NONFACTORIZATION THEOREMS
IN WEIGHTED BERGMAN AND HARDY SPACES
ON THE UNIT BALL OF C^n (n > 1)

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ABSTRACT. Let \( A^{p,a}(B) \), \( A^{a,q}(B) \) and \( A^{l,q}(B) \) be weighted Bergman spaces on the unit ball of \( C^n \) (\( n > 1 \)). We prove:

THEOREM 1. If \( 1/l = 1/p + 1/q \) then \( A^{p,a}(B) \cdot A^{q,a}(B) \) is of first category in \( A^{l,q}(B) \).

THEOREM 2. Theorem 1 holds for Hardy spaces in place of weighted Bergman spaces.

We also show that Theorems 1 and 2 hold for the polydisc \( U^n \) in place of \( B \).

1. Introduction. Let \( U \) be the unit disc in \( C \). For \( 0 < r < \infty \) and \( -1 < \alpha < \infty \), let \( H^r(U) \) be the Hardy space of all holomorphic functions \( f \) on \( U \) satisfying

\[
\sup_{0 < r < 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^r \, d\theta < \infty,
\]

and let \( A^{l,a}(U) \) be the weighted Bergman space of all holomorphic functions \( f \) on \( U \) satisfying

\[
\int_{U} |f(z)|^r \left( 1 - |z|^2 \right)^\alpha \, dm(z) < \infty,
\]

where \( dm(z) \) denotes the Lebesgue measure on \( U \). If \( 0 < p, q, l < \infty \) and \( 1/p + 1/q = 1/l \), then it is well known that \( H^p(U) \cdot H^q(U) = H^l(U) \), where the left-hand side consists of all products of the form \( f \cdot g \) with \( f \in H^p(U) \) and \( g \in H^q(U) \). Horowitz [3] proved that \( A^{p,a}(U) \cdot A^{q,a}(U) = A^{l,a}(U) \) whenever \( \alpha > 0 \) and \( 1/p + 1/q = 1/l \).

In \( C^n \) (\( n > 1 \)), the above results are no longer valid. Rudin [6] and Miles [4] showed that \( H^2(U^n) \cdot H^2(U^n) \) is a proper subset of \( H^l(U^n) \) for \( n \geq 3 \). (Here \( U^n \) denotes the unit polydisc in \( C^n \).) Rosay [5] showed that \( H^2(U^n) \cdot H^2(U^n) \) is of first category in \( H^l(U^n) \) for \( n \geq 2 \), thereby completely solving the Factorization Problem (see [6, 4.2]) in Hardy spaces of the polydisc. In [7, Problem 19.3.1], Rudin asked whether \( H^2(B) \cdot H^2(B) \) is properly contained in \( H^l(B) \), where \( B \) denotes the unit ball of \( C^n \) (\( n > 1 \)). In this paper we show that \( H^p(B) \cdot H^q(B) \) is of first category in \( H^l(B) \) whenever \( 0 < p, q, l < \infty \) and \( 1/p + 1/q = 1/l \). We prove a similar result.
(Theorem 1) for the weighted Bergman spaces on the unit ball $B$ (see §2 for notations and terminology). Essential ideas required to prove these results come from Rosay [5].

Coifman, Rochberg and Weiss [1] proved that any function in $H^1(B)$ is an infinite sum of the form $\sum_{i=1}^{\infty} f_i g_i$ where $f_i$ and $g_i$ belong to $H^2(B)$ for all $i$. We do not know if the infinite sum can be replaced by a finite sum (see Remark 4).

2. Preliminaries. Notations are as in [7]. For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in $C^n$, let $\langle z, w \rangle = \sum_{i=1}^{n} z_i \bar{w}_i$ and $|z|^2 = \sum_{i=1}^{n} |z_i|^2$; let $B = B_n = \{z \in C^n: |z| < 1\}$ and $S = \{z \in C^n: |z| = 1\}$. For $z \in C^n$ we sometimes write $z = (z_1, z')$ where $z' = (z_2, z_3, \ldots, z_n)$, $e_1 = (1, 0, 0, \ldots, 0)$.

Let $a$, $z \in B$ and $a \neq 0$, let

$$\phi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{1/2} Q_a z}{1 - \langle z, a \rangle}$$

where $P_a z = \langle z, a \rangle a / \langle a, a \rangle$ and $Q_a z = z - P_a z$. $\phi_a(z)$ is a holomorphic automorphism of $B$ satisfying $\phi_a(\phi_a(z)) = z$.

