TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 277, Number 1, May 1983

NONFACTORIZATION THEOREMS IN WEIGHTED BERGMAN AND HARDY SPACES ON THE UNIT BALL OF $\mathbf{C}^n$ $(n > 1)$

BY

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ABSTRACT. Let $A^{p,\alpha}(B)$, $A^{q,\alpha}(B)$ and $A^{l,\alpha}(B)$ be weighted Bergman spaces on the unit ball of $\mathbf{C}^n$ $(n > 1)$. We prove:

THEOREM 1. *If* $1/l = 1/p + 1/q$ *then* $A^{p,\alpha}(B) \cdot A^{q,\alpha}(B)$ *is of first category in* $A^{l,\alpha}(B)$.

THEOREM 2. *Theorem* 1 *holds for Hardy spaces in place of weighted Bergman spaces.*

We also show that Theorems 1 and 2 hold for the polydisc U^n in place of B.

1. Introduction. Let U be the unit disc in $\mathbf{C}$. For $0 < t < \infty$ and $-1 < \alpha < \infty$, let $H^t(U)$ be the Hardy space of all holomorphic functions f on U satisfying

$$\sup_{0 \leq r < 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^t \, d\theta < \infty,$$

and let $A^{t,\alpha}(U)$ be the weighted Bergman space of all holomorphic functions f on U satisfying

$$\int_U |f(z)|^t \left(1 - |z|^2\right)^\alpha dm(z) < \infty,$$

where $dm(z)$ denotes the Lebesgue measure on U. If $0 < p, q, l < \infty$ and $1/p + 1/q = 1/l$, then it is well known that $H^p(U) \cdot H^q(U) = H^l(U)$, where the left-hand side consists of all products of the form $f \cdot g$ with $f \in H^p(U)$ and $g \in H^q(U)$. Horowitz [**3**] proved that $A^{p,\alpha}(U) \cdot A^{q,\alpha}(U) = A^{l,\alpha}(U)$ whenever $\alpha \geq 0$ and $1/p + 1/q = 1/l$.

In $\mathbf{C}^n$ $(n > 1)$, the above results are no longer valid. Rudin [**6**] and Miles [**4**] showed that $H^2(U^n) \cdot H^2(U^n)$ is a proper subset of $H^1(U^n)$ for $n \geq 3$. (Here U^n denotes the unit polydisc in $\mathbf{C}^n$.) Rosay [**5**] showed that $H^2(U^n) \cdot H^2(U^n)$ is of first category in $H^1(U^n)$ for $n \geq 2$, thereby completely solving the Factorization Problem (see [**6**, 4.2]) in Hardy spaces of the polydisc. In [**7**, Problem 19.3.1], Rudin asked whether $H^2(B) \cdot H^2(B)$ is properly contained in $H^1(B)$, where B denotes the unit ball of $\mathbf{C}^n$ $(n > 1)$. In this paper we show that $H^p(B) \cdot H^q(B)$ is of first category in $H^l(B)$ whenever $0 < p, q, l < \infty$ and $1/p + 1/q = 1/l$. We prove a similar result

Received by the editors March 10, 1982.

1980 *Mathematics Subject Classification*. Primary 32A35.

Key words and phrases. Weighted Bergman space, Hardy space, unit ball, unit polydisc.

[1]This research was partially supported by a grant from the National Science Foundation at the University of Wisconsin, Madison. This paper was presented to the American Mathematical Society at the 798th meeting held in Baton Rouge, Louisiana, November 12–13, 1982. This paper forms a part of the author's doctoral thesis, done under the supervision of Professor Walter Rudin and submitted to University of Wisconsin-Madison.

0002-9947/82/0000-0593/$03.25

(Theorem 1) for the weighted Bergman spaces on the unit ball B (see §2 for notations and terminology). Essential ideas required to prove these results come from Rosay [5].

Coifman, Rochberg and Weiss [1] proved that any function in $H^1(B)$ is an infinite sum of the form $\sum_{i=1}^{\infty} f_i g_i$ where f_i and g_i belong to $H^2(B)$ for all i. We do not know if the infinite sum can be replaced by a finite sum (see Remark 4).

2. Preliminaries. Notations are as in [7]. For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in $\mathbf{C}^n$, let $\langle z, w\rangle = \sum_{i=1}^n z_i \bar{w}_i$ and $|z|^2 = \sum_{i=1}^n |z_i|^2$; let $B = B_n = \{z \in \mathbf{C}^n : |z| < 1\}$ and $S = \{z \in \mathbf{C}^n : |z| = 1\}$. For $z \in \mathbf{C}^n$ we sometimes write $z = (z_1, z')$ where $z' = (z_2, z_3, \ldots, z_n)$. $e_1 = (1, 0, 0, \ldots, 0)$.

