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NONFACTORIZATION THEOREMS IN WEIGHTED BERGMAN AND HARDY SPACES ON THE UNIT BALL OF $\mathbf{C}^n$ ($n > 1$)

BY

M. SEETHARAMA GOWDA[1]

ABSTRACT. Let $A^{p,\alpha}(B)$, $A^{q,\alpha}(B)$ and $A^{l,\alpha}(B)$ be weighted Bergman spaces on the unit ball of $\mathbf{C}^n$ ($n > 1$). We prove:

THEOREM 1. *If* $1/l = 1/p + 1/q$ *then* $A^{p,\alpha}(B) \cdot A^{q,\alpha}(B)$ *is of first category in* $A^{l,\alpha}(B)$.

THEOREM 2. *Theorem* 1 *holds for Hardy spaces in place of weighted Bergman spaces.*

We also show that Theorems 1 and 2 hold for the polydisc U^n in place of B.

1. Introduction. Let U be the unit disc in $\mathbf{C}$. For $0 < t < \infty$ and $-1 < \alpha < \infty$, let $H^t(U)$ be the Hardy space of all holomorphic functions f on U satisfying

$$\sup_{0 \leqslant r < 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^t \, d\theta < \infty,$$

and let $A^{t,\alpha}(U)$ be the weighted Bergman space of all holomorphic functions f on U satisfying

$$\int_U |f(z)|^t \left(1 - |z|^2\right)^\alpha dm(z) < \infty,$$

where $dm(z)$ denotes the Lebesgue measure on U. If $0 < p, q, l < \infty$ and $1/p + 1/q = 1/l$, then it is well known that $H^p(U) \cdot H^q(U) = H^l(U)$, where the left-hand side consists of all products of the form $f \cdot g$ with $f \in H^p(U)$ and $g \in H^q(U)$. Horowitz [**3**] proved that $A^{p,\alpha}(U) \cdot A^{q,\alpha}(U) = A^{l,\alpha}(U)$ whenever $\alpha \geqslant 0$ and $1/p + 1/q = 1/l$.

In $\mathbf{C}^n$ ($n > 1$), the above results are no longer valid. Rudin [**6**] and Miles [**4**] showed that $H^2(U^n) \cdot H^2(U^n)$ is a proper subset of $H^1(U^n)$ for $n \geqslant 3$. (Here U^n denotes the unit polydisc in $\mathbf{C}^n$.) Rosay [**5**] showed that $H^2(U^n) \cdot H^2(U^n)$ is of first category in $H^1(U^n)$ for $n \geqslant 2$, thereby completely solving the Factorization Problem (see [**6**, 4.2]) in Hardy spaces of the polydisc. In [**7**, Problem 19.3.1], Rudin asked whether $H^2(B) \cdot H^2(B)$ is properly contained in $H^1(B)$, where B denotes the unit ball of $\mathbf{C}^n$ ($n > 1$). In this paper we show that $H^p(B) \cdot H^q(B)$ is of first category in $H^l(B)$ whenever $0 < p, q, l < \infty$ and $1/p + 1/q = 1/l$. We prove a similar result

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(Theorem 1) for the weighted Bergman spaces on the unit ball B (see §2 for notations and terminology). Essential ideas required to prove these results come from Rosay [**5**].

Coifman, Rochberg and Weiss [**1**] proved that any function in $H^1(B)$ is an infinite sum of the form $\sum_{i=1}^{\infty} f_i g_i$ where f_i and g_i belong to $H^2(B)$ for all i. We do not know if the infinite sum can be replaced by a finite sum (see Remark 4).

2. Preliminaries. Notations are as in [**7**]. For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in $\mathbf{C}^n$, let $\langle z, w\rangle = \sum_{i=1}^n z_i \overline{w}_i$ and $|z|^2 = \sum_{i=1}^n |z_i|^2$; let $B = B_n = \{z \in \mathbf{C}^n\colon |z| < 1\}$ and $S = \{z \in \mathbf{C}^n\colon |z| = 1\}$. For $z \in \mathbf{C}^n$ we sometimes write $z = (z_1, z')$ where $z' = (z_2, z_3, \ldots, z_n)$. $e_1 = (1, 0, 0, \ldots, 0)$.

Let a, $z \in B$ and $a \neq 0$, let

$$\phi_a(z) = \frac{a - P_a z - \left(1 - |a|^2\right)^{1/2} Q_a z}{1 - \langle z, a\rangle}$$

where $P_a z = \langle z, a\rangle a / \langle a, a\rangle$ and $Q_a z = z - P_a z$. $\phi_a(z)$ is a holomorphic automorphism of B satisfying $\phi_a(\phi_a(z)) = z$.

