink the elements of G, it suffices to produce a homeomorphism h: $S^3 \to S^3$, restricted by a saturated open cover, so that for each element $g \in G$, $h(g)$ does not intersect nonadjacent 2-simplexes in the 2-skeleton of T. In [**S1**], the following Disjoint Disks Property was defined, reflecting the facts that disjoint 2-simplexes of $T^{(2)}$ could be handled in pairs, and that moving the 2-simplexes off common elements of G is as good as doing the reverse.

DEFINITION 1. A disk D' in S^3 is obtained from a disk D by a *simple replacement of subdisks* if there are disjoint subdisks $\{E_i\}_{i=1}^n$ on D and $\{E_i'\}_{i=1}^n$ on D' where $D' - \bigcup_{i=1}^n E_i' = D - \bigcup_{i=1}^n E_i$.

DEFINITION 2. The union of the nondegenerate elements of a decomposition G will be denoted by N_G.

The following theorem is a criterion for shrinkability of 0-dimensional decompositions of S^3 proved in [**S1**].

THEOREM 2.1. *Let G be a 0-dimensional, cell-like (or cellular), usc decomposition of S^3. Then S^3/G is homeomorphic to S^3 if and only if for each saturated open cover of N_G and pair of disjoint tame disks D_1, D_2 with* $(\text{Bd } D_1 \cup \text{Bd } D_2) \cap N_G = \varnothing$, *there are disks D_1', D_2' such that:*

(1) *D_1' and D_2' are obtained from D_1 and D_2, respectively, by simple replacement of subdisks;*

(2) *all replacement subdisks of D_1', D_2' are in the given saturated open cover of N_G; and*

(3) *no element $g \in G$ intersects both D_1' and D_2'.*

The following corollary of Theorem 2.1 reflects how Theorem 2.1 is used in dealing with countable decompositions. Its proof [**S1**, Theorem 4.1] is based on the fact that the union of the elements of G that intersect two disjoint disks is compact.

COROLLARY 2.2. *Let G be a countable, usc cellular decomposition of S^3. Then G is shrinkable if and only if each element g in G satisfies the following Property* (a) ("a" *stands for the ability to cut off a pair of disks from g without creating new intersections of the pair of disks with other elements of G*):

PROPERTY (a). *For each 3-cell C containing g in its interior, each pair D_1, D_2 of disjoint tame disks whose boundaries miss g, and $\varepsilon > 0$, there are disks D_1', D_2' such that:*

(1) *D_1' and D_2' are obtained from D_1 and D_2, respectively, by an ε-approximation followed by simple replacement of subdisks;*

(2) *all replacement subdisks of D_1', D_2' are in C;*

(3) *not both D_1' and D_2' intersect g; and*

(4) *if an element $\gamma \in G$ intersects both D_1' and D_2' then γ also intersects both D_1 and D_2.*

3. Examples concerning Property (a). Under what conditions does a cellular set satisfy Property (a) of Corollary 2.2? Providing an answer to this question is a major portion of this paper. We begin.

First note that no set of conditions on the topology of a cellular set g above can insure that g has Property (a). This fact follows from the result below.

THEOREM [**S2**, MAIN THEOREM]. *Let $\{g_i\}_{i\in\omega}$ be a collection of nondegenerate continua, each of which admits a cellular embedding in S^3. Then there are cellular, disjoint embeddings $\{h_i\colon g_i \to S^3\}_{i\in\omega}$ so that the decomposition G of S^3 whose nondegenerate elements are $\{h_i(g_i)\}_{i\in\omega}$ is a cellular, usc decomposition of S^3 for which S^3/G is not homeomorphic to S^3. In fact, $\{h_i(g_i)\}_{i\in\omega}$ could be chosen to be a null sequence.*

Therefore, conditions on a cellular set g designed to ensure that g has Property (a) must be conditions on the embedding of g. The conditions formulated in this paper are conditions on a defining sequence of 3-cells for g. Let us begin by defining the standard set-up associated with Property (a) as follows:

Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$. Let g be an element of a cellular usc decomposition G of S^3. Let $\{C_i\}_{i\in\omega}$ be a sequence of PL 3-cells containing g and let A be a PL ray in

$C_0 - g$ so that:

(1) $g = \bigcap_{i\in\omega} C_i$;

(2) for each $i \in \omega$, $C_{i+1} \subset \operatorname{Int} C_i$;

(3) for each element γ in G, if $\gamma \cap \operatorname{Bd} C_i \neq \varnothing$, then $\gamma \cap C_{i+1} = \varnothing$; and

(4) for each $i \in \omega$, $A \cap \operatorname{Bd} C_i =$ one point.

In addition, we will be concerned with the components of $(D_1 \cup D_2) \cap (C_0 - g)$ that intersect $\operatorname{Bd} C_0$. Hence, let $\{P_j\}_{j=1}^r$ be PL, connected, disjoint, relatively closed subsets of $C_0 - g - A$ so that for each $j = 1, \dots, r$:

(i) $P_j \cap \operatorname{Bd} C_0 \neq \varnothing$;

(ii) P_j is a 2-manifold with boundary where the boundary equals $P_j \cap \operatorname{Bd} C_0$;

(iii) P_j can be embedded in a disk;

(iv) P_j is in general position with respect to $\bigcup_{i\in\omega} \operatorname{Bd} C_i$;

(v) if J, K are two components of $P_j \cap \operatorname{Bd} C_i$ and J, K are in the same component of $P_j \cap C_i$, then J, K are in the same component of $P_j \cap (C_i - C_{i+1})$;

(vi) if $\gamma \in G$, H is a component of $P_j \cap (C_i - C_{i+1})$, $\gamma \cap \operatorname{Bd} C_i \neq \varnothing$ and $\gamma \cap H \neq \varnothing$, then $H \cap \operatorname{Bd} C_i \neq \varnothing$.

Each component P_j is called a *principal component*.

This completes the definition of Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$.

We will use Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ to help us in finding replacement disks as required in Property (a). Our method will be to find simple closed curves J of $P_j \cap \operatorname{Bd} C_i$ and cap them off on $\operatorname{Bd} C_i$. In order to accomplish this while satisfying Property (a), it will be useful to see how J separates P_j.