$d\sigma$ denotes the rotation invariant probability measure on $S$. $d\nu(z) = d\nu(z) = 2n\pi^{n-1} dr d\sigma(\xi)$ is the normalized Lebesgue measure on $B$. Here $z = r \zeta$, $r = |z|$ and $\xi \in S$.

$H(B)$ denotes the space of all holomorphic functions on $B$.
$C(B)$ denotes the space of all continuous functions on $B$.
$A(B) = H(B) \cap C(B)$ is the ball algebra.

For $0 < t < \infty$, $H^t(B)$ is the Hardy space of all $f \in H(B)$ satisfying

$$\|f\|_{t,a} = \left( \sup_{0 < r < 1} \int_S |f(r \zeta)|^t d\sigma(\zeta) \right)^{1/t} < \infty.$$ 

Let

$$d\mu_a(z) = (1 - |z|^2)^{\alpha} d\nu(z) / nB(n, \alpha + 1)$$

where $-1 < \alpha < \infty$ and $B(n, \alpha + 1)$ denotes the Beta function. For $-1 < \alpha < \infty$ and $0 < t < \infty$, we write $A^{\alpha}(B)$ to denote the space of all $f \in H(B)$ satisfying

$$\|f\|_{t,a} = \left( \int_B |f|^t d\mu_a \right)^{1/t} < \infty.$$ 

We note that $d\mu_a$ is a probability measure on $B$ and

$$\lim_{\alpha \to -1} \int_B f(z) d\mu_a(z) = \int_S f(\xi) d\sigma(\xi)$$

for all $f \in C(B)$ (The above relation holds for monomials $z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n} \bar{z}_1^{\gamma_1} \cdots \bar{z}_n^{\gamma_n}$ and hence for linear combinations of monomials. The Stone-Weierstrass theorem proves (1) for any $f \in C(B)$.)
Because of (1) we can think of $H^p(B)$ as a “limiting” case of $A^{p,\alpha}(B)$ for $\alpha = -1$. Let $f \in H(B)$. Then from [7, Theorem 7.2.5],

$$|f(z)|^p \left(1 - \frac{|z|}{r}\right)^{\frac{n}{2}} \leq 2^n \int_S |f(r \xi)|^p \, d\sigma(\xi)$$

for $|z| < r < 1$ and $0 < p < \infty$.

Let $K$ be a compact subset of $B$. If we multiply the above inequality by $(1 - r^2)^\alpha r^{2n-1} \, dr$ and integrate over the interval $(1 + |z|)/2 < r < 1$, we get

$$|f(z)| \leq C_{n, \alpha, p, K} \|f\|_{p, \alpha}(1 - |z|)^{-n/p} \quad (\forall z \in K)$$

where $C_{n, \alpha, p, K}$ is a constant depending only on its subscripts.

The above two inequalities, together with a normality argument, give

**Fact 1.** Every bounded sequence in $H^p(B)$ (or in $A^{p,\alpha}(B)$) has a subsequence which converges uniformly on compact subsets of $B$.

From this it follows that $A^{p,\alpha}(B)$ and $H^p(B)$ are $F$-spaces.

Let $f \in H^p(B)$ ($A^{p,\alpha}(B)$) and $f_r(z) = f(rz)$ for $0 < r < 1$. Then $f_r \to f$ in $H^p(B)$ (in $A^{p,\alpha}(B)$) as $r \to 1$. For a suitable $r$ and $\delta$ ($0 < \delta < 1$), $(1 - z)/(1 - rz)$ is close to $f$ in $H^p(B)$ (in $A^{p,\alpha}(B)$) and vanishes at $e_1$. Hence we have

**Fact 2.** The set of all $f \in A(B)$, $f(e_1) = 0$, is dense in $H^p(B)$ ($A^{p,\alpha}(B)$).

We need the following identities [7, Proposition 1.4.7]:

$$\int_S f(\xi) \, d\sigma(\xi) = \frac{1}{2\pi} \int_0^{\pi} f(e^{i\theta} \xi) \, d\theta \, d\sigma(\xi),$$

$$\int_S f(\xi_1, \xi') \, d\sigma(\xi) = \frac{1}{2\pi} \int_0^{\pi} f(e^{i\theta} \xi_1, \xi') \, d\theta \, d\sigma(\xi).$$

3. Lemmas.

**Lemma 1.** Let $0 < t < \infty$ and $\alpha > -1$. Then $\|z_2^N\|_{t, \alpha}^t \sim N^{-(n+\alpha)}$ as $N \to \infty$.