Let a, $z \in B$ and $a \neq 0$, let

$$\phi_a(z) = \frac{a - P_a z - \left(1 - |a|^2\right)^{1/2} Q_a z}{1 - \langle z, a\rangle}$$

where $P_a z = \langle z, a\rangle a / \langle a, a\rangle$ and $Q_a z = z - P_a z$. $\phi_a(z)$ is a holomorphic automorphism of B satisfying $\phi_a(\phi_a(z)) = z$.

$d\sigma$ denotes the rotation invariant probability measure on S. $d\nu(z) = d\nu_n(z) = 2nr^{2n-1}\, dr\, d\sigma(\zeta)$ is the normalized Lebesgue measure on B. Here $z = r\zeta$, $r = |z|$ and $\zeta \in S$.

$H(B)$ denotes the space of all holomorphic functions on B.

$C(\bar{B})$ denotes the space of all continuous functions on $\bar{B}$.

$A(B) = H(B) \cap C(\bar{B})$ is the ball algebra.

For $0 < t < \infty$, $H^t(B)$ is the Hardy space of all $f \in H(B)$ satisfying

$$\| f \|_{t,\sigma} = \left(\sup_{0 \leq r < 1} \int_S |f(r\zeta)|^t \, d\sigma(\zeta) \right)^{1/t} < \infty.$$

Let

$$d\mu_\alpha(z) = \left(1 - |z|^2\right)^\alpha d\nu(z) / nB(n, \alpha + 1)$$

where $-1 < \alpha < \infty$ and $B(n, \alpha + 1)$ denotes the Beta function. For $-1 < \alpha < \infty$ and $0 < t < \infty$, we write $A^{t,\alpha}(B)$ to denote the space of all $f \in H(B)$ satisfying

$$\| f \|_{t,\alpha} = \left(\int_B |f|^t \, d\mu_\alpha \right)^{1/t} < \infty.$$

We note that $d\mu_\alpha$ is a probability measure on B and

$$\lim_{\alpha \to -1} \int_B f(z)\, d\mu_\alpha(z) = \int_S f(\zeta)\, d\sigma(\zeta) \tag{1}$$

for all $f \in C(\bar{B})$. (The above relation holds for monomials $z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n} \cdot \bar{z}_1^{\gamma_1} \cdots \bar{z}_n^{\gamma_n}$ and hence for linear combinations of monomials. The Stone-Weierstrass theorem proves (1) for any $f \in C(\bar{B})$.)

Because of (1) we can think of $H^p(B)$ as a "limiting" case of $A^{p,\alpha}(B)$ for $\alpha = -1$. Let $f \in H(B)$. Then from [7, Theorem 7.2.5],

$$|f(z)|^p \left(1 - \frac{|z|}{r}\right)^n \leqslant 2^n \int_S |f(r\zeta)|^p \, d\sigma(\zeta)$$

for $|z| < r < 1$ and $0 < p < \infty$.

Let K be a compact subset of B. If we multiply the above inequality by $(1 - r^2)^\alpha r^{2n-1}\, dr$ and integrate over the interval $(1 + |z|)/2 < r < 1$, we get

$$|f(z)| \leqslant C_{n,\alpha,p,K} \|f\|_{p,\alpha} (1 - |z|)^{-n/p} \quad (\forall z \in K)$$

where $C_{n,\alpha,p,K}$ is a constant depending only on its subscripts.

The above two inequalities, together with a normality argument, give

Fact 1. *Every bounded sequence in $H^p(B)$ (or in $A^{p,\alpha}(B)$) has a subsequence which converges uniformly on compact subsets of B.*

From this it follows that $A^{p,\alpha}(B)$ and $H^p(B)$ are F-spaces.

Let $f \in H^p(B)$ $(A^{p,\alpha}(B))$ and $f_r(z) = f(rz)$ for $0 < r < 1$. Then $f_r \to f$ in $H^p(B)$ (in $A^{p,\alpha}(B)$) as $r \nearrow 1$. For a suitable r and δ $(0 < \delta < 1)$, $(1 - z_1) f_r(z)/(1 - \delta z_1)$ is close to f in $H^p(B)$ (in $A^{p,\alpha}(B)$) and vanishes at e_1. Hence we have

Fact 2. *The set of all $f \in A(B)$, $f(e_1) = 0$, is dense in $H^p(B)$ $(A^{p,\alpha}(B))$.*

We need the following identities [7, Proposition 1.4.7]:

$$\int_S f(\zeta)\, d\sigma(\zeta) = \int_S \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}\zeta)\, d\theta \right) d\sigma(\zeta), \tag{2}$$

$$\int_S f(\zeta_1, \zeta')\, d\sigma(\zeta) = \int_{B_{n-1}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}\zeta_1, \zeta')\, d\theta \right) d\nu_{n-1}(\zeta'). \tag{3}$$