DEFINITION. Let Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ be given. Let J be a simple closed curve in $P_j \cap \operatorname{Bd} C_i$. Then E_J is the disk on $\operatorname{Bd} C_i$ bounded by J that does not contain $A \cap \operatorname{Bd} C_i$. Also F_J is the closure of the component of $P_j - J$ that does not contain $P_j \cap \operatorname{Bd} C_0$. (Note that condition (v) on P_j guarantees that F_J is well defined.) Finally, if $\operatorname{Int} E_J \cap F_J = \varnothing$, let $\hat{F}_J$ be the component of $(C_0 - g) - (E_J \cup F_J)$ that does not contain the arc A.

We now give two examples which illustrate features of a Set-up that are associated with the ability or inability to do disk replacements as required for Property (a).

Good Case. Suppose Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ is given with the following property:

For each simple closed curve J of $P_j \cap \operatorname{Bd} C_i$ ($j = 1, \dots, r$; $i \in \omega$):

(i) $\operatorname{Int} E_J \cap P_j = \varnothing$ and, for emphasis, the redundant;

(ii) for each curve K in $F_J \cap \operatorname{Bd} C_{i+m}$ ($m \geqslant 1$), $\operatorname{Int} E_K \subset \hat{F}_J$ (see Figure 3.1).

In this situation, there is a new Set-up $(\{C_i\}_{i\in\omega}, \{P_j'\}_{j=1}^r, A)$ so that

(a) each P_j' is obtained from P_j by the process of replacing F_J's by E_J's and

(b) if an element γ of G intersects both P_j' and P_k', then γ intersects P_j and P_k.

PROOF. Denote by $\hat{P}_j$ the component of $C_0 - g - P_j$ that does not contain A. Order the $\{P_j\}_{j=1}^r$ so that for each $j = 1, \dots, r-1$, $\hat{P}_j \cap (\bigcup_{k>j} P_k) = \varnothing$. Suppose $\{J_k\}_{k=1}^s$ is a collection of simple closed curves in $P_j \cap \operatorname{Bd} C_j$. Then let P_j' be the component of $(P_j - \bigcup_{k=1}^s F_{J_k}) \cup \bigcup_{k=1}^s E_{J_k}$ that contains $P_j \cap \operatorname{Bd} C_0$.

This collection of P_j''s satisfies the conclusion. In particular, conclusion (b) follows from hypothesis (ii).

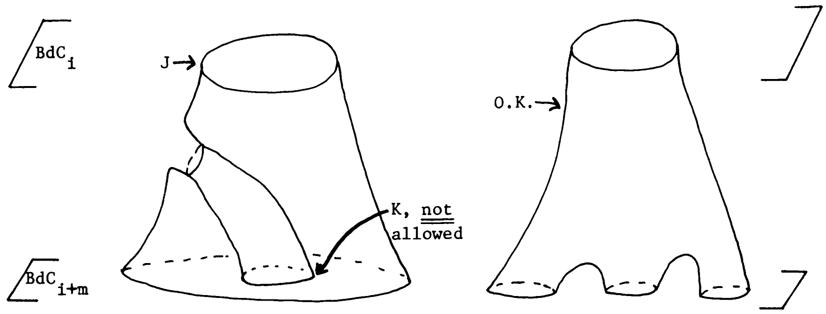

FIGURE 3.1

A Set-up with the properties above arises if g is a PL embedded complex in S^3, disks D_1, D_2 are PL in general position with respect to g and $\{C_i\}_{i\in\omega}$ are PL regular neighborhoods of g. Then the $\{P_j\}_{j=1}^r$ are the components of $(D_1 \cup D_2) \cap (C_0 - g)$ that contain a point on Bd C_0 which can be joined to Bd D_1 $\cup$ Bd D_2 by an arc on $D_1 \cup D_2 -$ Int C_0. Changing each P_j to P_j' can be seen to produce disks D_1', D_2' by replacement of subdisks in a manner satisfying Property (a). This example then contains the ingredients in showing that countable, usc cellular decompositions G of S^3 whose elements are tame polyhedra are shrinkable.

Bad Case. Consider the biggest element g in Bing's minimal example of a nonshrinkable, null sequence, cellular usc decomposition **[B2]**. Construct a Set-up $(\{C_i\}_{i\in\omega}, \{P_1, P_2\}, A)$ where the C_i's are standard neighborhoods of g and P_1 and P_2 are the intersections of disjoint meridional disks of the first defining torus with $C_0 - g$. Examine a curve J of $P_1 \cap$ Bd C_j and note how the curves of $F_J \cap$ Bd C_{j+1} are badly nested with each other and with curves from $P_2 \cap$ Bd C_{j+1}, demonstrating the failure here of hypothesis (ii) of the Good Case.

4. Separation properties of principal components $\{P_j\}_{j=1}^r$. In this section we investigate how principal components intersect the Bd C_i's and describe some separation and nesting properties of various E_J's, F_J's, and $\hat{F}_J$'s. In the Good Case of §3, we were allowed to cap off a principal component because of its separation properties in $C_0 - g$. The first lemma below specifies a circumstance in which a part of a principal component can be cut off on a Bd C_i without danger of having a nondegenerate element intersect a pair of principal components which it did not intersect before.

LEMMA 4.1. *Let Set-up* $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ *be given, let P be a principal component and let J be a simple closed curve of $P \cap$ Bd C_i such that* Int $E_J \cap (\bigcup_{j=1}^r P_j) = \varnothing$. *Suppose K is a component of $F_J \cap$ Bd C_{i+m} $(m \geq 1)$ such that* Int $E_K \subset \hat{F}_J$. *Then if an element γ of G intersects E_K and P_k for some k, then γ intersects F_J and P_k. (See Figure* 4.1.)

PROOF. By property (3) of Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$, $\gamma \cap E_J = \varnothing$. But $F_J \cup E_J$ separates Int E_K from $(\bigcup_{j=1}^r P_j - P)$. Therefore, the result follows from the connectedness of γ.