$d\sigma$ denotes the rotation invariant probability measure on S. $d\nu(z) = d\nu_n(z) = 2nr^{2n-1}\, dr\, d\sigma(\zeta)$ is the normalized Lebesgue measure on B. Here $z = r\zeta$, $r = |z|$ and $\zeta \in S$.

$H(B)$ denotes the space of all holomorphic functions on B.

$C(\overline{B})$ denotes the space of all continuous functions on $\overline{B}$.

$A(B) = H(B) \cap C(\overline{B})$ is the ball algebra.

For $0 < t < \infty$, $H^t(B)$ is the Hardy space of all $f \in H(B)$ satisfying

$$\|f\|_{t,\sigma} = \left(\sup_{0 \leq r < 1} \int_S |f(r\zeta)|^t \, d\sigma(\zeta)\right)^{1/t} < \infty.$$

Let

$$d\mu_\alpha(z) = \left(1 - |z|^2\right)^\alpha d\nu(z) / nB(n, \alpha + 1)$$

where $-1 < \alpha < \infty$ and $B(n, \alpha + 1)$ denotes the Beta function. For $-1 < \alpha < \infty$ and $0 < t < \infty$, we write $A^{t,\alpha}(B)$ to denote the space of all $f \in H(B)$ satisfying

$$\|f\|_{t,\alpha} = \left(\int_B |f|^t \, d\mu_\alpha\right)^{1/t} < \infty.$$

We note that $d\mu_\alpha$ is a probability measure on B and

$$\lim_{\alpha \to -1} \int_B f(z)\, d\mu_\alpha(z) = \int_S f(\zeta)\, d\sigma(\zeta) \tag{1}$$

for all $f \in C(\overline{B})$. (The above relation holds for monomials $z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n} \cdot \overline{z}_1^{\gamma_1} \cdots \overline{z}_n^{\gamma_n}$ and hence for linear combinations of monomials. The Stone-Weierstrass theorem proves (1) for any $f \in C(\overline{B})$.)

Because of (1) we can think of $H^p(B)$ as a "limiting" case of $A^{p,\alpha}(B)$ for $\alpha = -1$. Let $f \in H(B)$. Then from [7, Theorem 7.2.5],

$$|f(z)|^p \left(1 - \frac{|z|}{r}\right)^n \leq 2^n \int_S |f(r\zeta)|^p \, d\sigma(\zeta)$$

for $|z| < r < 1$ and $0 < p < \infty$.

Let K be a compact subset of B. If we multiply the above inequality by $(1 - r^2)^\alpha r^{2n-1}\, dr$ and integrate over the interval $(1 + |z|)/2 < r < 1$, we get

$$|f(z)| \leq C_{n,\alpha,p,K} \| f \|_{p,\alpha} (1 - |z|)^{-n/p} \quad (\forall z \in K)$$

where $C_{n,\alpha,p,K}$ is a constant depending only on its subscripts.

The above two inequalities, together with a normality argument, give

Fact 1. *Every bounded sequence in $H^p(B)$ (or in $A^{p,\alpha}(B)$) has a subsequence which converges uniformly on compact subsets of B.*

From this it follows that $A^{p,\alpha}(B)$ and $H^p(B)$ are F-spaces.

Let $f \in H^p(B)$ $(A^{p,\alpha}(B))$ and $f_r(z) = f(rz)$ for $0 < r < 1$. Then $f_r \to f$ in $H^p(B)$ (in $A^{p,\alpha}(B)$) as $r \nearrow 1$. For a suitable r and δ $(0 < \delta < 1)$, $(1 - z_1) f_r(z)/(1 - \delta z_1)$ is close to f in $H^p(B)$ (in $A^{p,\alpha}(B)$) and vanishes at e_1. Hence we have

Fact 2. *The set of all $f \in A(B)$, $f(e_1) = 0$, is dense in $H^p(B)$ $(A^{p,\alpha}(B))$.*

We need the following identities [7, Proposition 1.4.7]:

$$\int_S f(\zeta)\, d\sigma(\zeta) = \int_S \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}\zeta)\, d\theta \right) d\sigma(\zeta), \tag{2}$$

$$\int_S f(\zeta_1, \zeta')\, d\sigma(\zeta) = \int_{B_{n-1}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}\zeta_1, \zeta')\, d\theta \right) d\nu_{n-1}(\zeta'). \tag{3}$$

3. Lemmas.

LEMMA 1. *Let $0 < t < \infty$ and $\alpha > -1$. Then $\| z_2^N \|_{t,\alpha}^t \sim N^{-(n+\alpha)}$ as $N \to \infty$.*