3. Lemmas.

LEMMA 1. *Let $0 < t < \infty$ and $\alpha > -1$. Then $\|z_2^N\|_{t,\alpha}^t \sim N^{-(n+\alpha)}$ as $N \to \infty$.*

PROOF. We have

$$\begin{aligned} \|z_2^N\|_{t,\alpha}^t &= \frac{2}{B(n, \alpha+1)} \int_B |(r\zeta_2)^N|^t (1 - r^2)^\alpha r^{2n-1}\, dr\, d\sigma(\zeta) \\ &= \left(\int_S |\zeta_2^N|^t\, d\sigma(\zeta) \right) \left(\frac{2}{B(n, \alpha+1)} \int_0^1 r^{Nt+2n-1} (1 - r^2)^\alpha\, dr \right). \end{aligned}$$

The second integral, on putting $u = r^2$ becomes $\frac{1}{2} B(Nt/2 + n, \alpha + 1)$. By Stirling's formula this behaves like $1/N^{\alpha+1}$ as $N \to \infty$. For the first integral, we use the identity [7, 1.4.5, p. 15]

$$\int_S f(\langle \zeta, \eta \rangle)\, d\sigma(\zeta) = \frac{n-1}{\pi} \iint_U (1 - r^2)^{n-2} f(re^{i\theta}) r\, dr\, d\theta.$$

We get

$$\begin{aligned} \int_S |\zeta_2|^{Nt}\, d\sigma(\zeta) &= \frac{n-1}{\pi} \int_0^1 (1 - r^2)^{n-2} r^{Nt+1}\, dr = \frac{n-1}{\pi} B\left(\frac{Nt+1}{2}, n - 1 \right) \\ &\sim 1/N^{n-1} \quad \text{by Stirling's formula.} \end{aligned}$$

Hence

$$\|z_2^N\|_{t,\alpha}^t \sim 1/N^{n-1} \cdot 1/N^{\alpha+1} = N^{-(n+\alpha)}.$$

REMARK 1. $\|z_2^N\|_{t,\sigma}^t \sim N^{-(n-1)}$.

LEMMA 2. *Let* $K(z) = \sum_{i=N-1}^{\infty} K_i(z)$ *be holomorphic in* B, *where* $K_i(z)$ *is a homogeneous polynomial of degree* i *and* N *is a positive integer. Then for* $0 < t < \infty$, *there exists a constant* M (*depending only on* t) *such that*

$$\|K_N\|_{t,\sigma} \leqslant M \cdot \|K\|_{t,\sigma}, \tag{4}$$

$$\|K_N\|_{t,\alpha} \leqslant M \cdot \|K\|_{t,\alpha}. \tag{4'}$$

PROOF. For $0 < t < \infty$, there exists an M such that if $G(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots$ is in the disc algebra $A(U)$ then

$$|a_1|^t \leqslant M^t \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{i\theta})|^t \, d\theta.$$

(For $t \geqslant 1$ we can take $M = 1$. For $0 < t < 1$, see [**2**, Theorem 6.4, p. 98]. In fact, $M = 2^{1/t}$ works for any t.) Now for a fixed z, let

$$G(\lambda) = K(\lambda z)/\lambda^{N-1} = K_{N-1}(z) + \lambda K_N(z) + \cdots.$$

We get

$$|K_N(z)|^t \leqslant M^t \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} |K(e^{i\theta}z)|^t \, d\theta.$$

We let $z = r\zeta$, integrate both sides with respect to $d\sigma(\zeta)$ and use (2) to get

$$\int_S |K_N(r\zeta)|^t \, d\sigma(\zeta) \leqslant M^t \int_S |K(r\zeta)|^t \, d\sigma(\zeta). \tag{5}$$

Taking the supremum over r in the interval $0 < r < 1$ and tth roots, we get (4). To get (4′), we multiply both sides of (5) by $(2/B(n, \alpha + 1))\, r^{2n-1}(1 - r^2)^\alpha \, dr$, integrate over $0 < r < 1$ and take tth roots.

LEMMA 3. *Let* $0 < p, q, l < \infty$, $1/l = 1/p + 1/q$, $-1 < \alpha < \infty$ *and* $n > 1$. *Then the product map* $(h, k) \to h \cdot k$ *from* $A^{p,\alpha}(B) \times A^{q,\alpha}(B)$ *to* $A^{l,\alpha}(B)$ *is not open at the origin, i.e., for any constant* $C > 0$, *there exists* $f \in A^{l,\alpha}(B)$ *such that* $\|f\|_{l,\alpha} \leqslant 1$ *and if* $f = h \cdot k$ *with* $h \in A^{p,\alpha}(B)$, $k \in A^{q,\alpha}(B)$ *then at least one of* $\|h\|_{p,\alpha}$, $\|k\|_{q,\alpha}$ *is larger than* C.