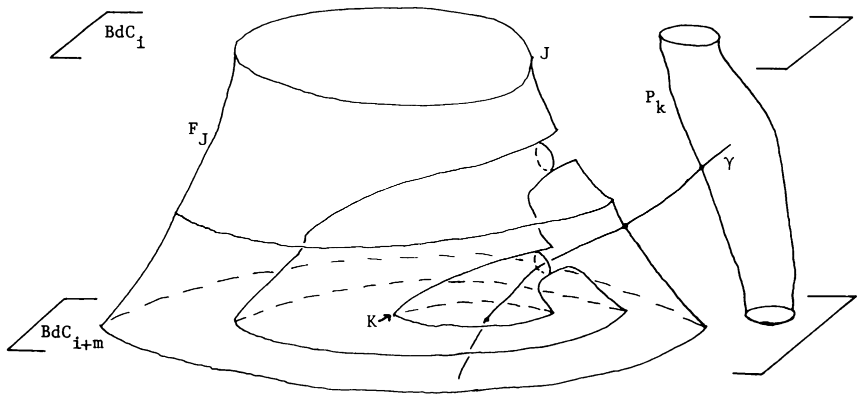

FIGURE 4.1

The next lemmas demonstrate how nesting of curves of $P \cap \operatorname{Bd} C_i$ are related to those of $P \cap \operatorname{Bd} C_{i+m}$. One of the difficulties we face in this process is the problem of handling the fact that a principal component does not typically go straight through a $\operatorname{Bd} C_i$. That is, $P \cap (C_0 - \operatorname{Int} C_i)$ is not generally connected, although by condition (v) of the Set-up, only one component of $P \cap (C_0 - \operatorname{Int} C_i)$ intersects $\operatorname{Bd} C_0$. In order to deal with this problem, we define first curves below.

DEFINITION. Given Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$, a curve J of $(\operatorname{Bd} C_i) \cap P_j$ is a *first curve* of $(\operatorname{Bd} C_i) \cap P_j$ if and only if J is on the component of $P_j \cap (C_0 - \operatorname{Int} C_i)$ that intersects $\operatorname{Bd} C_0$.

Because of this local oscillation problem of P around $\operatorname{Bd} C_i$, we will describe the nesting of a curve J of $P \cap \operatorname{Bd} C_i$ not by looking directly at J on $\operatorname{Bd} C_i$, but instead by looking at how first curves of $\bigcup P_j \cap \operatorname{Bd} C_{i+m}$ are nested in first curves of $F_J \cap \operatorname{Bd} C_{i+m}$.

REMARK In the Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ the words innermost, outermost, and nesting, for curves J on $\operatorname{Bd} C_i$, refer to relationships among the disks E_J.

LEMMA 4.2. *Let Set-up* $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ *be given, let P be a principal component, let J be a simple closed curve of $P \cap \operatorname{Bd} C_i$ where* $(\operatorname{Int} E_J) \cap F_J = \varnothing$, *and let K be outermost on* $\operatorname{Bd} C_{i+m}$ *among first curves of* $F_J \cap \operatorname{Bd} C_{i+m}$ $(m \geq 1)$. *Then a collar of K in E_K lies in $\hat{F}_J$.* (*See Figure* 4.2.)

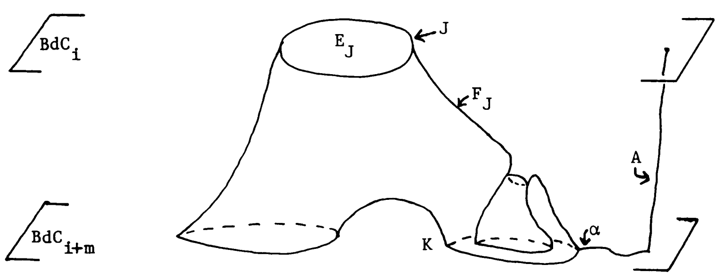

FIGURE 4.2

PROOF. Let F be the component of $(E_J \cup F_J) \cap (C_0 - \operatorname{Int} C_{i+m})$ which contains E_J. No component B of $(F_J - F) \cap (C_0 - \operatorname{Int} C_{i+m})$ separates F from the arc A because condition (v) of the Set-up ensures that $B \cap \operatorname{Bd} C_i = \varnothing$. Therefore an arc α on $\operatorname{Bd} C_{i+m}$ from $A \cap \operatorname{Bd} C_{i+m}$ to a point on K, whose interior misses F, must pierce each component of $F_J \cap (C_0 - \operatorname{Int} C_{i+m})$ an even number of times. Since A is not in $\hat{F}_J$, an exterior collar of K on $\operatorname{Bd} C_{i+m}$ must also be outside $\hat{F}_J$, therefore a collar of K in E_K must be inside $\hat{F}_J$.

The next lemma expresses the fact that pockets cause nesting of first curves.

LEMMA 4.3. *Let Set-up* $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ *be given, let P be a principal component, and let J be a curve in* $P \cap \operatorname{Bd} C_i$ *where* $(\operatorname{Int} E_J) \cap F_J = \varnothing$. *Let K be a curve of* $F_J \cap \operatorname{Bd} C_{i+m}$ $(m \geqslant 1)$ *with* $E_k \cap \hat{F}_J = \varnothing$. *Then for each first curve L of* $F_K \cap \operatorname{Bd} C_{i+n}$ $(n > m)$, *there is a first curve M of* $(F_J - F_K) \cap \operatorname{Bd} C_{i+n}$ *so that* $L \subset E_M$. (*See Figure* 4.3.)

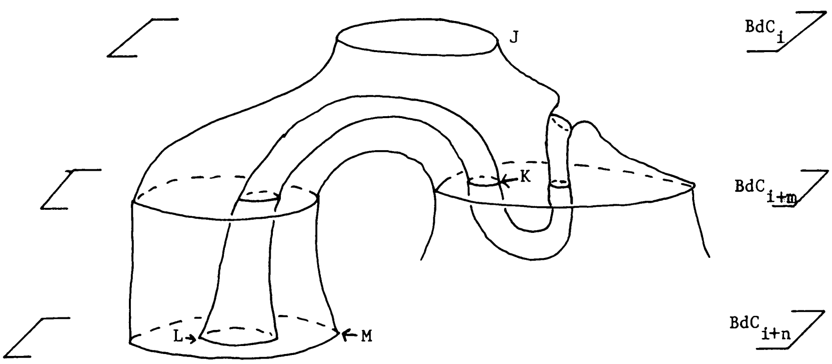

FIGURE 4.3

PROOF. Since $E_K \cap \hat{F}_J = \varnothing$, $\hat{F}_K \cap \hat{F}_J = \varnothing$. Assume that L is outermost on $\operatorname{Bd} C_{i+n}$ among the first curves of $F_K \cap \operatorname{Bd} C_{i+n}$. By Lemma 4.2, E_L contains a collar of L in $\hat{F}_K$, therefore an exterior collar of L on $\operatorname{Bd} C_{i+n}$ is in $\hat{F}_J$. Hence no collar of L in E_L is in $\hat{F}_J$, so by Lemma 4.2, L is not outermost on $\operatorname{Bd} C_{i+n}$ among the first curves of $F_J \cap \operatorname{Bd} C_{i+n}$. This proves the lemma.

The next lemma also deals with nesting of first curves, but this time with nonpocket curves.