PROOF. We have

$$\begin{aligned} \| z_2^N \|_{t,\alpha}^t &= \frac{2}{B(n, \alpha + 1)} \int_B |(r\zeta_2)^N|^t (1 - r^2)^\alpha r^{2n-1}\, dr\, d\sigma(\zeta) \\ &= \left(\int_S |\zeta_2^N|^t\, d\sigma(\zeta) \right) \left(\frac{2}{B(n, \alpha + 1)} \int_0^1 r^{Nt + 2n - 1} (1 - r^2)^\alpha\, dr \right). \end{aligned}$$

The second integral, on putting $u = r^2$ becomes $\frac{1}{2} B(Nt/2 + n, \alpha + 1)$. By Stirling's formula this behaves like $1/N^{\alpha+1}$ as $N \to \infty$. For the first integral, we use the identity [7, 1.4.5, p. 15]

$$\int_S f(\langle \zeta, \eta \rangle)\, d\sigma(\zeta) = \frac{n-1}{\pi} \iint_U (1 - r^2)^{n-2} f(re^{i\theta}) r\, dr\, d\theta.$$

We get

$$\begin{aligned} \int_S |\zeta_2|^{Nt}\, d\sigma(\zeta) &= \frac{n-1}{\pi} \int_0^1 (1 - r^2)^{n-2} r^{Nt+1}\, dr = \frac{n-1}{\pi} B\left(\frac{Nt+1}{2}, n - 1 \right) \\ &\sim 1/N^{n-1} \quad \text{by Stirling's formula.} \end{aligned}$$

Hence

$$\|z_2^N\|_{t,\alpha}^t \sim 1/N^{n-1} \cdot 1/N^{\alpha+1} = N^{-(n+\alpha)}.$$

REMARK 1. $\|z_2^N\|_{t,\sigma}^t \sim N^{-(n-1)}$.

LEMMA 2. *Let $K(z) = \sum_{i=N-1}^{\infty} K_i(z)$ be holomorphic in B, where $K_i(z)$ is a homogeneous polynomial of degree i and N is a positive integer. Then for $0 < t < \infty$, there exists a constant M (depending only on t) such that*

$$\|K_N\|_{t,\sigma} \leqslant M \cdot \|K\|_{t,\sigma}, \tag{4}$$

$$\|K_N\|_{t,\alpha} \leqslant M \cdot \|K\|_{t,\alpha}. \tag{4'}$$

PROOF. For $0 < t < \infty$, there exists an M such that if $G(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots$ is in the disc algebra $A(U)$ then

$$|a_1|^t \leqslant M^t \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{i\theta})|^t \, d\theta.$$

(For $t \geqslant 1$ we can take $M = 1$. For $0 < t < 1$, see [**2**, Theorem 6.4, p. 98]. In fact, $M = 2^{1/t}$ works for any t.) Now for a fixed z, let

$$G(\lambda) = K(\lambda z)/\lambda^{N-1} = K_{N-1}(z) + \lambda K_N(z) + \cdots.$$

We get

$$|K_N(z)|^t \leqslant M^t \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |K(e^{i\theta} z)|^t \, d\theta.$$

We let $z = r\zeta$, integrate both sides with respect to $d\sigma(\zeta)$ and use (2) to get

$$\int_S |K_N(r\zeta)|^t \, d\sigma(\zeta) \leqslant M^t \int_S |K(r\zeta)|^t \, d\sigma(\zeta). \tag{5}$$

Taking the supremum over r in the interval $0 < r < 1$ and tth roots, we get (4). To get (4′), we multiply both sides of (5) by $(2/B(n, \alpha + 1))\, r^{2n-1}(1 - r^2)^\alpha \, dr$, integrate over $0 < r < 1$ and take tth roots.

LEMMA 3. *Let $0 < p, q, l < \infty$, $1/l = 1/p + 1/q$, $-1 < \alpha < \infty$ and $n > 1$. Then the product map $(h, k) \to h \cdot k$ from $A^{p,\alpha}(B) \times A^{q,\alpha}(B)$ to $A^{l,\alpha}(B)$ is not open at the origin, i.e., for any constant $C > 0$, there exists $f \in A^{l,\alpha}(B)$ such that $\|f\|_{l,\alpha} \leqslant 1$ and if $f = h \cdot k$ with $h \in A^{p,\alpha}(B)$, $k \in A^{q,\alpha}(B)$ then at least one of $\|h\|_{p,\alpha}$, $\|k\|_{q,\alpha}$ is larger than C.*