PROOF. Let $F(z) = z_1^{N-1} + z_2^N$, $N > 1$. Suppose $F(z) = H(z) \cdot K(z)$ with H and K holomorphic in B. We expand $H(z)$ and $K(z)$ in terms of homogeneous polynomials: $H = H_i + H_{i+1} + \cdots$, $K = K_{N-1-i} + K_{N-i} + \cdots$. Here, as usual, subscript refers to the degree, $H_i \not\equiv 0$ and $K_{N-1-i} \not\equiv 0$. From $F = H \cdot K$ we get, by comparing degrees,

$$H_i \cdot K_{N-1-i} = z_1^{N-1} \tag{6}$$

and

$$H_i K_{N-i} + H_{i+1} K_{N-1-i} = z_2^N. \tag{7}$$

From (6) and (7) we get $i = 0$ or $N - 1$. *We assume for a moment that* $i = 0$. Then H_0 is a constant, say A. We have from (6) and (7),

$$AK_N(z) = z_2^N - (H_1(z) \cdot z_1^{N-1})/A.$$

Letting $z = r(e^{i\theta}\zeta_1, \zeta')$ we get

$$AK_N(e^{i\theta}\zeta_1, \zeta') = \zeta_2^N - A^{-1}H_1(e^{i\theta}\zeta_1, \zeta')e^{i(N-1)\theta}\zeta_1^{N-1}.$$

Therefore, ζ_2^N is the constant term in the polynomial $AK_N(\lambda\zeta_1, \zeta')$ in λ. By subharmonicity,

$$|\zeta_2^N|^t \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |AK_N(e^{i\theta}\zeta_1, \zeta')|^t \, d\theta \quad \text{for } 0 < t < \infty. \tag{8}$$

Now we multiply both sides of (8) by $d\nu_{n-1}(\zeta')$ and integrate over B_{n-1}. Using (3), we get

$$\int_S |\zeta_2^N|^t \, d\sigma(\zeta) \leqslant \int_S |AK_N(\zeta)|^t \, d\sigma(\zeta)$$

and

$$\int_S |(r\zeta_2)^N|^t \, d\sigma(\zeta) \leqslant \int_S |AK_N(r\zeta)|^t \, d\sigma(\zeta). \tag{9}$$

We multiply both sides of (9) by $(2/B(n, \alpha + 1))r^{2n-1}(1 - r^2)^\alpha \, dr$, integrate over $0 < r < 1$ and take tth roots to get

$$\|z_2^N\|_{t,\alpha} \leqslant |A| \, \|K_N\|_{t,\alpha}.$$

Since $|H(z)|^t$ is subharmonic and $A = H(0)$, we have $|A|^t \leqslant \int_S |H(r\zeta)|^t \, d\sigma(\zeta)$. From this we get $|A| \leqslant \|H\|_{t,\alpha}$. Hence

$$\|z_2^N\|_{t,\alpha} \leqslant |A| \, \|K_N\|_{t,\alpha} \leqslant \|H\|_{t,\alpha} \|K_N\|_{t,\alpha}.$$

Using Lemma 2, we get $\|z_2^N\|_{t,\alpha} \leqslant M \|H\|_{t,\alpha} \|K\|_{t,\alpha}$. *By symmetry, this inequality holds when* $i = N - 1$. Now let $f = F/\|F\|_{l,\alpha}$. Then $\|f\|_{l,\alpha} = 1$. Suppose $f = h \cdot k$ where $h \in A^{p,\alpha}(B)$ and $k \in A^{q,\alpha}(B)$. Then $F = H \cdot K$ where $H = \|F\|_{l,\alpha} h$ and $K = k$. Therefore

$$\|z_2^N\|_{t,\alpha} \leqslant M \|H\|_{t,\alpha} \cdot \|K\|_{t,\alpha} \leqslant M \|F\|_{l,\alpha} \|h\|_{t,\alpha} \cdot \|k\|_{t,\alpha}.$$

Now we take $t = \min(p, q)$. Then $\|h\|_{t,\alpha} \leqslant \|h\|_{p,\alpha}$ and $\|k\|_{t,\alpha} \leqslant \|k\|_{q,\alpha}$. We have

$$\|F\|_{l,\alpha}^l = \int_B |z_1^{N-1} + z_2^N|^l \, d\mu_\alpha \leqslant 2^l[\|z_1^{N-1}\|_{l,\alpha}^l + \|z_2^N\|_{l,\alpha}^l].$$

