ALMOST CONVERGENT AND WEAKLY ALMOST PERIODIC
FUNCTIONS ON A SEMIGROUP

BY
HENERI A. M. DZINOTIWEYI

ABSTRACT. Let \( S \) be a topological semigroup, \( \text{UC}(S) \) the set of all bounded uniformly continuous functions on \( S \), \( \text{WAP}(S) \) the set of all (bounded) weakly almost periodic functions on \( S \), \( \mathcal{E}_0(S) := \{ f \in \text{UC}(S): m(|f|) = 0 \text{ for each left and right invariant mean } m \text{ on } \text{UC}(S) \} \) and \( \mathcal{W}_0(S) := \{ f \in \text{WAP}(S): m(|f|) = 0 \text{ for each left and right invariant mean } m \text{ on } \text{WAP}(S) \} \).

Among other results, for a large class of noncompact locally compact topological semigroups \( S \), we show that the quotient space \( \mathcal{E}_0(S)/\mathcal{W}_0(S) \) contains a linear isometric copy of \( l^\infty \) and so is nonseparable.

1. Introduction. In this paper we study matters that continued to burn inside our mind during and after writing our paper [3]. In particular terms undefined here are as defined in [3] and we shall refer to [3] at various stages of the paper.

Let \( S \) be a (Hausdorff jointly) continuous topological semigroup, \( C(S) \) the set of all bounded real-valued continuous functions on \( S \), \( M(S) \) the set of all bounded real-valued Radon measures on \( S \) and \( M_a(S) := \{ \mu \in M(S): \text{the maps } x \to |\mu|((x^{-1}C) \text{ and } x \to |\mu|((C^{-1}x) \text{ of } S \text{ into } \mathbb{R} \text{ are continuous, for all compact } C \subseteq S \} \). \( S \) is said to be the foundation of \( M_a(S) \) if \( S \) coincides with the closure of \( \bigcup \{ \text{supp}(\mu): \mu \in M_a(S) \} \); where \( \text{supp}(\mu) \) stands for the support of \( \mu \). For each \( f \in C(S) \) and \( x \in S \) we define \( x_f \) and \( f_x \) in \( C(S) \) by

\[
x_f(y) := f(xy) \quad \text{and} \quad f_x(y) := f(yx) \quad (y \in S).
\]

Let \( \text{UC}(S) := \{ f \in C(S): \text{the maps } x \to x_f \text{ and } x \to f_x \text{ of } S \text{ into } C(S) \text{ are norm continuous} \} \) and \( \text{WAP}(S) := \{ f \in C(S): \text{the set } \{ x_f: x \in S \} \text{ is relatively weakly compact} \} \).

A functional \( m \in \text{UC}(S)^* \) is called a mean on \( \text{UC}(S) \) if \( \|m\| = m(1) = 1 \), where 1 stands for the constant function one on \( S \). A mean \( m \) is left (or right) invariant if \( m(x_f) = m(f) \) (or \( m(f_x) = m(f) \), respectively), for all \( f \in \text{UC}(S) \) and \( x \in S \). Similarly one defines left and right invariant means on other subspaces of \( C(S) \).

We label our most important definitions as 1.1, 1.2 and 1.3.

**Definition 1.1.** The following sets consist of special cases of the so-called almost convergent functions: \( \mathcal{E}_0(S) := \{ f \in \text{UC}(S): m(|f|) = 0 \text{ for every left or right invariant mean } m \text{ on } \text{UC}(S) \} \) and \( E_0(S) := \{ f \in \text{UC}(S): m(|f|) = 0 \text{ for every left and right invariant mean } m \text{ on } \text{WAP}(S) \} \).
and right invariant mean \( m \) on \( UC(S) \)). We also define \( W_0(S) := \{ f \in WAP(S): m(|f|) = 0 \) for every left and right invariant mean \( m \) on \( WAP(S) \)\).

The following definition is taken from [3].

**Definition 1.2.** For any subsets \( A_1, \ldots, A_n \) of a semigroup \( S \) we define

\[
A_1 \otimes A_2 := \{ A_1 A_2, A_1^{-1} A_2, A_1 A_2^{-1} \},
\]

\[
A_1 \otimes A_2 \otimes A_3 := \left( \bigcup \{ A_1 \otimes B: B \in A_2 \otimes A_3 \} \right) \cup \left( \bigcup \{ B \otimes A_3: B \in A_2 \otimes A_3 \} \right)
\]

and hence inductively define \( A_1 \otimes \cdots \otimes A_n \).