PROOF. Let $F(z) = z_1^{N-1} + z_2^N$, $N > 1$. Suppose $F(z) = H(z) \cdot K(z)$ with H and K holomorphic in B. We expand $H(z)$ and $K(z)$ in terms of homogeneous polynomials: $H = H_i + H_{i+1} + \cdots$, $K = K_{N-1-i} + K_{N-i} + \cdots$. Here, as usual, subscript refers to the degree, $H_i \not\equiv 0$ and $K_{N-1-i} \not\equiv 0$. From $F = H \cdot K$ we get, by comparing degrees,

$$H_i \cdot K_{N-1-i} = z_1^{N-1} \tag{6}$$

and

$$H_i K_{N-i} + H_{i+1} K_{N-1-i} = z_2^N. \tag{7}$$

From (6) and (7) we get $i = 0$ or $N - 1$. *We assume for a moment that* $i = 0$. Then H_0 is a constant, say A. We have from (6) and (7),

$$AK_N(z) = z_2^N - \left(H_1(z)\cdot z_1^{N-1}\right)/A.$$

Letting $z = r(e^{i\theta}\zeta_1, \zeta')$ we get

$$AK_N\left(e^{i\theta}\zeta_1, \zeta'\right) = \zeta_2^N - A^{-1}H_1\left(e^{i\theta}\zeta_1, \zeta'\right)e^{i(N-1)\theta}\zeta_1^{N-1}.$$

Therefore, ζ_2^N is the constant term in the polynomial $AK_N(\lambda\zeta_1, \zeta')$ in λ. By subharmonicity,

$$|\zeta_2^N|^t \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |AK_N\left(e^{i\theta}\zeta_1, \zeta'\right)|^t\, d\theta \quad \text{for } 0 < t < \infty. \tag{8}$$

Now we multiply both sides of (8) by $d\nu_{n-1}(\zeta')$ and integrate over B_{n-1}. Using (3), we get

$$\int_S |\zeta_2^N|^t\, d\sigma(\zeta) \leq \int_S |AK_N(\zeta)|^t\, d\sigma(\zeta)$$

and

$$\int_S |(r\zeta_2)^N|^t\, d\sigma(\zeta) \leq \int_S |AK_N(r\zeta)|^t\, d\sigma(\zeta). \tag{9}$$

We multiply both sides of (9) by $(2/B(n, \alpha + 1))r^{2n-1}(1 - r^2)^\alpha\, dr$, integrate over $0 < r < 1$ and take tth roots to get

$$\|z_2^N\|_{t,\alpha} \leq |A|\, \|K_N\|_{t,\alpha}.$$

Since $|H(z)|^t$ is subharmonic and $A = H(0)$, we have $|A|^t \leq \int_S |H(r\zeta)|^t\, d\sigma(\zeta)$. From this we get $|A| \leq \|H\|_{t,\alpha}$. Hence

$$\|z_2^N\|_{t,\alpha} \leq |A|\, \|K_N\|_{t,\alpha} \leq \|H\|_{t,\alpha}\|K_N\|_{t,\alpha}.$$

Using Lemma 2, we get $\|z_2^N\|_{t,\alpha} \leq M\|H\|_{t,\alpha}\|K\|_{t,\alpha}$. *By symmetry, this inequality holds when* $i = N - 1$. Now let $f = F/\|F\|_{l,\alpha}$. Then $\|f\|_{l,\alpha} = 1$. Suppose $f = h\cdot k$ where $h \in A^{p,\alpha}(B)$ and $k \in A^{q,\alpha}(B)$. Then $F = H\cdot K$ where $H = \|F\|_{l,\alpha}h$ and $K = k$. Therefore

$$\|z_2^N\|_{t,\alpha} \leq M\|H\|_{t,\alpha}\cdot\|K\|_{t,\alpha} \leq M\|F\|_{l,\alpha}\|h\|_{t,\alpha}\cdot\|k\|_{t,\alpha}.$$

Now we take $t = \min(p, q)$. Then $\|h\|_{t,\alpha} \leq \|h\|_{p,\alpha}$ and $\|k\|_{t,\alpha} \leq \|k\|_{q,\alpha}$. We have

$$\|F\|_{l,\alpha}^l = \int_B |z_1^{N-1} + z_2^N|^l\, d\mu_\alpha \leq 2^l\left[\|z_1^{N-1}\|_{l,\alpha}^l + \|z_2^N\|_{l,\alpha}^l\right].$$

By Lemma 1, the right side of the above inequality is like $N^{-(n+\alpha)}$ for large N. We see that

$$\|h\|_{p,\alpha}\cdot\|k\|_{q,\alpha} \geq \|z_2^N\|_{t,\alpha}/M\|F\|_{l,\alpha}.$$

Hence $\|h\|_{p,\alpha}\|k\|_{q,\alpha}$ is bigger than a constant times $N^{-(n+\alpha)(1/t-1/l)}$ which goes to ∞ as $N \to \infty$ (recall $t = \min(p, q) > l$). Therefore, for any constant C, we can find a large N so that $\|h\|_{p,\alpha}\cdot\|k\|_{q,\alpha} > C^2$. This completes the proof.