LEMMA 4.4. *Let Set-up* $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ *be given, and let P be a principal component. Suppose* $J \subset (\operatorname{Bd} C_i \cap P)$ *where* $(\operatorname{Int} E_J) \cap F_J = \varnothing$. *Let K be a curve of* $F_J \cap \operatorname{Bd} C_{i+m}$ $(m \geqslant 1)$ *such that* $\operatorname{Int} E_K \subset \hat{F}_J$ *and let L be a curve of* $E_K \cap (\bigcup_{j=1}^r P_j)$ *such that* $L \neq K$.

Then for every first curve M of $F_L \cap \operatorname{Bd} C_{i+n}$ $(n > m)$, *there is a first curve N of* $(F_J - F_L) \cap \operatorname{Bd} C_{i+n}$ *so that* $M \subset E_N$. (*See Figure* 4.4.)

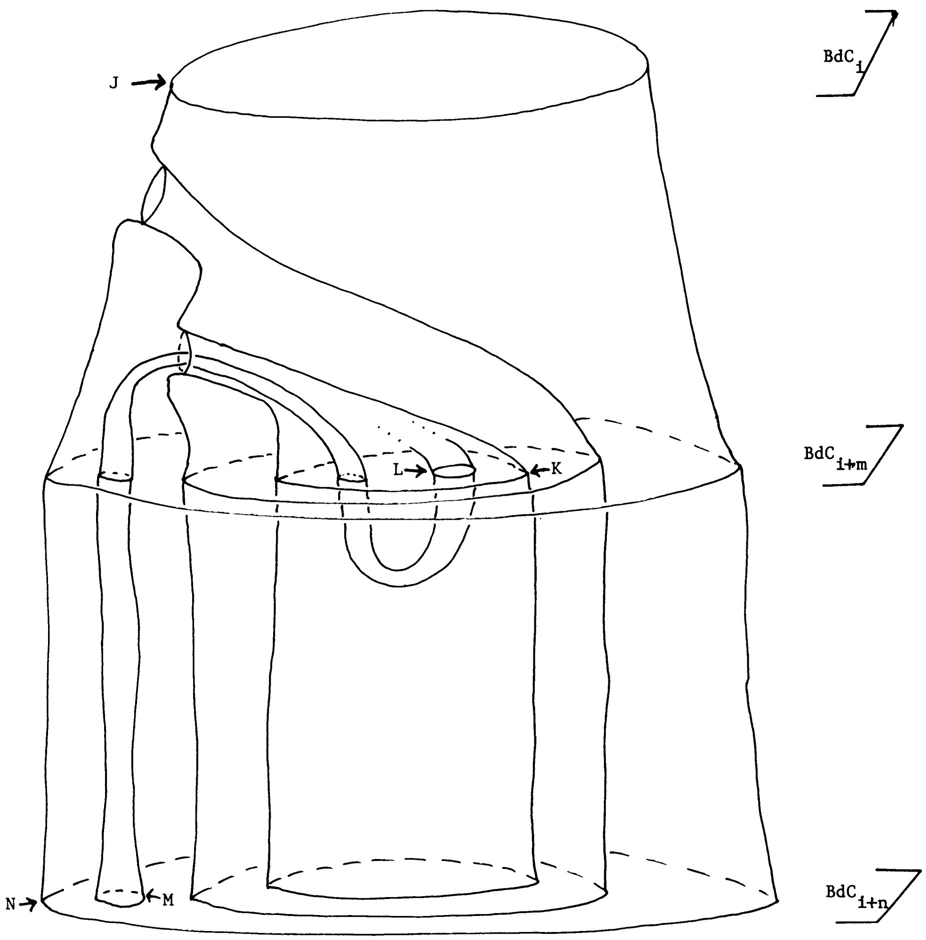

FIGURE 4.4

PROOF. Note first that $F_L \subset \hat{F}_J$. Therefore, the component of $F_L \cap (C_0 - \operatorname{Int} C_{i+n})$ which contains L must be separated from A by the component of $(E_J \cup F_J) \cap (C_0 - \operatorname{Int} C_{i+n})$ which contains E_J. This proves the lemma.

5. The Bounded Nesting Theorem. We are now ready to state and prove the Bounded Nesting Theorem. In the previous section, we saw how principal components intersect the Bd C_i's. Here we assert that if the nesting of first curves is bounded, the principal components can be cut off by repeated uses of Lemma 4.1. The nesting of first curves is formalized in the following definition.

DEFINITION. A Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ has *bounded nesting* if and only if there is an integer M such that for each $i \in \omega$ and set of first curves $\{J_k\}_{k=1}^n$ on Bd C_i with $E_{J_k} \subset E_{J_{k+1}}$ $(k = 1, 2, \ldots, n-1)$, $n < M$.

Our objective is to reduce the intersections of $\bigcup_{j=1}^r P_j$ with the Bd C_i's by changing both the P_j's and the C_i's. We measure the difficulty we have in removing a curve J from $P \cap$ Bd C_i by a notion of complexity defined below. The proof of the curve elimination Lemma 5.2 is accomplished by induction on complexity.

DEFINITION 1. Let $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ be a Set-up, P a principal component, and K a first curve of $P \cap \operatorname{Bd} C_k$. Then K has *girth* $g(K)$ equal to the maximum number n for which there exist first curves $\{J_i\}_{i=1}^n$ in Int E_K with $E_{J_i} \subset E_{J_{i+1}}$ $(i = 1,\dots,n-1)$.

REMARK. If J is a curve of $P \cap \operatorname{Bd} C_i$ $(i \geqslant 1)$, then F_J is not a disk if and only if $F_J \cap \operatorname{Bd} C_k \neq \varnothing$ for each $k > i$.

DEFINITION 2. Let Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ be given with bounded nesting. If J is a curve of $P \cap \operatorname{Bd} C_i$ $(i \geqslant 1)$ for which F_J is not a disk, then the *complexity* $c(J)$ *of* J equals $\min_{j>i} \max\{g(K) \mid K$ is a first curve of $F_J \cap \operatorname{Bd} C_k$ for some $k > j\}$. If F_J is a disk, we define $c(J) = -1$.

For a curve J above, let $m(J)$ be an integer for which the following statement is true: if K is a first curve of $F_J \cap \operatorname{Bd} C_k$ and $k > m(J)$, then $g(K) \leqslant c(J)$.

Lemma 5.1 provides the beginning step of the inductive proof of the curve elimination Lemma 5.2.