A subset \( B \) of \( S \) is said to be relatively neo-compact if \( B \) is contained in a (finite) union of sets in \( A_1 \otimes \cdots \otimes A_n \) for some compact subsets \( A_1, \ldots, A_n \) of \( S \).

We note in particular that if \( S \) is such that \( C^{-1} D \) and \( D C^{-1} \) are compact sets for all compact subsets \( C \) and \( D \) of \( S \), then \( B \subset S \) is relatively neo-compact if and only if \( B \) is relatively compact.

In the following definition we urge the reader to note that in the case of a topological group all the sets mentioned in (b) are compact.

**Definition 1.3.** A locally compact topological semigroup \( S \) with an identity element 1 is said to have property \((E)\) if \( S \) contains a nonrelatively neo-compact subset (also denoted by the letter \( E \)) such that if \( U \) is a neighbourhood of 1, then for all \( x \in E \) we have that

1. there is a neighbourhood \( V \) of 1 such that \( xV \subseteq Ux \) and \( Vx \subseteq Ux \);

2. if \( C \) and \( D \) are compact subsets and \( p, t \in S \), then

\[
\mathcal{L}[C, xD; t] := \bigcup \left\{ (yCt^{-1})(yxD): y \in S \right\},
\]

\[
\mathcal{L}_1[xD; t, p] := \bigcup \left\{ y^{-1}\left((yxD)t^{-1}\right)p: y \in S \right\},
\]

\[
\mathcal{A}[C, Dx; t] := \bigcup \left\{ (Dxy)(t^{-1}Cy)^{-1}: y \in S \right\},
\]

\[
\mathcal{A}_1[Dx; t, p] := \bigcup \left\{ \left(p(t^{-1}(Dxy))\right)y^{-1}: y \in S \right\}
\]

are relatively neo-compact subsets of \( S \).

For convenience of notation we shall write

\[
L[xD] := \mathcal{L}[\{1\}, xD; 1], \quad \mathcal{L}[C, FD, T] := \bigcup \{ \mathcal{L}[C, xD, t]: x \in F \text{ and } t \in T \}
\]

and similarly define \( R[DX], \mathcal{A}[C, DF; T], \mathcal{L}_1[FD; T, P] \) and \( \mathcal{A}_1[DF; T, P] \) for finite subsets \( F, T \) and \( P \) of \( S \).

When \( S \) is a noncompact locally compact topological group then our definition of property \((E)\) coincides with that of property \((E)\) as defined in [2]. Further, one can prove that if \( S \) is an infinite cancellative discrete topological semigroup with identity element, then indeed \( S \) has property \((E)\). Also if \( S \) is a noncompact locally compact topological semigroup that is cancellative, commutative and has an identity element, then \( S \) has property \((E)\).

Burckel [1, Theorem 3.19] proved that \( W_0(\mathbb{R}) \) is a proper subset of \( E_0(\mathbb{R}) \) where \( \mathbb{R} \) is the additive group of real numbers with the usual topology. Generalising this result, Ching Chou [2, Theorem 5.1] proved that if \( G \) is a locally compact topological group with property \((E)\) and such that \( \{xax^{-1}: x \in E \cup E^{-1} \} \) is relatively compact
for each \( a \in G \), then the quotient space \( F_0(G) / W_0(G) \) contains a linear isometric copy of \( l^\infty \). Also Ching Chou [2, p. 177] asked whether the conclusion of his Theorem 5.1 can be extended to every discrete topological group.

In this paper our main result says that if \( S \) is a locally compact topological semigroup such that \( S \) is the foundation of \( M_\alpha(S) \) and \( S \) has property (\( E \)) then \( E_0(S) / W_0(S) \) contains an isometric copy of \( l^\infty \).

So as to maintain clarity we collect together some preliminary results in §2. Our main result is proved in §3. Finally we examine property (\( E \)) in §4.

We are indebted to the referee for helpful criticisms.