By Lemma 1, the right side of the above inequality is like $N^{-(n+\alpha)}$ for large N. We see that

$$\|h\|_{p,\alpha} \cdot \|k\|_{q,\alpha} \geqslant \|z_2^N\|_{t,\alpha}/M \|F\|_{l,\alpha}.$$

Hence $\|h\|_{p,\alpha} \|k\|_{q,\alpha}$ is bigger than a constant times $N^{-(n+\alpha)(1/t - 1/l)}$ which goes to ∞ as $N \to \infty$ (recall $t = \min(p, q) > l$). Therefore, for any constant C, we can find a large N so that $\|h\|_{p,\alpha} \cdot \|k\|_{q,\alpha} > C^2$. This completes the proof.

REMARK 2. By considering H^p-norms instead of $A^{p,\alpha}$-norms, one can get the nonopenness of the product map (at the origin) for H^p-spaces.

LEMMA 4. *For $a \in B$ and $z \in \bar{B}$, let*

$$K(a, z) = \left[(1 - |a|^2)/|1 - \langle z, a\rangle|^2\right]^{(n+1+\alpha)}.$$

Then:

(i) $K(a, \phi_a(z)) \cdot K(a, z) = 1$.

(ii) $\int_B f(\omega)\, d\mu_\alpha(\omega) = \int_B f(\phi_a(z)) K(a, z)\, d\mu_\alpha(z)$ *for all* $f \in C(\bar{B})$.

(iii) $\int_B f(\phi_a(z))\, d\mu_\alpha(z) \to f(e_1)$ *as* $a \to e_1$ *for all* $f \in C(\bar{B})$.

PROOF. From [7, Theorem 2.2.5], we have

$$1 - \langle \phi_a(z), a\rangle = (1 - |a|^2)/(1 - \langle z, a\rangle).$$

Taking absolute values and using the definition of K, we get (i). From [7, Theorem 2.2.6] we have

$$\int_B f(\omega)(1 - |\omega|^2)^\alpha\, d\nu(\omega)$$
$$= \int_B f(\phi_a(z))(1 - |\phi_a(z)|^2)^\alpha \left(\frac{1 - |a|^2}{|1 - \langle z, a\rangle|^2}\right)^{n+1} d\nu(z).$$

Using

$$1 - |\phi_a(z)|^2 = (1 - |a|^2)(1 - |z|^2)/|1 - \langle z, a\rangle|^2 \quad \text{(see [7, Theorem 2.2.5])},$$

$$\int_B f(\omega)(1 - |\omega|^2)^\alpha\, d\nu(\omega) = \int_B f(\phi_a(z)) \frac{(1 - |a|^2)^{n+1+\alpha}(1 - |z|^2)^\alpha}{|1 - \langle z, a\rangle|^{2(n+1+\alpha)}}\, d\nu(z).$$

Hence

$$\int_B f(\omega)\, d\mu_\alpha(\omega) = \int_B f(\phi_a(z)) K(a, z)\, d\mu_\alpha(z).$$

This is (ii). Since $\lim_{a \to e_1} \phi_a(z) = e_1$, an application of the Bounded Convergence Theorem gives (iii).

REMARK 3. In the above lemma we assumed that $\alpha > -1$. The following statements hold when $\alpha = -1$.

(i) $K(a, \phi_a(z)) \cdot K(a, z) = 1$.

(ii) $\int_S f(\eta)\, d\sigma(\eta) = \int_S f(\phi_a(\zeta)) K(a, \zeta)\, d\sigma(\zeta)$ for all $f \in C(S)$.

(iii) $\int_S f(\phi_a(\zeta))\, d\sigma(\zeta) \to f(e_1)$ as $a \to e_1$ for all $f \in C(S)$.

We observe that when $\alpha = -1$, $K(a, z)$ is the Poisson kernel and statements (i) and (ii) are well known. Since $\int_S f(\phi_a(\zeta))\, d\sigma(\zeta)$ is the Poisson integral of f, (iii) follows (see, e.g., [7, Theorem 3.3.4(a)]).

LEMMA 5. *Let*

$$\psi_a(z) = \left[1 + \sqrt{1 - |a|^2}/(1 - \langle z, a\rangle)\right]^{2(n+1+\alpha)}.$$

Then

$$\max\{1, K(a, z)\} \leqslant |\psi_a(z)| \leqslant 2^{2(n+\alpha)+1}\{1 + K(a, z)\}$$

for all $z \in \bar{B}$, $a \in B$ *and* $\alpha \geqslant -1$.