**Proof.** We have

$$\|z_2^N\|_{t, \alpha}^t = \frac{2}{B(n, \alpha + 1)} \int_B |(r_2^N)^t |(1 - r^2)^\alpha r^{2n-1} \, dr \, d\sigma(\xi)$$

$$= \left(\int_S |\xi_2^N|^t \, d\sigma(\xi)\right) \left(\frac{2}{B(n, \alpha + 1)} \int_0^1 r^{Nt + 2n-1}(1 - r^2)^\alpha \, dr\right).$$

The second integral, on putting $u = r^2$ becomes $\frac{1}{2}B(Nt/2 + n, \alpha + 1)$. By Stirling’s formula this behaves like $1/N^{\alpha+1}$ as $N \to \infty$. For the first integral, we use the identity [7, 1.4.5, p. 15]

$$\int_S f(\xi, \eta) \, d\sigma(\xi) = \frac{n-1}{\pi} \int_U (1 - r^2)^{n-2} f(re^{i\theta}) r \, dr \, d\theta.$$

We get

$$\int_S |\xi_2^N|^t \, d\sigma(\xi) = \frac{n-1}{\pi} \int_0^1 (1 - r^2)^{n-2} r^{Nt+1} \, dr = \frac{n-1}{\pi} B\left(\frac{Nt + 1}{2}, n - 1\right)$$

$\sim 1/N^{\alpha+1}$ by Stirling’s formula.
Hence
\[ \| z_2^N \|_{t,\alpha} \sim 1/N^{n-1} \cdot 1/N^{\alpha+1} = N^{-(n+\alpha)}. \]

**Remark 1.** \( \| z_2^N \|_{t,\alpha} \sim N^{-(n-1)}. \)

**Lemma 2.** Let \( K(z) = \sum_{i=N-1}^{\infty} K_i(z) \) be holomorphic in \( B \), where \( K_i(z) \) is a homogeneous polynomial of degree \( i \) and \( N \) is a positive integer. Then for \( 0 < t < \infty \), there exists a constant \( M \) (depending only on \( t \)) such that
\[
\| K_N \|_{t,\alpha} \leq M \cdot \| K \|_{t,\alpha},
\]
\[
\| K_N \|_{t,\alpha} \leq M \cdot \| K \|_{t,\alpha}.
\]

**Proof.** For \( 0 < t < \infty \), there exists an \( M \) such that if \( G(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots \) is in the disc algebra \( A(U) \) then
\[
| a_1 | t \leq M \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} | G(e^{i\theta}) | t \ d\theta.
\]
(For \( t \geq 1 \) we can take \( M = 1 \). For \( 0 < t < 1 \), see [2, Theorem 6.4, p. 98]. In fact, \( M = 2^{1/t} \) works for any \( t \).)

Now for a fixed \( z \), let
\[ G(z) = K(z^N) / N^{(n-1)} = K_{N-1}(z) + \lambda K_N(z) + \cdots. \]

We get
\[
| K_N(z) | t \leq M \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} | K(e^{i\theta}z) | t \ d\theta.
\]
We let \( z = \xi \), integrate both sides with respect to \( d\sigma(\xi) \) and use (2) to get
\[
\int_{S} | K_N(r\xi) | t \ d\sigma(\xi) \leq M \int_{S} | K(r\xi) | t \ d\sigma(\xi).
\]

Taking the supremum over \( r \) in the interval \( 0 < r < 1 \) and \( t \) th roots, we get (4). To get (4'), we multiply both sides of (5) by \((2/B(n, \alpha + 1)) r^{2n-1}(1-r^2)^{\alpha/2} \ dr\), integrate over \( 0 < r < 1 \) and take \( t \) th roots.

**Lemma 3.** Let \( 0 < p, q, l < \infty, 1/l = 1/p + 1/q \), \( -1 < \alpha < \infty \) and \( n > 1 \). Then the product map \((h, k) \to h \cdot k \) from \( A^{p,\alpha}(B) \times A^{q,\alpha}(B) \) to \( A^{l,\alpha}(B) \) is not open at the origin, i.e., for any constant \( C > 0 \), there exists \( f \in A^{l,\alpha}(B) \) such that \( \| f \|_{l,\alpha} \leq 1 \) and if \( f = h \cdot k \) with \( h \in A^{p,\alpha}(B), k \in A^{q,\alpha}(B) \) then at least one of \( \| h \|_{p,\alpha}, \| k \|_{q,\alpha} \) is larger than \( C \).