2. Preliminaries. We need the following result which is a special case of [4, Corollary 1.4].

**Proposition 2.1.** Let \( S \) be a locally compact topological semigroup with identity element and \( WUC(S) := \{ f \in C(S) : \text{the maps } x \to f_x \text{ and } x \to x f \text{ of } S \text{ into } C(S) \text{ are weakly continuous} \} \). Then if \( S \) is the foundation of \( M_\alpha(S) \), we have that \( WUC(S) = UC(S) \).

From [6, Theorems 2.2 and 2.4] we have

**Proposition 2.2.** Let \( S \) be a locally compact topological semigroup with identity element \( 1 \) and such that \( S \) is the foundation of \( M_\alpha(S) \). Then every compact neighbourhood \( U \) contains an element \( u \) such that \( U^{-1} u \cap u U^{-1} \) is a neighbourhood of \( 1 \).

3. The main result. First we prove a lemma. Item (i) of the following lemma is a special form of [3, Key Lemma 3] and is only mentioned here for completeness without reproducing the proof. Thus the proof of our next lemma is essentially a proof for item (ii) of the lemma.

**Lemma 3.1.** Let \( S \) be a locally compact topological semigroup with identity element \( 1 \), such that \( S \) is the foundation of \( M_\alpha(S) \). Then if \( S \) has property (\( E \)) we can find a compact neighbourhood \( V \) of \( 1 \), sequences \( \{ x_1, x_2, \ldots \} \), \( \{ y_1, y_2, \ldots \} \), \( \{ w_1, w_2, \ldots \} \) and \( \{ z_1, z_2, \ldots \} \) in \( E \), with \( w_1 = z_1 = 1 \); such that

\[
V^{-1}(V_{x_1} V V^{-1} \cap V^{-1}(V_{x_n} y_m V) V^{-1} = \emptyset
\]

if any one of the following conditions holds:

(a) \( n \leq m \) and \( i > j \);
(b) \( n > m \), \( i > j \) and \( n \neq i \);
(c) \( n \leq m \), \( i < j \) and \( m \neq j \).

(ii) If \( n \neq m \) then

(a) \( i > j \) and \( k > l \) imply \( V^{-1}(V_{x_i} y_j V)(w_n V)^{-1} \cap V^{-1}(V_{x_k} y_l V)(w_m V)^{-1} = \emptyset \);
(b) \( i < j \) and \( k < l \) imply \( (V^{-1}(V_{z_n} V V^{-1} \cap (V_{z_m})^{-1}(V_{x_k} y_l V)V^{-1} = \emptyset \),

for all \( i, j, k, l, n, m \) in \( \mathbb{N} \).

**Proof.** Without losing generality we assume that \( E \) is a subsemigroup of \( S \). By Proposition 2.2 we fix compact neighbourhoods \( C, D \) of \( 1 \) and \( c \in C, d \in D \) such that \( C \subset D^{-1}d \cap dd^{-1} \) and \( C^{-1}c \cap cc^{-1} \) is a neighbourhood of \( 1 \). Next, by property (\( E \)) and Proposition 2.2, we choose compact neighbourhoods \( U_1, U_2, U_3, U_4 \) and \( V \) of \( 1 \) and \( u \in U_4 \) meeting the following inclusion relations.
For convenience let $X_p := \{x_1, \ldots, x_p\}$, $Y_p := \{y_1, \ldots, y_p\}$, $W_p := \{w_1, \ldots, w_p\}$, $Z_p := \{z_1, \ldots, z_p\}$ and $N_p := \{1, 2, \ldots, p\}$ for $p \in \mathbb{N}$.

Now suppose, by the inductive hypothesis, we have the lemma (i.e. items (i) and (ii)) and item (iii) $n < m$ implies that

- $(a)$ $w_m u C \cap L[W_m u C] = \emptyset$,
- $(b)$ $C u z_m \cap R[C u Z_n] = \emptyset$,
- $(c)$ $w_m u C \cap L[W_m u C; Y_m c, Y_m c] = \emptyset$,
- $(d)$ $C u z_m \cap R[C u Z_n; c X_n, c X_n] = \emptyset$,
- $(e)$ $y_m c \notin E[W_m u C, W_m u C; Y_m c]$,
- $(f)$ $c x_m \notin E[C u Z_m, C u Z_m; c X_n]$,

for all $i$, $j$, $k$, $l$, $m$, $n$ in $N_p$.