REMARK 2. By considering H^p-norms instead of $A^{p,\alpha}$-norms, one can get the nonopenness of the product map (at the origin) for H^p-spaces.

LEMMA 4. *For $a \in B$ and $z \in \bar{B}$, let*

$$K(a, z) = \left[\left(1 - |a|^2\right) / |1 - \langle z, a\rangle|^2\right]^{(n+1+\alpha)}.$$

Then:

(i) $K(a, \phi_a(z)) \cdot K(a, z) = 1$.

(ii) $\int_B f(\omega)\, d\mu_\alpha(\omega) = \int_B f(\phi_a(z)) K(a, z)\, d\mu_\alpha(z)$ *for all* $f \in C(\bar{B})$.

(iii) $\int_B f(\phi_a(z))\, d\mu_\alpha(z) \to f(e_1)$ *as* $a \to e_1$ *for all* $f \in C(\bar{B})$.

PROOF. From [7, Theorem 2.2.5], we have

$$1 - \langle \phi_a(z), a\rangle = \left(1 - |a|^2\right) / \left(1 - \langle z, a\rangle\right).$$

Taking absolute values and using the definition of K, we get (i). From [7, Theorem 2.2.6] we have

$$\int_B f(\omega)\left(1 - |\omega|^2\right)^\alpha d\nu(\omega)$$

$$= \int_B f(\phi_a(z))\left(1 - |\phi_a(z)|^2\right)^\alpha \left(\frac{1 - |a|^2}{|1 - \langle z, a\rangle|^2}\right)^{n+1} d\nu(z).$$

Using

$$1 - |\phi_a(z)|^2 = \left(1 - |a|^2\right)\left(1 - |z|^2\right) / |1 - \langle z, a\rangle|^2 \qquad \text{(see [7, Theorem 2.2.5])},$$

$$\int_B f(\omega)\left(1 - |\omega|^2\right)^\alpha d\nu(\omega) = \int_B f(\phi_a(z)) \frac{\left(1 - |a|^2\right)^{n+1+\alpha}\left(1 - |z|^2\right)^\alpha}{|1 - \langle z, a\rangle|^{2(n+1+\alpha)}} d\nu(z).$$

Hence

$$\int_B f(\omega)\, d\mu_\alpha(\omega) = \int_B f(\phi_a(z)) K(a, z)\, d\mu_\alpha(z).$$

This is (ii). Since $\lim_{a \to e_1} \phi_a(z) = e_1$, an application of the Bounded Convergence Theorem gives (iii).

REMARK 3. In the above lemma we assumed that $\alpha > -1$. The following statements hold when $\alpha = -1$.

(i) $K(a, \phi_a(z)) \cdot K(a, z) = 1$.

(ii) $\int_S f(\eta)\, d\sigma(\eta) = \int_S f(\phi_a(\zeta)) K(a, \zeta)\, d\sigma(\zeta)$ for all $f \in C(S)$.

(iii) $\int_S f(\phi_a(\zeta))\, d\sigma(\zeta) \to f(e_1)$ as $a \to e_1$ for all $f \in C(S)$.

We observe that when $\alpha = -1$, $K(a, z)$ is the Poisson kernel and statements (i) and (ii) are well known. Since $\int_S f(\phi_a(\zeta))\, d\sigma(\zeta)$ is the Poisson integral of f, (iii) follows (see, e.g., [7, Theorem 3.3.4(a)]).

LEMMA 5. *Let*

$$\psi_a(z) = \left[1 + \sqrt{1 - |a|^2} / \left(1 - \langle z, a\rangle\right)\right]^{2(n+1+\alpha)}.$$

Then

$$\max\{1, K(a, z)\} \leqslant |\psi_a(z)| \leqslant 2^{2(n+\alpha)+1}\{1 + K(a, z)\}$$

for all $z \in \bar{B}$, $a \in B$ and $\alpha \geqslant -1$.

PROOF. For $\lambda \in \mathbf{C}$ and $\operatorname{Re}\lambda \geq 0$, we have $\max\{1, |\lambda|\} \leq |1+\lambda|$. This can be seen by plotting λ and $1+\lambda$ in the complex plane. Also, $|1+\lambda|^m \leq (1+|\lambda|)^m \leq 2^{m-1}(1+|\lambda|^m)$ for $m \geq 1$. Taking $\lambda = \sqrt{1-|a|^2}/(1-\langle z, a\rangle)$ and $m = 2(n+1+\alpha)$, we get the lemma.