LEMMA 5.1. *Let* $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ *be a Set-up with bounded nesting and let* J *be a curve of* $P \cap \operatorname{Bd} C_i$ $(i \geqslant 1)$ *with* F_J *a disk. Then there is a new Set-up* $(\{C_i'\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ *with bounded nesting such that*:

(a) $J \not\subset \operatorname{Bd} C_i'$;

(b) *for* $0 \leqslant k < i$, $C_k' = C_k$;

(c) $(\bigcup_{j=1}^r P_j \cap \operatorname{Bd} C_i') \subset ((\bigcup_{j=1}^r P_j) \cap \operatorname{Bd} C_i)$;

(d) *for* $k > i$, $C_k' = C_{k+m}$ *for some fixed* $m \geqslant 1$; *and*

(e) *the new complexity* $c'(L) = c(L)$ *for each curve* L *of* $\bigcup P_j \cap \bigcup_{k\in\omega} \operatorname{Bd} C_k'$.

PROOF. Since F_J is a disk, there is some integer $m \geqslant 1$ so that $F_J \cap C_{i+m-1} = \varnothing$, so define $C_k' = C_{k+m}$ for $k > i$, and define $C_k' = C_k$ for $0 \leqslant k < i$. Let K be a curve which is innermost on the disk F_J among the curves of $F_J \cap \operatorname{Bd} C_i$. Alter $\operatorname{Bd} C_i$ to obtain $\operatorname{Bd} \tilde{C}_i$ by replacing E_K by a disk parallel to F_K and sufficiently close such that $(\{C_j', \tilde{C}_i\}_{j\neq i, j\in\omega}, \{P_j\}_{j=1}^r, A)$ is a Set-up. Now $\operatorname{Bd} \tilde{C}_i$ has fewer intersections with F_J than $\operatorname{Bd} C_i$ had. Note that the 3-cell $\tilde{C}_i$ bounded by $\operatorname{Bd} \tilde{C}_i$ contains g since $(F_K \cup E_K) \cap A = \varnothing$. Repeat this process until we obtain a 3-cell C_i' such that $\operatorname{Bd} C_i' \cap F_J = \varnothing$.

Note that since for $k > i$, $C_k' = C_{k+m}$, the complexity of any curve L of $\bigcup P_j \cap \bigcup \operatorname{Bd} C_k'$ is unchanged and the Set-up has bounded nesting.

LEMMA 5.2 (CURVE ELIMINATION). *Let Set-up* $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ *be given with bounded nesting. Let* P *be a principal component and* J *be a component of* $P \cap \operatorname{Bd} C_i$ $(i \geqslant 1)$ *where* Int $E_J \cap F_J = \varnothing$. *Then there exists a new Set-up* $(\{C_i'\}_{i\in\omega}, \{P_j'\}_{j=1}^r, A)$ *such that*:

(1) $(\{C_i'\}_{i\in\omega}, \{P_j'\}_{j=1}^r, A)$ *has bounded nesting*;

(2) *for each* j, $P_j' \cap (C_0 - C_1) = P_j \cap (C_0 - C_1)$;

(3) $J \not\subset P_j' \cap \operatorname{Bd} C_i'$;

(4) *each* P_j' *is obtained from* P_j *by repeatedly finding a curve* K *as in Lemma* 4.1, *and replacing* F_K *with* E_K;

(5) *if* $\gamma \in G$ *intersects* P_k' *and* P_s', *then* γ *intersects* P_k *and* P_s;

(6) *for* $k < i$, $C_k' = C_k$;

(7) $(\bigcup_{j=1}^r P_j') \cap \operatorname{Bd} C_i' \subset (\bigcup_{j=1}^r P_j) \cap \operatorname{Bd} C_i$;

(8) *the new complexity* $c'(L) \leqslant c(L)$ *for each curve* L *of* $\bigcup P_j' \cap \bigcup_{k\in\omega} \operatorname{Bd} C_k'$.

Proof. The proof proceeds by induction on the complexity of J. Suppose $c(J) = -1$. Then F_J is a disk and Lemma 5.1 implies the desired conclusions.

Assume that Lemma 5.2 is true for each curve K which satisfies the hypotheses of Lemma 5.2 and with $c(K) < c(J)$. We show that it is true for J.

We prove this by eliminating all curves of $F_J \cap \operatorname{Bd} C_{i+1}$ by applications of Lemma 5.2 as applied to curves on $\operatorname{Bd} C_k$ for $k > i$. After several such applications, the new F_J will not intersect the new $\operatorname{Bd} C_{i+1}$ at all, so F_J will be a disk, which is a previously considered case.

Case 1. $\operatorname{Int} E_J \cap \bigcup P_j = \varnothing$.

Let K be an innermost curve of $F_J \cap \operatorname{Bd} C_{i+1}$, thereby satisfying the hypotheses of this lemma.

If $E_K \subset \hat{F}_J$, then replace F_K by E_K and adjust $\operatorname{Bd} C_{i+1}$ slightly near E_K to eliminate the curve K from $F_J \cap \operatorname{Bd} C_{i+1}$. Lemma 4.1 implies that the new Set-up satisfies conclusion (5). The other conclusions of Lemma 5.2 as applied to K are easily checked.

Claim. If $E_K \cap \hat{F}_J = \varnothing$, then $c(K) < c(J)$.

Proof of Claim. Let $k > \max\{m(J), m(K)\}$ and let $L \subset (F_K \cap \operatorname{Bd} C_k)$ be a first curve with $g(L) = c(K)$. By Lemma 4.3, there exists a first curve M of $(F_J - F_K) \cap \operatorname{Bd} C_k$ such that $L \subset E_M$. So $g(M) > g(L)$ and the Claim is proved.

Therefore, K can be eliminated by induction. Case 1 is completed by repeatedly removing innermost curves K as above until $F_J \cap \operatorname{Bd} C_{i+1} = \varnothing$. Case 1 is proved.

Case 2. $\operatorname{Int} E_J \cap \bigcup P_j \neq \varnothing$.

Let K be an innermost curve of $F_J \cap \operatorname{Bd} C_{i+1}$. If $E_K \cap \hat{F}_J = \varnothing$, then $c(K) < c(J)$ as was shown in Case 1. In this event K can be eliminated by induction. Suppose, therefore, that $\operatorname{Int} E_K \subset \hat{F}_J$. Let L be an innermost curve of $\operatorname{Int} E_K \cap \bigcup P_j$.

Claim. $c(L) < c(J)$.