By the definition of relative neo-compactness we can choose $w_{p+1}$ and $z_{p+1}$ in $E$ such that

- $(1a)$ $w_{p+1} \notin (V^{-1}(V X_p Y_p V)(W_p V)\cdot V^{-1}(V X_p Y_p V) V^{-1})$,
- $(1b)$ $w_{p+1} \notin (L[W_p u C] D)(u d)^{-1}$,
- $(1c)$ $z_{p+1} \notin (Y_p c)(u c)^{-1}$,
- $(1'a)$ $z_{p+1} \notin (V^{-1}(V X_p Y_p V) V^{-1})(V Z_p)^{-1}(V X_p Y_p V) V^{-1}$,
- $(1'b)$ $z_{p+1} \notin (u d)^{-1}(D R[C u Z_p])$,
- $(1'c)$ $z_{p+1} \notin (C u)^{-1}(R[C u Z_p; c X_p, c X_p])$.

Next we choose $x_{p+1}$ and $y_{p+1}$ in $E$ such that item (i) is met for $i$, $j$, $m$, $n$ in $N_{p+1}$ which is possible by our proof of [3, Lemma 3] (and by our definition of a relatively neo-compact set),

- $(2a)$ $x_{p+1} \notin V^{-1}(V V^{-1}(V X_p Y_p V)(W_{p+1} V)\cdot V^{-1}(V X_p Y_p V) V^{-1}) W_{p+1}(V Y_p V)^{-1}$,
- $(2b)$ $x_{p+1} \notin (V - R(C u Z_{p+1}, C u Z_{p+1}; c X_p)$,
- $(3a)$ $y_{p+1} \notin (V X_{p+1})^{-1}(V Z_{p+1})^{-1}(V X_{p+1} Y_{p+1} V) V^{-1}) V^{-1}$,
- $(3b)$ $y_{p+1} \notin (w_{p+1} u C, W_{p+1} u C; Y_p c)$.

Now for $i$, $j$, $k$, $l$, $m$, $n$ in $N_{p+1}$ we have from $(*)$

\[
V^{-1}(V x_i y_j V) V^{-1} w_{n}^{-1} \cap V^{-1}(V x_k y_j V) V^{-1} w_{m}^{-1} \\
\subseteq u^{-1}(U_{i} V x_i y_j V U_{i}) u^{-1} w_{n}^{-1} \cap u^{-1}(U_{k} V x_k y_j V U_{k}) u^{-1} w_{m}^{-1} \\
\subseteq u^{-1}\left((U_{i} x_i y_j U_{i}) (w_n u)^{-1} \cap (U_{k} x_k y_j U_{k}) (w_m u)^{-1}\right) \\
\subseteq u^{-1}\left((x_i y_j U_{i}) (w_n u)^{-1} \cap (x_k y_j U_{k}) (w_m u)^{-1}\right) \\
\subseteq u^{-1}\left((x_i y_j U_{i}) (w_n u)^{-1} \cap (x_k y_j U_{i}) (w_m u)^{-1}\right) \\
\subseteq u^{-1}\left((x_i y_j U_{i}) (w_n u)^{-1} \cap (x_k y_j U_{i}) (w_m u)^{-1}\right).
\]
Hence setting \( A(i, j, n; k, l, m) := (x_i y_j c)(w_n u C)^{-1} \cap (x_k y_l c)(u_m u C)^{-1} \) we have
\[
(4) \quad V^{-1}(V X_y y V)^{-1} w_n^{-1} \cap V^{-1}(V X_k y_l V)^{-1} w_m^{-1} \subseteq u^{-1} A(i, j, n; k, l, m).
\]
Similarly setting \( B(i, j, n; k, l, m) := (C u_z y_j)^{-1} \cap (C u_m y_l)^{-1} \) we obtain
\[
(5) \quad z_n^{-1} V^{-1}(V X_y y V)^{-1} z_m^{-1} \cap z_m^{-1} V^{-1}(V X_k y_l V)^{-1} \subseteq B(i, j, n; k, l, m) u^{-1}.
\]