4. Main theorem.

THEOREM 1. *Let $n > 1$, $-1 < \alpha < \infty$, $0 < p, q, l < \infty$ and $1/l = 1/p + 1/q$. Then $A^{p,\alpha}(B)\cdot A^{q,\alpha}(B)$ is of first category in $A^{l,\alpha}(B)$.*

PROOF. Let V and W be the closed unit balls in $A^{p,\alpha}(B)$ and $A^{q,\alpha}(B)$, respectively. We claim that $V\cdot W$ is closed in $A^{l,\alpha}(B)$. Let $g_m \in V$, $h_m \in W$ such that $g_m \cdot h_m \to f$ in $A^{l,\alpha}(B)$. By Fact 1 (of §2) we may assume, without loss of generality, that $g_m \to g$ and $h_m \to h$ uniformly on compact subsets of B. By Fatou's Lemma, $g \in V$ and $h \in W$. Since $g_m \cdot h_m \to f$ uniformly on compact subsets of B, $f = g\cdot h \in V\cdot W$. Hence the claim. We have $A^{p,\alpha}(B)\cdot A^{q,\alpha}(B) = \bigcup_{m=1}^{\infty}(mV\cdot W)$. We show that $mV\cdot W$ has empty interior in $A^{l,\alpha}(B)$ for each $m \geq 1$. Assume the contrary. Then some $mV\cdot W$ will have an interior point in $A^{l,\alpha}(B)$. There exist an $R \in A^{l,\alpha}(B)$ and a constant C such that

$$(10)\qquad \begin{cases} \int_B |R-F|^l\,d\mu_\alpha \leq 4^{n+1+\alpha},\ F \in A^{l,\alpha}(B), \text{ implies} \\ F = g\cdot h \text{ with } \|g\|_{p,\alpha} \leq C \text{ and } \|h\|_{q,\alpha} \leq C. \end{cases}$$

By Fact 2 (of §2), we may assume that R is a function in $A(B)$ vanishing at e_1. Now by Lemma 3, for the constant C there is an $f \in A^{l,\alpha}(B)$ such that $\|f\|_{l,\alpha} \leq 1$ and $f = g\cdot h$, $g \in A^{p,\alpha}(B)$, $h \in A^{q,\alpha}(B)$ imply that at least one of $\|g\|_{p,\alpha}$, $\|h\|_{q,\alpha}$ is larger than C. There is an $\varepsilon > 0$ such that

$$(11)\qquad \begin{cases} \|f-f_1\|_{l,\alpha} \leq \varepsilon \text{ and } f_1 = g_1\cdot h_1 \in A^{p,\alpha}(B)\cdot A^{q,\alpha}(B) \\ \quad\text{implies either } \|g_1\|_{p,\alpha} > C \text{ or } \|h_1\|_{q,\alpha} > C. \end{cases}$$

We may assume, after Fact 2, that f is a function in $A(B)$ vanishing at e_1.

We now come to perhaps the most important single step in the proof (see **[5]**). Let $F(z) = f(\phi_a(z))\psi_a^{1/l}(z) + R(z)$. ($\phi_a(z)$ is defined in §2 and $\psi_a(z)$ is defined in Lemma 5.) Now

$$\begin{aligned}\int_B |F-R|^l\,d\mu_\alpha &= \int |f(\phi_a(z))|^l\,|\psi_a(z)|\,d\mu_\alpha \\ &\leq 2^{2(n+\alpha)+1}\left[\int_B |f(\phi_a)|^l\,d\mu_\alpha + \int_B |f(\phi_a(z))|^l K(a,z)\,d\mu_a(z)\right]\end{aligned}$$

by Lemma 5. The second integral in the above inequality is $\int_B |f|^l\,d\mu_\alpha$ by (ii) of Lemma 4 and the first integral goes to zero as $a \to e_1$, by (iii) of Lemma 4 (recall that $f(e_1)=0$). Hence when a is close to e_1,

$$\begin{aligned}\int_B |F-R|^l\,d\mu_\alpha &\leqslant 2^{2(n+\alpha)+1}\left[1+\int_B |f|^l\,d\mu_\alpha\right]\\ &\leqslant 2^{2(n+\alpha)+1}[1+1] \quad (\text{since } \|f\|_{l,\alpha}\leqslant 1)\\ &= 4^{n+\alpha+1}.\end{aligned}$$