PROOF. For $\lambda \in \mathbf{C}$ and $\operatorname{Re}\lambda \geqslant 0$, we have $\max\{1, |\lambda|\} \leqslant |1+\lambda|$. This can be seen by plotting λ and $1+\lambda$ in the complex plane. Also, $|1+\lambda|^m \leqslant (1+|\lambda|)^m \leqslant 2^{m-1}(1+|\lambda|^m)$ for $m \geqslant 1$. Taking $\lambda = \sqrt{1-|a|^2}/(1-\langle z, a\rangle)$ and $m = 2(n+1+\alpha)$, we get the lemma.

4. Main theorem.

THEOREM 1. *Let $n > 1$, $-1 < \alpha < \infty$, $0 < p, q, l < \infty$ and $1/l = 1/p + 1/q$. Then $A^{p,\alpha}(B) \cdot A^{q,\alpha}(B)$ is of first category in $A^{l,\alpha}(B)$.*

PROOF. Let V and W be the closed unit balls in $A^{p,\alpha}(B)$ and $A^{q,\alpha}(B)$, respectively. We claim that $V \cdot W$ is closed in $A^{l,\alpha}(B)$. Let $g_m \in V$, $h_m \in W$ such that $g_m \cdot h_m \to f$ in $A^{l,\alpha}(B)$. By Fact 1 (of §2) we may assume, without loss of generality, that $g_m \to g$ and $h_m \to h$ uniformly on compact subsets of B. By Fatou's Lemma, $g \in V$ and $h \in W$. Since $g_m \cdot h_m \to f$ uniformly on compact subsets of B, $f = g \cdot h \in V \cdot W$. Hence the claim. We have $A^{p,\alpha}(B) \cdot A^{q,\alpha}(B) = \bigcup_{m=1}^{\infty}(mV \cdot W)$. We show that $mV \cdot W$ has empty interior in $A^{l,\alpha}(B)$ for each $m \geqslant 1$. Assume the contrary. Then some $mV \cdot W$ will have an interior point in $A^{l,\alpha}(B)$. There exist an $R \in A^{l,\alpha}(B)$ and a constant C such that

$$\tag{10} \begin{cases} \int_B |R - F|^l \, d\mu_\alpha \leqslant 4^{n+1+\alpha}, F \in A^{l,\alpha}(B), \text{implies} \\ F = g \cdot h \text{ with } \|g\|_{p,\alpha} \leqslant C \text{ and } \|h\|_{q,\alpha} \leqslant C. \end{cases}$$

By Fact 2 (of §2), we may assume that R is a function in $A(B)$ vanishing at e_1. Now by Lemma 3, for the constant C there is an $f \in A^{l,\alpha}(B)$ such that $\|f\|_{l,\alpha} \leqslant 1$ and $f = g \cdot h$, $g \in A^{p,\alpha}(B)$, $h \in A^{q,\alpha}(B)$ imply that at least one of $\|g\|_{p,\alpha}$, $\|h\|_{q,\alpha}$ is larger than C. There is an $\varepsilon > 0$ such that

$$\tag{11} \begin{cases} \|f - f_1\|_{l,\alpha} \leqslant \varepsilon \text{ and } f_1 = g_1 \cdot h_1 \in A^{p,\alpha}(B) \cdot A^{q,\alpha}(B) \\ \quad \text{implies either } \|g_1\|_{p,\alpha} > C \text{ or } \|h_1\|_{q,\alpha} > C. \end{cases}$$

We may assume, after Fact 2, that f is a function in $A(B)$ vanishing at e_1.

We now come to perhaps the most important single step in the proof (see **[5]**). Let $F(z) = f(\phi_a(z))\psi_a^{1/l}(z) + R(z)$. ($\phi_a(z)$ is defined in §2 and $\psi_a(z)$ is defined in Lemma 5.) Now

$$\begin{aligned} \int_B |F - R|^l \, d\mu_\alpha &= \int |f(\phi_a(z))|^l |\psi_a(z)| \, d\mu_\alpha \\ &\leqslant 2^{2(n+\alpha)+1}\left[\int_B |f(\phi_a)|^l \, d\mu_\alpha + \int_B |f(\phi_a(z))|^l K(a, z) \, d\mu_a(z)\right] \end{aligned}$$

by Lemma 5. The second integral in the above inequality is $\int_B |f|^l \, d\mu_\alpha$ by (ii) of Lemma 4 and the first integral goes to zero as $a \to e_1$, by (iii) of Lemma 4 (recall that $f(e_1) = 0$). Hence when a is close to e_1,

$$\int_B |F - R|^l \, d\mu_\alpha \leqslant 2^{2(n+\alpha)+1}\left[1 + \int_B |f|^l \, d\mu_\alpha\right]$$
$$\leqslant 2^{2(n+\alpha)+1}[1 + 1] \quad (\text{since } \|f\|_{l,\alpha} \leqslant 1)$$
$$= 4^{n+\alpha+1}.$$