We are now in a position to verify the inductive step. Already item (i) is done as remarked before. Items (iii)(a) and (b), for \( n, m \) in \( N_{p+1} \), should be clear in view of (1b), (1'b) and the inclusion relationships
\[
L[W_p u C](u C)^{-1} \subseteq (L[W_p u C] D)(u d)^{-1},
\]
\[
(C u)^{-1} R[C u Z_p] \subseteq (d u)^{-1} (D R[C u Z_p]).
\]
Also with \( N_{p+1} \) in place of \( N_p \) items (iii)(c),(d),(e) and (f) follow from items (1c), (1'c), (3b) and (2b) (respectively) and the inductive hypothesis.

Next we prove (ii)(a) for \( i, j, k, l, m, n \) in \( N_{p+1} \). To this end we assume that \( n < m \) and consider the following cases:

Case (a): \( i, k < p \).

If \( m < p \), the result follows by the inductive hypothesis, so we assume \( m = p + 1 \). Now item (1a) is equivalent to
\[
v^{-1}(v x p y p v)^{-1} \cap v^{-1}(v x p y p v) v^{-1} w_p^{-1} = \emptyset,
\]
which in turn implies (ii)(a) (under the present case).

Case (b): \( i = p + 1 \) or \( k = p + 1 \) and \( i \neq k \).

The reader can easily deduce (ii)(a), for this case, from item (2a).

Case (c): \( i = k = p + 1 \) and \( j, l < m \).

From item (4) it is sufficient to show that \( A = A(p + 1, j, n; p + 1, l, m) = \emptyset \). Suppose on the contrary there exists \( a \in A \). Then from the definition of \( A \) we have
\[
aw_n u c_1 = x_{p+1} y_j c \quad \text{and} \quad aw_m u c_2 = x_{p+1} y_1 c
\]
for some \( c_1 \) and \( c_2 \) in \( C \).

Hence
\[
w_m \in a^{-1} \left( \left((a W_n u C)(Y_j c)^{-1}\right) Y_1 c \right)(u C)^{-1} \subseteq \mathcal{C}_1 \left[W_n u C; Y_j c, Y_1 c\right](u C)^{-1}
\]
where \( t := \max\{j, l\} \). This contradicts (iii)(c) (with \( N_{p+1} \) in place of \( N_p \)). By this conflict we have (ii)(a) under the present case.

Case (d): \( i = k = p + 1 \), \( j \neq 1 \) and \( m = \max\{j, l\} \).

We assume that \( l < j \) and suppose there exists \( a \in A(p + 1, j, n; p + 1, l, m) \).

Then by the definition of the latter set we have
\[
y_j c \in \left((a W_n u C)(Y_j c)^{-1}\right)^{-1} (a W_n u C) \subseteq \mathcal{C} \left[W_n u C, W_m u C; Y_j c\right]
\]
which contradicts item (iii)(e) (with \( N_{p+1} \) in place of \( N_p \)). By this conflict \( A = \emptyset \) and item (4) gives the result.

Case (e): \( (i, j) = (k, l) \).

Suppose there exists \( a \in A(i, j, n; k, l, m) \). Then
\[
aw_n u c_1 = x_i y_j c = x_k y_l c = aw_m u c_2 \quad \text{for some} \ c_1, c_2 \in C.
\]
Hence
\[ w_m u C_2 \subseteq a^{-1}(aw_n u C) \cap w_m u C \subseteq L(W_n u C) \cap w_m u C, \]
which contradicts item (iii)(a) (with \( N_{p+1} \) in place of \( N_p \)). By this conflict \( A = \emptyset \) and item (4) implies the result.

This completes our proof for the inductive step for (ii)(a).

Similarly from items (iii)(b), (d) and (f); (I'a), (I'c); (2b), (3a) and (5), we obtain the inductive step for (ii)(b).

Repeating the argument countably many times we obtain our lemma.

We now give our main result.

**Theorem 3.2.** Let \( S \) be a locally compact topological semigroup with identity element 1 and such that \( S \) is the foundation of \( M_a(S) \). Then if \( S \) has property (E) we have that the quotient space \( E_0(S)/W_0(S) \) contains a linear isometric copy of \( l^\infty \) and so is nonseparable.