**Proof.** Let \( F(z) = z_1^{N-1} + z_2^N, N > 1 \). Suppose \( F(z) = H(z) \cdot K(z) \) with \( H \) and \( K \) holomorphic in \( B \). We expand \( H(z) \) and \( K(z) \) in terms of homogeneous polynomials: \( H = H_i + H_{i+1} + \cdots, K = K_{N-1-i} + K_{N-1} + \cdots \). Here, as usual, subscript refers to the degree, \( H_i \equiv 0 \) and \( K_{N-1-i} \equiv 0 \). From \( F = H \cdot K \) we get, by comparing degrees,
\[
H_i \cdot K_{N-1-i} = z_1^{N-1}
\]
and
\[
H_i K_{N-i} + H_{i+1} K_{N-1-i} = z_2^N.
\]
From (6) and (7) we get \( i = 0 \) or \( N - 1 \). We assume for a moment that \( i = 0 \). Then \( H_0 \) is a constant, say \( A \). We have from (6) and (7),

\[
AK_N(z) = z_N^N - \left( H_1(z) \cdot z_1^{N-1} \right)/A.
\]

Letting \( z = r(e^{i\theta}, \xi) \) we get

\[
AK_N(e^{i\theta_1}, \xi') = z_N^N - A^{-1}H_1(e^{i\theta_1}, \xi')e^{i(N-1)\theta_1^{N-1}}.
\]

Therefore, \( z_N^N \) is the constant term in the polynomial \( AK_N(\lambda \xi, \xi') \) in \( \lambda \). By subharmonicity,

\[
|z_N^N| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |AK_N(e^{i\theta_1}, \xi')| d\theta \quad \text{for } 0 < \theta < \infty.
\]

Now we multiply both sides of (8) by \( dv_{n-1}(\xi') \) and integrate over \( B_{n-1} \). Using (3), we get

\[
\int_S |z_N^N| d\sigma(\xi) \leq \int_S |AK_N(\xi)| d\sigma(\xi)
\]

and

\[
\int_S |(r_N^N)| d\sigma(\xi) \leq \int_S |AK_N(r_\xi)| d\sigma(\xi).
\]

We multiply both sides of (9) by \((2/B(n, \alpha + 1))(1 - r^2)^{\alpha} \) integrate over \( 0 < r < 1 \) and take \( \alpha \)th roots to get

\[
|z_N^N|_{t, \alpha} \leq |A| ||K_N||_{t, \alpha}.
\]

Since \( |H(z)|^t \) is subharmonic and \( A = H(0) \), we have \( |A|^t \leq \int_S |H(r_\xi)|^t d\sigma(\xi) \). From this we get \( |A| \leq \|H\|_{t, \alpha} \). Hence

\[
|z_N^N|_{t, \alpha} \leq |A| \|K_N\|_{t, \alpha} \leq ||H||_{t, \alpha} \|K_N\|_{t, \alpha}.
\]

Using Lemma 2, we get \( ||z_N^N||_{t, \alpha} \leq M \|H\|_{t, \alpha} \|K\|_{t, \alpha} \). By symmetry, this inequality holds when \( i = N - 1 \). Now let \( f = F/\|F\|_{t, \alpha} \). Then \( \|f\|_{t, \alpha} = 1 \). Suppose \( f = h \cdot k \) where \( h \in A^{p, \alpha}(B) \) and \( k \in A^{q, \alpha}(B) \). Then \( F = H \cdot K \) where \( H = \|F\|_{t, \alpha} h \) and \( K = k \). Therefore

\[
||z_N^N||_{t, \alpha} \leq M \|H\|_{t, \alpha} \cdot \|K\|_{t, \alpha} \leq M \|F\|_{t, \alpha} \|h\|_{t, \alpha} \cdot \|k\|_{t, \alpha}.
\]

Now we take \( t = \min(p, q) \). Then \( \|h\|_{t, \alpha} \leq \|h\|_{p, \alpha} \) and \( \|k\|_{t, \alpha} \leq \|k\|_{q, \alpha} \). We have

\[
\|F\|_{t, \alpha} \leq \int_B \left| z_1^{N-1} + z_2^{N} \right| d\mu_{t, \alpha} \leq 2 \left( \|z_1^{N-1}\|_{t, \alpha} + \|z_2^{N}\|_{t, \alpha} \right).
\]