By (10), $F = g \cdot h$ with $\|g\|_{p,\alpha} \leqslant C$ and $\|h\|_{q,\alpha} \leqslant C$. Therefore $f(\phi_a(z)) \cdot \psi_a^{1/l}(z) + R(z) = g(z) \cdot h(z)$. Replacing z by $\phi_a(z)$ and using $\phi_a(\phi_a(z)) = z$, we get

$$f(z) + \frac{R(\phi_a(z))}{\psi_a^{1/l}(\phi_a(z))} = \frac{g(\phi_a(z)) \cdot h(\phi_a(z))}{\psi_a^{1/l}(\phi_a(z))} = \frac{g(\phi_a(z))}{\psi_a^{1/p}(\phi_a(z))} \cdot \frac{h(\phi_a(z))}{\psi_a^{1/q}(\phi_a(z))}.$$

We have

$$\int_B \left| \frac{R(\phi_a)}{\psi_a^{1/l}(\phi_a)} \right|^l d\mu_\alpha \leqslant \int_B |R(\phi_a)|^l \, d\mu_\alpha$$

by Lemma 5. Since $R(e_1) = 0$, the right side integral in the above inequality goes to zero as $a \to e_1$ by (iii) of Lemma 4. Hence if a is close to e_1, (11) holds with $f_1 = g_1 \cdot h_1$ where

$$g_1 = g(\phi_a)/\psi_a^{1/p}(\phi_a) \quad \text{and} \quad h_1 = h(\phi_a)/\psi_a^{1/q}(\phi_a).$$

Therefore either $\|g_1\|_{p,\alpha} > C$ or $\|h_1\|_{q,\alpha} > C$. Suppose $\|g_1\|_{p,\alpha} > C$. Then

$$C^p < \int_B \left| \frac{g(\phi_a)}{\psi_a^{1/p}(\phi_a)} \right|^p d\mu_\alpha$$
$$\leqslant \int_B \frac{|g(\phi_a(z))|^p}{K(a, \phi_a(z))} \, d\mu_\alpha(z) \quad (\text{by Lemma 5})$$
$$= \int_B |g(\phi_a(z))|^p K(a, z) \, d\mu_\alpha(z) \quad (\text{by (i) of Lemma 4})$$
$$= \int_B |g|^p \, d\mu_\alpha \quad (\text{by (ii) of Lemma 4})$$
$$\leqslant C^p \quad (\text{since } \|g\|_{p,\alpha} \leqslant C).$$

We reach a contradiction. Similarly $\|h_1\|_{q,\alpha} > C$ gives a contradiction. Hence all $m(V \cdot W)$ have empty interiors. So

$$A^{p,\alpha}(B) \cdot A^{q,\alpha}(B) = \bigcup_{m=1}^{\infty} m(V \cdot W)$$

is of first category in $A^{l,\alpha}(B)$.

5. Other results. Here is a nonfactorization theorem for Hardy spaces.

THEOREM 2. *Let $n > 1$ and $0 < p, q, l < \infty$. If $1/l = 1/p + 1/q$ then $H^p(B) \cdot H^q(B)$ is of first category in $H^l(B)$.*

The proof of this theorem is very similar to that of Theorem 1. One has to integrate functions in the Hardy class $H^t(B)$ (for $t = p$, q and l) with respect to $d\sigma$ over S. α should be replaced by -1 (relation (1) can also be used at appropriate places). We omit the details. Theorem 2 can also be proved, for $n > 2$, using Theorem 1 (with $\alpha = 0$) and Theorem 7.2.4 in [7].

REMARK 4. Let T be the mapping $(f_1, g_1, f_2, g_2, \dots, f_k, g_k) \to \sum_{i=1}^k f_i g_i$. The proof of Theorem 1 shows that

$$T\colon A^{p,\alpha}(B) \times A^{q,\alpha}(B) \times \cdots \times A^{p,\alpha}(B) \times A^{q,\alpha}(B) \to A^{l,\alpha}(B)$$

$(1/l = 1/p + 1/q)$ is onto if and only if it is open at the origin. Nonopenness of T at the origin would imply the existence of a function in $A^{l,\alpha}(B)$ which is not of the form $\sum_{i=1}^k f_i g_i$ with $f_i \in A^{p,\alpha}(B)$ and $g_i \in A^{q,\alpha}(B)$. However, any function F in $A^{1,\alpha}(B)$ (for $\alpha = 0, 1, 2 \dots$) can be written as $F = \sum_{i=1}^\infty G_i H_i$ where G_i and H_i belong to $A^{2,\alpha}(B)$ (see [**1**, Theorem IV]). Similar statements can be made for Hardy spaces.