By (10), $F=g\cdot h$ with $\|g\|_{p,\alpha}\leqslant C$ and $\|h\|_{q,\alpha}\leqslant C$. Therefore $f(\phi_a(z))\cdot\psi_a^{1/l}(z)+R(z)=g(z)\cdot h(z)$. Replacing z by $\phi_a(z)$ and using $\phi_a(\phi_a(z))=z$, we get

$$f(z)+\frac{R(\phi_a(z))}{\psi_a^{1/l}(\phi_a(z))}=\frac{g(\phi_a(z))\cdot h(\phi_a(z))}{\psi_a^{1/l}(\phi_a(z))}=\frac{g(\phi_a(z))}{\psi_a^{1/p}(\phi_a(z))}\cdot\frac{h(\phi_a(z))}{\psi_a^{1/q}(\phi_a(z))}.$$

We have

$$\int_B\left|\frac{R(\phi_a)}{\psi_a^{1/l}(\phi_a)}\right|^l d\mu_\alpha\leqslant\int_B|R(\phi_a)|^l\,d\mu_\alpha$$

by Lemma 5. Since $R(e_1)=0$, the right side integral in the above inequality goes to zero as $a\to e_1$ by (iii) of Lemma 4. Hence if a is close to e_1, (11) holds with $f_1=g_1\cdot h_1$ where

$$g_1=g(\phi_a)/\psi_a^{1/p}(\phi_a)\quad\text{and}\quad h_1=h(\phi_a)/\psi_a^{1/q}(\phi_a).$$

Therefore either $\|g_1\|_{p,\alpha}>C$ or $\|h_1\|_{q,\alpha}>C$. Suppose $\|g_1\|_{p,\alpha}>C$. Then

$$\begin{aligned}C^p &< \int_B\left|\frac{g(\phi_a)}{\psi_a^{1/p}(\phi_a)}\right|^p d\mu_\alpha\\ &\leqslant \int_B\frac{|g(\phi_a(z))|^p}{K(a,\phi_a(z))}\,d\mu_\alpha(z) \quad (\text{by Lemma 5})\\ &= \int_B |g(\phi_a(z))|^p K(a,z)\,d\mu_\alpha(z) \quad (\text{by (i) of Lemma 4})\\ &= \int_B |g|^p\,d\mu_\alpha \quad (\text{by (ii) of Lemma 4})\\ &\leqslant C^p \quad (\text{since } \|g\|_{p,\alpha}\leqslant C).\end{aligned}$$

We reach a contradiction. Similarly $\|h_1\|_{q,\alpha}>C$ gives a contradiction. Hence all $m(V\cdot W)$ have empty interiors. So

$$A^{p,\alpha}(B)\cdot A^{q,\alpha}(B)=\bigcup_{m=1}^{\infty} m(V\cdot W)$$

is of first category in $A^{l,\alpha}(B)$.

5. Other results. Here is a nonfactorization theorem for Hardy spaces.

THEOREM 2. *Let* $n>1$ *and* $0<p,q,l<\infty$. *If* $1/l=1/p+1/q$ *then* $H^p(B)\cdot H^q(B)$ *is of first category in* $H^l(B)$.

The proof of this theorem is very similar to that of Theorem 1. One has to integrate functions in the Hardy class $H^t(B)$ (for $t = p$, q and l) with respect to $d\sigma$ over S. α should be replaced by -1 (relation (1) can also be used at appropriate places). We omit the details. Theorem 2 can also be proved, for $n > 2$, using Theorem 1 (with $\alpha = 0$) and Theorem 7.2.4 in [7].

REMARK 4. Let T be the mapping $(f_1, g_1, f_2, g_2, \dots, f_k, g_k) \to \sum_{i=1}^k f_i g_i$. The proof of Theorem 1 shows that

$$T\colon A^{p,\alpha}(B) \times A^{q,\alpha}(B) \times \cdots \times A^{p,\alpha}(B) \times A^{q,\alpha}(B) \to A^{l,\alpha}(B)$$

$(1/l = 1/p + 1/q)$ is onto if and only if it is open at the origin. Nonopenness of T at the origin would imply the existence of a function in $A^{l,\alpha}(B)$ which is not of the form $\sum_{i=1}^k f_i g_i$ with $f_i \in A^{p,\alpha}(B)$ and $g_i \in A^{q,\alpha}(B)$. However, any function F in $A^{1,\alpha}(B)$ (for $\alpha = 0, 1, 2 \dots$) can be written as $F = \sum_{i=1}^\infty G_i H_i$ where G_i and H_i belong to $A^{2,\alpha}(B)$ (see [**1**, Theorem IV]). Similar statements can be made for Hardy spaces.