By Lemma 1, the right side of the above inequality is like \( N^{-(n+\alpha)} \) for large \( N \). We see that

\[
\|h\|_{p, \alpha} \cdot \|k\|_{q, \alpha} \geq \|z_2^{N}\|_{t, \alpha}/M \|F\|_{t, \alpha}.
\]

Hence \( \|h\|_{p, \alpha} \cdot \|k\|_{q, \alpha} \) is bigger than a constant times \( N^{-(n+\alpha)(1/t - 1/l)} \) which goes to \( \infty \) as \( N \to \infty \) (recall \( t = \min(p, q) > 1 \)). Therefore, for any constant \( C \), we can find a large \( N \) so that \( \|h\|_{p, \alpha} \cdot \|k\|_{q, \alpha} > C^2 \). This completes the proof.

**Remark 2.** By considering \( H^p \)-norms instead of \( A^{p, \alpha} \)-norms, one can get the nonopenness of the product map (at the origin) for \( H^p \)-spaces.
Lemma 4. For $a \in B$ and $z \in \overline{B}$, let

$$K(a, z) = \left[ \frac{1 - |a|^2}{1 - \langle z, a \rangle^2} \right]^{(n+1)+\alpha}.$$ 

Then:

(i) $K(a, \phi_a(z)) \cdot K(a, z) = 1$.

(ii) $\int_B f(\omega) \, d\mu_a(\omega) = \int_B f(\phi_a(z))K(a, z) \, d\mu_a(z)$ for all $f \in C(\overline{B})$.

(iii) $\int_B f(\phi_a(z)) \, d\mu_a(z) \to f(e_1)$ as $a \to e_1$ for all $f \in C(\overline{B})$.

Proof. From [7, Theorem 2.2.5], we have

$$1 - \langle \phi_a(z), a \rangle = \frac{1 - |a|^2}{1 - \langle z, a \rangle^2}. $$

Taking absolute values and using the definition of $K$, we get (i). From [7, Theorem 2.2.6] we have

$$\int_B f(\omega)(1 - |\omega|^2)^a \, d\nu(\omega) = \int_B f(\phi_a(z))(1 - |\phi_a(z)|^2)^a \left( \frac{1 - |a|^2}{1 - \langle z, a \rangle^2} \right)^{n+1} \, d\nu(z).$$

Using

$$1 - |\phi_a(z)|^2 = \frac{1 - |a|^2}{1 - \langle z, a \rangle^2},$$

we obtain

$$\int_B f(\omega)(1 - |\omega|^2)^a \, d\nu(\omega) = \int_B f(\phi_a(z)) \frac{(1 - |a|^2)^{n+1} + a(1 - |z|^2)^a}{1 - \langle z, a \rangle^2} \, d\nu(z).$$

Hence

$$\int_B f(\omega) \, d\mu_a(\omega) = \int_B f(\phi_a(z))K(a, z) \, d\mu_a(z).$$

This is (ii). Since $\lim_{a \to e_1} \phi_a(z) = e_1$, an application of the Bounded Convergence Theorem gives (iii).

Remark 3. In the above lemma we assumed that $\alpha > -1$. The following statements hold when $\alpha = -1$.

(i) $K(a, \phi_a(z)) \cdot K(a, z) = 1$.

(ii) $\int_S f(\eta) \, d\sigma(\eta) = \int_B f(\phi_a(\xi))K(a, \xi) \, d\sigma(\xi)$ for all $f \in C(S)$.

(iii) $\int_B f(\phi_a(\xi)) \, d\sigma(\xi) \to f(e_1)$ as $a \to e_1$ for all $f \in C(S)$.

We observe that when $\alpha = -1$, $K(a, z)$ is the Poisson kernel and statements (i) and (ii) are well known. Since $\int_S f(\phi_a(\xi)) \, d\sigma(\xi)$ is the Poisson integral of $f$, (iii) follows (see, e.g., [7, Theorem 3.3.4(a))].

Lemma 5. Let

$$\psi_a(z) = \left[ 1 + \sqrt{1 - |a|^2} \right]^{2(n+1)+\alpha}.$$

Then

$$\max\{1, K(a, z)\} \leq |\psi_a(z)| \leq 2^{2(n+\alpha)} \{1 + K(a, z)\}$$

for all $z \in \overline{B}$, $a \in B$ and $\alpha \geq -1$. 