Proof. Let $k > \max\{m(L), m(J)\}$. Let M be a first curve of $F_L \cap \operatorname{Bd} C_k$ for which $g(M) = c(L)$. By Lemma 4.4, there is a first curve N of $(F_J - F_L) \cap \operatorname{Bd} C_k$ with $M \subset E_N$. Therefore $g(M) < g(N)$ and so $c(L) < c(J)$ and the Claim is proved.

Therefore, all curves in $\operatorname{Int} E_K$ can be eliminated by induction. If K remains, then $\operatorname{Int} E_K \cap \bigcup P_j = \varnothing$. Since $K \subset F_J$, $c(K) \leqslant c(J)$. So Case 1 allows us to eliminate K.

Case 2 is completed by repeatedly removing innermost curves K as above until $F_J \cap \operatorname{Bd} C_{i+1} = \varnothing$. This finishes the proof of Case 2 and the lemma.

This lemma allows us now to state and prove the Bounded Nesting Theorem. First we give a definition of what it means for an element to have the bounded nesting property. The definition below is more complicated than is ordinarily necessary; however, it is used for one of the applications. The simpler version, which is more important to understand, is obtained from the definition below by replacing (ii) and (iii) below with the following condition:

(ii)′ $\{P_j\}_{j=1}^r$ equals the set of components of $(\tilde{D}_1 \cup \tilde{D}_2) \cap (C_0 - g)$ that contain a point on $\operatorname{Bd} C_0$ which can be connected to $\operatorname{Bd} \tilde{D}_1 \cup \operatorname{Bd} \tilde{D}_2$ by an arc in $\tilde{D}_1 \cup \tilde{D}_2 - \operatorname{Int} C_0$.

The fancy definition below is used only in the mapping cylinder application, Theorem 6.3.

DEFINITION. Let G be a cellular, usc decomposition of S^3. An element g in G has the *bounded nesting property* if and only if for each pair of disjoint tame disks D_1, D_2 with $(\text{Bd } D_1 \cup \text{Bd } D_2) \cap g = \varnothing$ and $\varepsilon > 0$, there are PL disks $\tilde{D}_1$, $\tilde{D}_2$ and a Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ with bounded nesting such that:

(i) $\tilde{D}_1$, $\tilde{D}_2$ are ε-approximations of D_1, D_2;

(ii) $\bigcup_{j=1}^r P_j \cap \text{Bd } C_0$ contains every point of $(\tilde{D}_1 \cup \tilde{D}_2) \cap \text{Bd } C_0$ which can be joined to $\text{Bd } \tilde{D}_1 \cup \text{Bd } \tilde{D}_2$ by an arc on $\tilde{D}_1 \cup \tilde{D}_2 - \text{Int } C_0$;

(iii) for each component B of $(\tilde{D}_1 \cup \tilde{D}_2) \cap (C_0 - g)$ that intersects $\text{Bd } C_0$ and can be joined to $\text{Bd } \tilde{D}_1 \cup \text{Bd } \tilde{D}_2$ by an arc on $\tilde{D}_1 \cup \tilde{D}_2 - \text{Int } C_0$, there is a P_j so that $B \cap \text{Bd } C_0 = P_j \cap \text{Bd } C_0$ and P_j is ε-homeomorphic to a subset of B.

DEFINITION. A cellular, usc decomposition G of S^3 has the *bounded nesting property* if and only if each element of G has the bounded nesting property.

BOUNDED NESTING THEOREM 5.3. *Let G be a countable, cellular usc decomposition of S^3 with the bounded nesting property. Then S^3/G is homeomorphic to S^3.*

PROOF. It suffices to prove that each element g of G has Property (a) of Theorem 2.2. Let D_1, D_2 be tame disks with $(\text{Bd } D_1 \cup \text{Bd } D_2) \cap g = \varnothing$. Let $\tilde{D}_1$, $\tilde{D}_2$, and Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ be those guaranteed by the fact that g has the bounded nesting property. Repeated use of Lemma 5.2 allows us to produce a new Set-up $(\{C_i'\}_{i\in\omega}, \{P_j'\}_{j=1}^r, A)$ which is obtained from Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ satisfying the conclusions of Lemma 5.2 such that $\bigcup_{j=1}^r P_j' \cap g = \varnothing$. Let the disks D_i' $(i = 1, 2)$ required in Property (a) be obtained from $\tilde{D}_i$ $(i = 1, 2)$ by first deleting those points of $\tilde{D}_i$ which are separated from $\text{Bd } \tilde{D}_i$ on $\tilde{D}_i$ by $\text{Bd } C_0$ and then adding in each P_j' that intersects what remains. Since, by the conclusion of Lemma 5.2, each P_j' is obtained from P_j by replacing F_J's by E_J's, one sees that the D_i''s are obtained from the $\tilde{D}_i$'s by simple replacement of subdisks. The other requirements for Property (a) are also incorporated in Lemma 5.2, so the theorem is proved.

6. Star-like equivalent and other applications. Many theorems about shrinkability of countable, cellular decompositions of S^3 follow from the Bounded Nesting Theorem. One new result, established independently by E. Woodruff, is the star-like equivalent case.

References below to "straight" in S^3 exploit the natural correspondence between R^3 and S^3 minus a point away from the relevant sets.

DEFINITIONS. 1. A compact set g in S^3 is *star-like* if and only if there is a point $p \in g$ so that for each straight ray $\overrightarrow{px}$ starting at p, $\overrightarrow{px} \cap g$ is an interval or a point.

2. A compact set g in S^3 is *star-like equivalent* if and only if there is a homeomorphism $h: S^3 \to S^3$ so that $h(g)$ is star-like.

STAR-LIKE EQUIVALENT THEOREM 6.1. *Let G be a countable usc decomposition of S^3, each element of which is star-like equivalent. Then S^3/G is homeomorphic to S^3.*