**Proof.** We now indicate how the proof of [3, Theorem 2.1] can be extended to yield our result.

Let \( \rho \) be a positive measure in \( M_a(S) \) with \( \| \rho \| = 1 \) and \( \text{supp}(\rho) \subseteq V \); where \( V \) is as stated in Lemma 3.1. We assume the notation of Lemma 3.1 throughout this proof. Take the measures \( \nu \) and \( \mu \) used in the proof of [3, Theorem 2.1] to be equal to \( \rho \) and define the sequence of functions \( \{ f_k \} \) as done there (i.e. in the proof of [3, Theorem 2.1]). Now Proposition 2.1 says that \( WUC(S) = UC(S) \), consequently \( WAP(S) \subseteq UC(S) \).

So the proof of [3, Theorem 2.1] would show that the mapping
\[
\{ c_k \} \rightarrow \sum_{k=1}^{\infty} c_k f_k + W_0(S)
\]
is a linear isometric map of \( l^\infty \) into \( E_0(S)/W_0(S) \) if we can show that the function \( f := \sum_{k=1}^{\infty} c_k f_k \) is in \( E_0(S) \). Already \( f \in WUC(S) = UC(S) \), so it remains to show that \( m(\| f \|) = 0 \) for every left and right invariant mean \( m \) on \( UC(S) \).

From the definition of the \( f_k \)'s we can write \( f = h - g \) for some \( h, g \) in \( UC(S) \) such that

\[
(1) \quad \text{supp}(z_n | h |) \subseteq \bigcup_{i=1}^{\infty} \bigcup_{i \leq j} (Vz_n)^{-1}(Vx_i,y)V^{-1},
\]
\[
(2) \quad \text{supp}(| g |_{w_n} | h |) \subseteq \bigcup_{j=1}^{\infty} \bigcup_{i \geq j} V^{-1}(Vx_i,y)V(Vw_n)^{-1},
\]
for all \( n \in \mathbb{N} \). From (1) and Lemma 3.1(ii)(b) we have that any two members of the sequence \( \{ z_n | h | \} \) have disjoint supports. So if \( m \) is any left and right invariant mean on \( UC(S) \), then for any \( n \in \mathbb{N} \) we have that
\[
\text{nm}(\| h \|) = m(z_1 | h |) + m(z_2 | h |) + \cdots + m(z_n | h |) = m(1) = \| h \|.
\]
Hence \( m(|h|) = 0 \). Similarly by using (2) and Lemma 3.1(ii)(a) we obtain that \( m(|g|) = 0 \) and hence \( m(|f|) = 0 \) and \( f \in E_0(S) \). This completes our proof.

As an immediate consequence we have

**Corollary 3.3.** If \( G \) is a locally compact topological group with property \((E)\) then \( E_0(G)/W_0(G) \) contains an isometric linear copy of \( l^\infty \).

### 4. Some remarks on property \((E)\).

4.1. Every infinite discrete cancellative semigroup with identity element has property \((E)\). Also if \( T \) is a (noncompact) cancellative commutative locally compact topological semigroup with identity element and \( H \) any compact semigroup with identity, then clearly the product semigroup \( T \times H \) has property \((E)\). In this way one can construct various examples of noncancellative and noncommutative topological semigroups with property \((E)\).

4.2. In this item we sharpen [3, Remark 7.5]. Let \( G \) be a locally compact topological group throughout this item and \( C_00(G) \) be the set of all functions in \( C(G) \) with compact support. Following [2], a set \( X \subseteq G \) is called an \( E \)-set if given a neighbourhood \( U \) of \( 1 \) the set \( \{ x^{-1}Ux: x \in X \} \) is again a neighbourhood of \( G \). It is shown in [2] that an \( E \)-set \( X \subseteq G \) has the property that, for a compact neighbourhood \( U \) of \( 1 \) such that \( xU \cap yU = \emptyset \) for distinct \( x, y \in X \) and function \( f \in C_00(G) \) with \( \text{supp}(f) \subseteq U \), we have that \( g := \sum_{x \in X} f \) belongs to \( UC(G) \). Such sums of translates of a function with compact support are useful in finding functions in \( UC(G) \setminus WAP(G) \). Our next proposition teaches us that, conversely, if \( X \subseteq G \) is a set such that “such functions” \( g := \sum_{x \in X} f \) are in \( UC(G) \), then the set \( X \) must be an \( E \)-set.