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Proof. For \( \lambda \in \mathbb{C} \) and \( \text{Re} \lambda \geq 0 \), we have \( \max\{1, |\lambda|\} \leq |1 + \lambda| \). This can be seen by plotting \( \lambda \) and \( 1 + \lambda \) in the complex plane. Also, \( |1 + \lambda|^m \leq (1 + |\lambda|)^m \leq 2^{m-1}(1 + |\lambda|^m) \) for \( m \geq 1 \). Taking \( \lambda = \sqrt{1 - |a|^2/(1 - \langle z, a \rangle)} \) and \( m = 2(n + 1 + \alpha) \), we get the lemma.

4. Main theorem.

Theorem 1. Let \( n > 1 \), \(-1 < \alpha < \infty \), \( 0 < p, q \), \( l < \infty \) and \( 1/l = 1/p + 1/q \). Then \( A^{p, \alpha}(B) \cdot A^{q, \alpha}(B) \) is of first category in \( A^{l, \alpha}(B) \).

Proof. Let \( V \) and \( W \) be the closed unit balls in \( A^{p, \alpha}(B) \) and \( A^{q, \alpha}(B) \), respectively. We claim that \( V \cdot W \) is closed in \( A^{l, \alpha}(B) \). Let \( g_m \in V \), \( h_m \in W \) such that \( g_m \cdot h_m \to f \) in \( A^{l, \alpha}(B) \). By Fact 1 (of §2) we may assume, without loss of generality, that \( g_m \to g \) and \( h_m \to h \) uniformly on compact subsets of \( B \). By Fatou's Lemma, \( g \in V \) and \( h \in W \). Since \( g_m \cdot h_m \to f \) uniformly on compact subsets of \( B \), \( f = g \cdot h \in V \cdot W \). Hence the claim. We have \( A^{p, \alpha}(B) \cdot A^{q, \alpha}(B) = \bigcup_{m=1}^{\infty} (mV \cdot W) \). We show that \( mV \cdot W \) has empty interior in \( A^{l, \alpha}(B) \) for each \( m \geq 1 \). Assume the contrary. Then some \( mV \cdot W \) will have an interior point in \( A^{l, \alpha}(B) \). There exist an \( R \in A^{l, \alpha}(B) \) and a constant \( C \) such that

\[
\int_B |R - F|^l d\mu_a \leq A^{n+1, \alpha}, \quad F \in A^{l, \alpha}(B),
\]

\[
F = g \cdot h \text{ with } \|g\|_{p, \alpha} \leq C \text{ and } \|h\|_{q, \alpha} \leq C.
\]

By Fact 2 (of §2), we may assume that \( R \) is a function in \( A(B) \) vanishing at \( e_1 \). Now by Lemma 3, for the constant \( C \) there is an \( f \in A^{l, \alpha}(B) \) such that \( \|f\|_{l, \alpha} \leq 1 \) and \( f = g \cdot h \), \( g \in A^{p, \alpha}(B) \), \( h \in A^{q, \alpha}(B) \) imply that at least one of \( \|g\|_{p, \alpha} \), \( \|h\|_{q, \alpha} \) is larger than \( C \). There is an \( \epsilon > 0 \) such that

\[
\|f - f_i\|_{l, \alpha} \leq \epsilon \text{ and } f_i = g_i \cdot h_i \in A^{p, \alpha}(B) \cdot A^{q, \alpha}(B)
\]

implies either \( \|g_i\|_{p, \alpha} > C \) or \( \|h_i\|_{q, \alpha} > C \).

We may assume, after Fact 2, that \( f \) is a function in \( A(B) \) vanishing at \( e_1 \).

We now come to perhaps the most important single step in the proof (see [5]). Let \( F(z) = f(\phi_a(z))\psi_a^{1/l}(z) + R(z) \). (\( \phi_a(z) \) is defined in §2 and \( \psi_a(z) \) is defined in Lemma 5.) Now

\[
\int_B |F - R|^l d\mu_a = \int |f(\phi_a(z))|^{l'/l} |\psi_a(z)| d\mu_a
\]

\[
\leq 2^{2(n+1)} \left[ \int_B |f(\phi_a)|^{l'/l} d\mu_a + \int_B |f(\phi_a(z))|^{l} K(a, z) d\mu_a(z) \right]
\]

by Lemma 5. The second integral in the above inequality is \( \int_B |f|^l d\mu_a \) by (ii) of Lemma 4 and the first integral goes to zero as \( a \to e_1 \), by (iii) of Lemma 4 (recall
that \(f(e_1) = 0\). Hence when \(a\) is close to \(e_1\),

\[
\int_B |F - R|' d\mu_a \leq 2^{(n+a)+1} \left[ 1 + \int_B |f|' d\mu_a \right]
\]

\[
\leq 2^{(n+a)+1} [1 + 1] \quad \text{(since } \|f\|_{1,a} \leq 1)\]

\[
= 4^{n+a+1}.
\]