PROOF. We show that G has the bounded nesting property. Let g be a nondegenerate element of G, let D_1 and D_2 be disjoint tame disks with $(\text{Bd } D_1 \cup \text{Bd } D_2) \cap g = \varnothing$, and let $\varepsilon > 0$. Let $h\colon S^3 \to S^3$ be a homeomorphism so that $h(g)$ is star-like and $h \mid S^3 - g$ is PL, and let $\delta > 0$ correspond to ε via the uniform continuity of h^{-1}. Let p be the point from which $h(g)$ is star-like, and let $\{C_i\}_{i\in\omega}$ be a defining sequence of ideally star-like 3-cells for $h(g)$, that is, each ray from p intersects Bd C_i in exactly one point, $C_{i+1} \subset \text{Int } C_i$ for each $i \in \omega$, and $g = \bigcap_{i\in\omega} C_i$. Assume that $C_0 \cap h(\text{Bd } D_1 \cup \text{Bd } D_2) = \varnothing$. Let $\tilde{D}_1$ and $\tilde{D}_2$ be disjoint PL disks that are δ-approximations of $h(D_1)$ and $h(D_2)$, respectively, and are in general position with respect to Bd C_i for each $i \in \omega$ so that $(\text{Bd } \tilde{D}_1 \cup \text{Bd } \tilde{D}_2) \cap C_0 = \varnothing$. Let $\{P_j\}_{j=1}^r$ be the set of components of $(\tilde{D}_1 \cup \tilde{D}_2) \cap (C_0 - g)$ that contain a point on Bd C_0 which can be joined to Bd $\tilde{D}_1 \cup$ Bd $\tilde{D}_2$ by an arc on $\tilde{D}_1 \cup \tilde{D}_2$ missing Int C_0. We assume that $\tilde{D}_1 \cup \tilde{D}_2$ miss an endpoint q of $h(g)$ by some distance d, and we also assume that C_0 is inside a d-neighborhood of $h(g)$. Let A be the half open interval $(q, r]$ where $r = \overrightarrow{pq} \cap \text{Bd } C_0$. The triple $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ satisfies properties (1) and (2) of a Set-up as defined in §3 and we can assume property (3) by upper semicontinuity. Property (4) holds since each C_i is ideally star-like and properties (i)–(iv) are obvious. Properties (v) and (vi) can be obtained by eliminating cells from the defining sequence. Therefore, we have a Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ as defined in §3. Furthermore, properties (i) and (ii′) of the bounded nesting property (see §5) are obvious so it remains to exhibit bounded nesting.

Suppose $\{J_k\}_{k=1}^n$ is a set of first curves on Bd C_i with $E_{J_k} \subset \text{Int } E_{J_{k+1}}$ for $k = 1, \ldots, n-1$. Choose a point x of Int E_{J_1} and a point y of Bd $C_i - E_{J_n}$, so that the rays $\overrightarrow{px}$ and $\overrightarrow{py}$ each intersect each 2-simplex of $\tilde{D}_1 \cup \tilde{D}_2$ in at most one point. Let a be a point on the ray $\overrightarrow{px}$ and b be a point on $\overrightarrow{py}$ so that a and b are outside a ball containing $\tilde{D}_1 \cup \tilde{D}_2 \cup C_0$, and let Q be a polygonal path from a to b outside this ball. The polygonal simple closed curve $S = \overline{pxa} \cup Q \cup \overline{byp}$ intersects each disk E_{J_k} exactly once and, therefore, links each curve J_k. Since the J_k's are first curves on P_j's which can be joined to Bd $\tilde{D}_1 \cup$ Bd $\tilde{D}_2$ missing Int C_0, they bound disjoint disks on $\tilde{D}_1 \cup \tilde{D}_2$. Therefore S intersects $\tilde{D}_1 \cup \tilde{D}_2$ at least n times. On the other hand, all intersections of S with $\tilde{D}_1 \cup \tilde{D}_2$ are on the segments $\overline{pxa}$ and $\overline{pyb}$, each of which can intersect each 2-simplex of $\tilde{D}_1 \cup \tilde{D}_2$ at most once. Therefore, n is bounded by twice the total number of 2-simplexes in $\tilde{D}_1 \cup \tilde{D}_2$. This fact proves that Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$ has bounded nesting.

We finish by observing that the Set-up $(\{h^{-1}(C_i)\}_{i\in\omega}, \{h^{-1}(P_j)\}_{j=1}^r, h^{-1}(A))$ also has bounded nesting since h is a homeomorphism. Also observe that $h^{-1}(\tilde{D}_1)$, $h^{-1}(\tilde{D}_2)$ are ε-approximations of D_1 and D_2 because of the choice of δ.

This proof is modified below to deal with bird-like equivalent elements (defined below). Since every star-like set or polyhedral cellular set is bird-like, the following theorem is a proper generalization of the Star-like Equivalent Theorem and the tame cellular polyhedra theorem [**E**; **S1**, Theorem 4.1; **S-W**, Theorem 1].

DEFINITIONS. 1. A compact set g in S^3 is *bird-like* (see Figure 6.1) if and only if it is definable by PL 3-cells $\{C_i\}_{i\in\omega}$ with the properties that:

(a) there is an integer m such that given two points x and y on Bd C_i, there is a polygonal arc from x to y in C_i with at most m 1-simplexes;

(b) for each pair of disjoint tame disks D_1, D_2 with $(\mathrm{Bd}\, D_1 \cup \mathrm{Bd}\, D_2) \cap g = \varnothing$ and $\varepsilon > 0$, there are PL, disjoint ε-approximations $\tilde{D}_1$, $\tilde{D}_2$ of D_1, D_2, respectively, and an integer n such that there exists a PL ray A in $C_n - (g \cup \tilde{D}_1 \cup \tilde{D}_2)$ such that for each $k > n$, $A \cap \mathrm{Bd}\, C_k$ is one point.

2. A compact set g in S^3 is *bird-like equivalent* if and only if there is a homeomorphism $h: S^3 \to S^3$ so that $h(g)$ is bird-like.

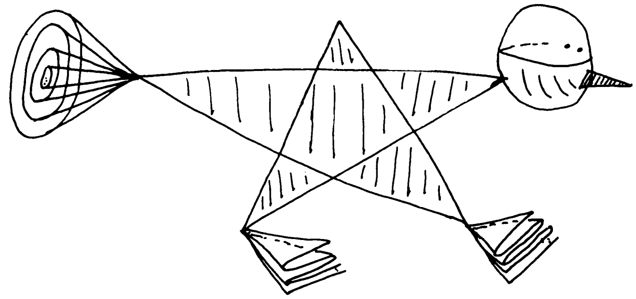

FIGURE 6.1

BIRD-LIKE EQUIVALENT THEOREM 6.2. *Let G be a countable usc decomposition of S^3, each element of which is bird-like equivalent. Then S^3/G is homeomorphic to S^3.*

PROOF. Proceed as in Theorem 6.1 to obtain Set-up $(\{C_i\}_{i\in\omega}, \{P_j\}_{j=1}^r, A)$, omitting a finite number of C_i's in order to find the ray A, and renumbering the C_i's to start at 0. We proceed below to show that this Set-up has bounded nesting.