REMARK 5. Let $0 < t < \infty$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_i > -1$. Let $A^{t,\alpha}(U^n)$ be the space of all holomorphic functions f satisfying $\| f \|_{t,\alpha} = (\int_{U^n} |f|^t \, d\mu_\alpha)^{1/t} < \infty$ where $d\mu_\alpha(z) = \prod_{i=1}^n (1 - |z_i|^2)^{\alpha_i} \, dm_i(z_i)$, $dm_i(z_i)$ being the Lebesgue measure on U for all $i = 1, 2, \dots, n$. Then *Theorem* 1 *holds for* U_n *in place of* B. We sketch a proof of this statement. If $K(z) = \sum_{i=N-1}^\infty K_i(z)$ is as in Lemma 2 then

$$\int_{-\pi}^{\pi} |K_N(r_1 e^{i\theta}, e^{i\theta} z')|^t \, d\theta \leqslant C_t \int_{-\pi}^{\pi} |K(r_1 e^{i\theta}, e^{i\theta} z')^t \, d\theta$$

and hence $\| K_N \|_{t,\alpha} \leqslant M_t \| K \|_{t,\alpha}$ where C_t and M_t are constants depending only on t. Without loss of generality let $\alpha_1 \geqslant \alpha_2$. We have

$$\| z_i^N \|_{t,\alpha}^t \sim N^{-(1+\alpha_i)} \qquad (i = 1, 2),$$

by Lemma 1 and using $F = z_1^{N-1} + z_2^N$ we get (imitating the proof of Lemma 3) the nonopenness of the product map from $A^{p,\alpha}(U^n) \times A^{q,\alpha}(U^n)$ to $A^{l,\alpha}(U^n)$ where $1/p + 1/q = 1/l$. (If

$$AK_N(z) = z_2^N - \left(H_1(z) \cdot z_1^{N-1}\right)/A$$

then

$$AK_N(r_1 e^{i\theta}, z') = z_2^N - \frac{H_1(r_1 e^{i\theta}, z')}{A} e^{i(N-1)\theta} r_1^{N-1}$$

and

$$\int_{-\pi}^{\pi} |z_2^N|^t \, d\theta \leqslant C_t |A|^t \int_{-\pi}^{\pi} |K_N(r_1 e^{i\theta}, z')|^t \, d\theta \quad \text{etc.})$$

For $0 < r < 1$, let

$$a = (r, 0, 0, \dots, 0), \qquad \phi_a(z) = ((r - z_1)/(1 - r z_1), z_2, z_3, \dots, z_n),$$
$$K(a, z) = \left((1 - r^2)/|1 - r z_1|^2\right)^{2+\alpha_1}$$

and

$$\phi_a(z) = \left(1 + \sqrt{1 - r^2}/(1 - r z_1)\right)^{2(2+\alpha_1)}.$$

We note that as $r \to 1$, $a \to (1, 0, \ldots, 0)$ and $\phi_a(z) \to (1, z')$. Observe that functions f in $A(U^n)$ with $f(1, z') \equiv 0$ form a dense subset of $A^{p,\alpha}(U^n)$. With minor changes, one can get results similar to Lemmas 4 and 5. By imitating the proof of Theorem 1, we get the polydisc version of Theorem 1.

REMARK 6. Let $H^t(U^n)$ be the Hardy space of all holomorphic functions f in U^n satisfying

$$\| f \|_{t,\sigma} = \left(\sup_{0 \le r < 1} \int_{T^n} |f(r\zeta)|^t \, d\sigma(\zeta) \right)^{1/t} < \infty$$

where T^n is the torus in $\mathbf{C}^n$ and $d\sigma$ is the normalized Haar measure on T^n.

Then *Theorem* 2 *holds for* U^n *in place of* B. Rosay [5] proved this for $p = q = 2$ and $l = 1$.

To sketch a proof, let $P = (z_1 + z_2)^N - z_1^N - N z_1^{N-1} z_2$. Then $\| P \|_{t,\sigma} / \| P \|_{l,\sigma} \to \infty$ as $N \to \infty$ whenever $t > l$. There exists a constant C_t such that if $AK_N = P(z) + z_1^{N-1} Q(z)$, where $Q(z)$ is any linear polynomial in z, then $\| P \|_{t,\sigma} \le C_t |A| \, \| K_N \|_{t,\sigma}$ (use subharmonicity in z_2). The function $f = (P + z_1^{N-1}) / \| P + z_1^{N-1} \|_{l,\sigma}$ gives the nonopenness of the product map. Changing α_1 to -1 and making other minor changes in the proof of Remark 5, we get Theorem 2 for U^n.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

Current address: Department of Mathematics, The University of Mississippi, University, Mississippi 38677