**Proposition A.** Let \( V \) and \( U \) be a compact neighbourhoods of the identity of \( G \), let \( f \in C_00(G) \) be such that \( f(1) = 1 \) and \( \text{supp}(f) \subseteq V \), \( V^2 \subseteq U \) and let \( X \subseteq G \) be such that for all \( x, y \in X \) with \( x \neq y \) we have that \( xU \cap yU = \emptyset \). Then if \( g := \sum_{x \in X} f \) is in \( UC(G) \), we have that \( \cap \{ x^{-1}Ux: x \in X \} \) is a neighbourhood of \( 1 \).

**Proof.** Assuming \( g \in UC(G) \), we can find a neighbourhood \( W \) of \( 1 \) such that \( W \subseteq V \) and

\[
| g(ay) - g(y) | \leq \| a - g \| < 1 \quad (a \in W \text{ and } y \in G).
\]

Thus \( | \sum_{x \in X} (f(xay) - f(xy)) | < 1 \). Taking \( y = x_0^{-1} \) (for any fixed \( x_0 \) in \( X \)) and recalling that \( xU \cap yU = \emptyset \) for distinct \( x, y \in X \) we get \( |f(x_0ax_0^{-1}) - f(1)| < 1 \). Hence \( x_0ax_0^{-1} \in V \), for all \( a \in W \), or \( W \subseteq x_0^{-1}Vx_0 \). Thus \( W \subseteq \cap \{ x^{-1}Vx: x \in X \} \) and our proposition follows.

Note that by choosing \( V \)'s contracting to \( 1 \), the preceding proposition says that, for every neighbourhood \( V \) of \( 1 \) we have that \( \cap \{ x^{-1}Vx: x \in X \} \) is a neighbourhood of \( 1 \) and thus \( X \) is an \( E \)-set.

**Definitions.** A set \( B \subseteq G \) is said to be right uniformly discrete if there is a neighbourhood \( U \) of \( 1 \) such that \( Ux \cap Uy = \emptyset \) for distinct \( x, y \in B \).

\( G \) is said to be \( \alpha \)-compact if \( G \) can be written as a union of \( \alpha \) compact sets and cannot be written as a union of \( \beta \) compact sets if \( \beta < \alpha \).
**Proposition B** (cf. [5, Theorem 2.3]). If \( G \) is \( \alpha \)-compact, then \( G \) has equivalent left and right uniformities if and only if for each right-uniformly discrete set \( B \subseteq G \) satisfying \( \text{card}(B) < \alpha \) and each neighbourhood \( U \) of \( 1 \), we have \( \cap \{ x^{-1}Ux : x \in B \} \) a neighbourhood of \( 1 \).

Hence from Propositions A and B we have

**Proposition C.** If \( G \) is \( \alpha \)-compact then every right-uniformly discrete \( B \subseteq G \) yields \( g := \sum_{x \in B} f \) in \( UC(G) \) if and only if \( G \) has equivalent left and right uniformities; where \( f \) is chosen such that \((\text{supp}(f))x \cap (\text{supp}(f))y = \emptyset \) for distinct \( x, y \) in \( B \).

Equivalently (to Proposition C) we have that every right uniformly discrete set in \( G \) is an \( E \)-set if and only if \( G \) has equivalent left and right uniformities.

4.3. In view of [2, Theorem 4.6], the following conjecture seems reasonable.

**Conjecture.** Let \( S \) be a locally compact semigroup with identity element such that \( x^{-1}K \) and \( Kx^{-1} \) are compact sets for all compact \( K \subseteq S \) and \( x \in S \) and with \( S \) the foundation of \( M_\alpha(S) \). Then if \( S \) has property \( (E) \) we have that \( W_0(S)/C_0(S) \) is nonseparable. (Here \( C_0(S) \) is the set of functions in \( C(S) \) vanishing at infinity.)

**References**


**Department of Mathematics, University of Zimbabwe, Salisbury, Zimbabwe**