REMARK 5. Let $0 < t < \infty$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_i > -1$. Let $A^{t,\alpha}(U^n)$ be the space of all holomorphic functions f satisfying $\| f \|_{t,\alpha} = (\int_{U^n} |f|^t \, d\mu_\alpha)^{1/t} < \infty$ where $d\mu_\alpha(z) = \prod_{i=1}^n (1 - |z_i|^2)^{\alpha_i} dm_i(z_i)$, $dm_i(z_i)$ being the Lebesgue measure on U for all $i = 1, 2, \dots, n$. Then *Theorem* 1 *holds for* U_n *in place of* B. We sketch a proof of this statement. If $K(z) = \sum_{i=N-1}^\infty K_i(z)$ is as in Lemma 2 then

$$\int_{-\pi}^{\pi} |K_N(r_1 e^{i\theta}, e^{i\theta} z')|^t \, d\theta \leq C_t \int_{-\pi}^{\pi} |K(r_1 e^{i\theta}, e^{i\theta} z')^t \, d\theta$$

and hence $\| K_N \|_{t,\alpha} \leq M_t \| K \|_{t,\alpha}$ where C_t and M_t are constants depending only on t. Without loss of generality let $\alpha_1 \geq \alpha_2$. We have

$$\| z_i^N \|_{t,\alpha}^t \sim N^{-(1+\alpha_i)} \qquad (i = 1, 2),$$

by Lemma 1 and using $F = z_1^{N-1} + z_2^N$ we get (imitating the proof of Lemma 3) the nonopenness of the product map from $A^{p,\alpha}(U^n) \times A^{q,\alpha}(U^n)$ to $A^{l,\alpha}(U^n)$ where $1/p + 1/q = 1/l$. (If

$$AK_N(z) = z_2^N - \left(H_1(z) \cdot z_1^{N-1}\right)/A$$

then

$$AK_N(r_1 e^{i\theta}, z') = z_2^N - \frac{H_1(r_1 e^{i\theta}, z')}{A} e^{i(N-1)\theta} r_1^{N-1}$$

and

$$\int_{-\pi}^{\pi} |z_2^N|^t \, d\theta \leq C_t |A|^t \int_{-\pi}^{\pi} |K_N(r_1 e^{i\theta}, z')|^t \, d\theta \quad \text{etc.})$$

For $0 < r < 1$, let

$$a = (r, 0, 0, \dots, 0), \qquad \phi_a(z) = ((r - z_1)/(1 - rz_1), z_2, z_3, \dots, z_n),$$
$$K(a, z) = \left((1 - r^2)/|1 - rz_1|^2\right)^{2+\alpha_1}$$

and

$$\phi_a(z) = \left(1 + \sqrt{1 - r^2}/(1 - rz_1)\right)^{2(2+\alpha_1)}.$$

We note that as $r \to 1$, $a \to (1, 0, \ldots, 0)$ and $\phi_a(z) \to (1, z')$. Observe that functions f in $A(U^n)$ with $f(1, z') \equiv 0$ form a dense subset of $A^{p,\alpha}(U^n)$. With minor changes, one can get results similar to Lemmas 4 and 5. By imitating the proof of Theorem 1, we get the polydisc version of Theorem 1.

REMARK 6. Let $H^t(U^n)$ be the Hardy space of all holomorphic functions f in U^n satisfying

$$\|f\|_{t,\sigma} = \left(\sup_{0 \le r < 1} \int_{T^n} |f(r\zeta)|^t \, d\sigma(\zeta) \right)^{1/t} < \infty$$

where T^n is the torus in $\mathbf{C}^n$ and $d\sigma$ is the normalized Haar measure on T^n.

Then *Theorem* 2 *holds for* U^n *in place of* B. Rosay [5] proved this for $p = q = 2$ and $l = 1$.

To sketch a proof, let $P = (z_1 + z_2)^N - z_1^N - Nz_1^{N-1}z_2$. Then $\|P\|_{t,\sigma}/\|P\|_{l,\sigma} \to \infty$ as $N \to \infty$ whenever $t > l$. There exists a constant C_t such that if $AK_N = P(z) + z_1^{N-1}Q(z)$, where $Q(z)$ is any linear polynomial in z, then $\|P\|_{t,\sigma} \le C_t |A| \, \|K_N\|_{t,\sigma}$ (use subharmonicity in z_2). The function $f = (P + z_1^{N-1})/\|P + z_1^{N-1}\|_{l,\sigma}$ gives the nonopenness of the product map. Changing α_1 to -1 and making other minor changes in the proof of Remark 5, we get Theorem 2 for U^n.

ACKNOWLEDGEMENTS. I would like to thank Walter Rudin for suggesting the problem and for his invaluable suggestions during the preparation of this paper.

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