By (10), \(F = g \cdot h\) with \(\|g\|_{p,a} \leq C\) and \(\|h\|_{q,a} \leq C\). Therefore

\[
f(\phi_a(z)) = g(\phi_a(z)) \cdot h(\phi_a(z)) = g(\phi_a(z)) \cdot h(\phi_a(z)).
\]

Replacing \(z\) by \(\phi_a(z)\) and using \(\phi_a(\phi_a(z)) = z\), we get

\[
f(z) + \frac{R(\phi_a(z))}{\psi^{1/(q_a)}(\phi_a(z))} = g(\phi_a(z)) \cdot h(\phi_a(z)) = g(\phi_a(z)) \cdot h(\phi_a(z)).
\]

We have

\[
\int_B \left| \frac{R(\phi_a(z))}{\psi^{1/(q_a)}(\phi_a(z))} \right|' d\mu_a \leq \int_B |R(\phi_a)|' d\mu_a
\]

by Lemma 5. Since \(R(e_1) = 0\), the right side integral in the above inequality goes to zero as \(a \to e_1\) by (iii) of Lemma 4. Hence if \(a\) is close to \(e_1\), (11) holds with \(f_1 = g_1 \cdot h_1\) where

\[
g_1 = g(\phi_a) / \psi^{1/p}(\phi_a) \quad \text{and} \quad h_1 = h(\phi_a) / \psi^{1/q}(\phi_a).
\]

Therefore either \(\|g_1\|_{p,a} > C\) or \(\|h_1\|_{q,a} > C\). Suppose \(\|g_1\|_{p,a} > C\). Then

\[
C^p \leq \int_B \left| \frac{g(\phi_a)}{\psi^{1/p}(\phi_a)} \right|^p d\mu_a
\]

\[
\leq \int_B |g(\phi_a(z))|^p K(a, \phi_a(z)) d\mu_a(z) \quad \text{(by Lemma 5)}
\]

\[
= \int_B |g(\phi_a(z))|^p K(a, z) d\mu_a(z) \quad \text{(by (i) of Lemma 4)}
\]

\[
= \int_B |g|^p d\mu_a \quad \text{(by (ii) of Lemma 4)}
\]

\[
\leq C^p \quad \text{(since } \|g\|_{p,a} \leq C).\]

We reach a contradiction. Similarly \(\|h_1\|_{q,a} > C\) gives a contradiction. Hence all \(m(V \cdot W)\) have empty interiors. So

\[
A^{p,a}(B) \cdot A^{q,a}(B) = \bigcup_{m=1}^\infty m(V \cdot W)
\]

is of first category in \(A^{1,a}(B)\).

5. Other results. Here is a nonfactorization theorem for Hardy spaces.

**Theorem 2.** Let \(n > 1\) and \(0 < p, q, l < \infty\). If \(1/l = 1/p + 1/q\) then \(H^p(B) \cdot H^q(B)\) is of first category in \(H^l(B)\).
The proof of this theorem is very similar to that of Theorem 1. One has to integrate functions in the Hardy class $H^t(B)$ (for $t = p, q$ and $l$) with respect to $d\sigma$ over $S$. $\alpha$ should be replaced by $-1$ (relation (1) can also be used at appropriate places). We omit the details. Theorem 2 can also be proved, for $n > 2$, using Theorem 1 (with $\alpha = 0$) and Theorem 7.2.4 in [7].

**Remark 4.** Let $T$ be the mapping $(f_1, g_1, f_2, g_2, \ldots, f_k, g_k) \rightarrow \sum_{i=1}^{k} f_i g_i$. The proof of Theorem 1 shows that

$$T: A^{p,\alpha}(B) \times A^{q,\alpha}(B) \times \cdots \times A^{p,\alpha}(B) \times A^{q,\alpha}(B) \rightarrow A^l,\alpha(B)$$

$(1/l = 1/p + 1/q)$ is onto if and only if it is open at the origin. Nonopenness of $T$ at the origin would imply the existence of a function in $A^l,\alpha(B)$ which is not of the form $\sum_{i=1}^{k} f_i g_i$ with $f_i \in A^{p,\alpha}(B)$ and $g_i \in A^{q,\alpha}(B)$. However, any function $F$ in $A^l,\alpha(B)$ (for $\alpha = 0, 1, 2, \ldots$) can be written as $F = \sum_{i=1}^{\infty} G_i H_i$ where $G_i$ and $H_i$ belong to $A^{2,\alpha}(B)$ (see [1, Theorem IV]). Similar statements can be made for Hardy spaces.