Let B be a ball containing $C_0 \cup \tilde{D}_1 \cup \tilde{D}_2$, and let p be a point of $S^3 - B$. There is an integer s so that for each $x \in \mathrm{Bd}\, C_0 - (\tilde{D}_1 \cup \tilde{D}_2)$, there is a polygonal arc from x to p in $S^3 - \mathrm{Int}\, C_0$ that intersects $\tilde{D}_1 \cup \tilde{D}_2$ in at most s points.

As in the star-like case, suppose $\{J_k\}_{k=1}^n$ is a set of first curves on $\mathrm{Bd}\, C_i$ with $E_{J_k} \subset \mathrm{Int}\, E_{J_{k+1}}$ for $k = 1, \ldots, n-1$. We seek to produce a bound for n. Choose a point x of $\mathrm{Int}\, E_{J_1}$ very close to J_1, and choose a point y of $\mathrm{Bd}\, C_i - E_{J_n}$ very close to J_n. Let b and c be PL arcs in $S^3 - \mathrm{Int}\, C_i$ from x and y to points x' and y', respectively, on $\mathrm{Bd}\, C_0$ so that $(b \cup c) \cap (\tilde{D}_1 \cup \tilde{D}_2) = \varnothing$. These arcs can be constructed near the components of $(C_0 - \mathrm{Int}\, C_i) \cap (\tilde{D}_1 \cup \tilde{D}_2)$ since J_1 and J_n are first curves. Let d and e be PL arcs in $S^3 - \mathrm{Int}\, C_0$ from x' and y', respectively, to p, each of which intersects $(\tilde{D}_1 \cup \tilde{D}_2)$ in at most s points. Let f be the PL arc from x to y in C_i guaranteed by the fact that $h(g)$ is bird-like. In particular, f has at most m 1-simplexes each of which intersects a 2-simplex of $\tilde{D}_1 \cup \tilde{D}_2$ at most once. The polygonal simple closed curve $S = bdecf$ intersects each disk E_{J_k} exactly once and therefore links each curve J_k. The J_k's bound disjoint disks on $\tilde{D}_1 \cup \tilde{D}_2$, therefore S intersects $\tilde{D}_1 \cup \tilde{D}_2$ at least n times. On the other hand, all intersections of S with $\tilde{D}_1 \cup \tilde{D}_2$ lie on $d \cup e \cup f$, so n is bounded by $s + s + m \cdot j$, where j is the number of 2-simplexes in $\tilde{D}_1 \cup \tilde{D}_2$. This completes the proof that the nesting is bounded. Thus the Bounded Nesting Theorem 5.3 completes the proof of the bird-like theorem.

Another result, which includes the tame polyhedra case, is the following one, which can be derived from [E; **S1**, Theorem 4.1; **W**, Theorem 1].

MAPPING CYLINDER THEOREM 6.3. *Let G be a countable usc decomposition of S^3 each element of which has a mapping cylinder neighborhood. Then S^3/G is homeomorphic to S^3.*

PROOF. Let g be a nondegenerate element of G, let D_1 and D_2 be disjoint tame disks with $(\text{Bd}\, D_1 \cup \text{Bd}\, D_2) \cap g = \varnothing$, and let $\varepsilon > 0$. Let C_0 be a PL mapping cylinder neighborhood of g so that:

(1) $C_0 - g$ is PL homeomorphic (via h) to $S^2 \times (0, 1]$;

(2) the function f: $C_0 - g \to g$ defined by $f(h(z, t)) = \lim_{t\to 0} h(z, t)$ is well defined and continuous;

(3) $\text{diam}(h(\{z\} \times (0, 1])) < \varepsilon$ for each $z \in S^2$;

(4) there is an arc $A = h(\{z\} \times (0, 1])$ so that $A \cap (D_1 \cup D_2) = \varnothing$;

(5) for each element $\gamma \in G$ and for each integer $k \geq 1$, if $\gamma \cap h(S^2 \times \{1/k\}) \neq \varnothing$, then $\gamma \cap h(S^2 \times \{1/k + 1\}) = \varnothing$.

Let $\text{Bd}\, C_i = h(S^2 \times \{1/i + 1\})$ and let $\tilde{D}_1$ and $\tilde{D}_2$ be disjoint ε-PL approximations of D_1 and D_2, respectively, which miss A and are in general position with $\text{Bd}\, C_i$ for all $i \in \omega$. Let $\{P_j\}_{j=1}^r$ be the components of $(\tilde{D}_1 \cup \tilde{D}_2) \cap (C_0 - g)$ that intersect $\text{Bd}\, C_0$.

Let k be an integer ≥ 2 so that if J and K are components of $(\text{Bd}\, C_0 \cap P)$ for some P, then J and K are in the same component of $P \cap h(S^2 \times [1/k, 1])$. For each P_j and for each first curve L of $P_j \cap \text{Bd}\, C_{k-1}$, replace F_L by $h(h^{-1}(L) \times (0, 1])$ to obtain $\tilde{P}_j$. This change produces a new Set-up $(\{C_i\}_{i\in\omega}, \{\tilde{P}_j\}_{j=1}^r, A)$. The new Set-up has bounded nesting since the number of curves of $(\bigcup_{j=1}^r \tilde{P}_j) \cap \text{Bd}\, C_i$ $(i \in \omega)$ is bounded. Since the new Set-up was obtained from the original Set-up in accordance with the conditions in the definition of the bounded nesting property, we see that g has the bounded nesting property. Thus the Bounded Nesting Theorem 5.3 implies that G is shrinkable.

REFERENCES

[B1] R. H. Bing, *Upper semicontinuous decompositions of E^3*, Ann. of Math. (2) **65** (1957), 363–374.

[B2] ______, *Point like decompositions of E^3*, Fund. Math. **50** (1962), 431–453.

[E] Dan Everett, *Shrinking countable decompositions of E^3 into points and tame arcs*, Geometric Topology (J. Cantrell, Editor), Academic Press, New York, 1979, pp. 53–72.

[S1] Michael Starbird, *Cell-like, 0-dimensional, decompositions of E^3*, Trans. Amer. Math. Soc. **249** (1979), 203–215.

[S2] ______, *Null sequence cellular decompositions of S^3*, Fund. Math. **112** (1981), 81–87.

[S-W] Michael Starbird and Edythe P. Woodruff, *Decompositions of E^3 with countably many non-degenerate elements*, Geometric Topology (J. Cantrell, Editor), Academic Press, New York, 1979, pp. 239–252.

[W] Edythe P. Woodruff, *Decompositions of E^3 with countably many star-like equivalent elements*, preprint.

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