**Remark 5.** Let $0 < t < \infty$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \geq -1$. Let $\mathcal{A}^{t,\alpha}(U^n)$ be the space of all holomorphic functions $f$ satisfying $\|f\|_{t,\alpha} = \int_{U^n} |f|^t d\mu_{\alpha}$ (where $d\mu_{\alpha} = \prod_{i=1}^{n} (1 - |z_i|^2)^{\alpha_i} dm_i(z_i)$ being the Lebesgue measure on $U$ for all $i = 1, 2, \ldots, n$). Then Theorem 1 holds for $U^n$ in place of $B$. We sketch a proof of this statement. If $K(z) = \sum_{i=1}^{\infty} K_i(z)$ is as in Lemma 2 then

$$\int_{-\pi}^{\pi} |K_N(r_i e^{i\theta}, e^{i\theta} z')| \, d\theta \leq C |\int_{-\pi}^{\pi} |\sum_{i=1}^{\infty} K_N(r_i e^{i\theta}, e^{i\theta} z')| \, d\theta|,$$

and hence $\|K_N\|_{t,\alpha} \leq M \|K\|_{t,\alpha}$ where $C_i$ and $M_i$ are constants depending only on $t$. Without loss of generality let $a_1 = a_2$. We have

$$\|z_i^N\|_{t,\alpha} \sim N^{-1 + a_i} \quad (i = 1, 2),$$

by Lemma 1 and using $F = z_1^{N-1} + z_2^N$ we get (imitating the proof of Lemma 3) the nonopenness of the product map from $A^{p,\alpha}(U^n) \times A^{q,\alpha}(U^n)$ to $A^{l,\alpha}(U^n)$ where $1/p + 1/q = 1/l$. (If

$$AK_N(z) = z_2^N - \left(H_1(z) \cdot z_1^{N-1}\right) / A$$

then

$$AK_N(r_i e^{i\theta}, z') = z_2^N - \frac{H_1(r_i e^{i\theta}, z')}{A} e^{i(N-1)\theta} r_1^{N-1}$$

and

$$\int_{-\pi}^{\pi} |z_2^N| \, d\theta \leq C |A| \int_{-\pi}^{\pi} |\sum_{i=1}^{\infty} K_N(r_i e^{i\theta}, z')| \, d\theta \quad \text{etc.})$$

For $0 < r < 1$, let

$$a = (r, 0, 0, \ldots, 0), \quad \phi_a(z) = \left( (r - z_1) / (1 - rz), z_2, z_3, \ldots, z_n \right),$$

$$K(a, z) = \left( (1 - r^2) / |1 - rz_1|^2 \right)^{2+a_1}$$

and

$$\phi_a(z) = \left( 1 + \sqrt{|1 - r^2| / (1 - rz_1)} \right)^{2(2+a_1)}.$$
We note that as $r \to 1$, $a \to (1, 0, \ldots, 0)$ and $\phi_\alpha(z) \to (1, z')$. Observe that functions $f$ in $A(U^n)$ with $f(1, z') \equiv 0$ form a dense subset of $A^{p,\alpha}(U^n)$. With minor changes, one can get results similar to Lemmas 4 and 5. By imitating the proof of Theorem 1, we get the polydisc version of Theorem 1.

**Remark 6.** Let $H^p(U^n)$ be the Hardy space of all holomorphic functions $f$ in $U^n$ satisfying

$$
\|f\|_{t,\alpha} = \left( \sup_{0<r<1} \int_{T^n} \left| f(r\xi) \right|^t d\sigma(\xi) \right)^{1/t} < \infty
$$

where $T^n$ is the torus in $\mathbb{C}^n$ and $d\sigma$ is the normalized Haar measure on $T^n$.


To sketch a proof, let $P = (z_1 + z_2)^N - z_1^N - Nz_1^{N-1}z_2$. Then $\|P\|_{t,\alpha}/\|P\|_{t,\alpha} \to \infty$ as $N \to \infty$ whenever $t > l$. There exists a constant $C_t$ such that if $AK_N = P(z) + z_1^{N-1}Q(z)$, where $Q(z)$ is any linear polynomial in $z$, then $\|P\|_{t,\alpha} \leq C_t |A| \|K_N\|_{t,\alpha}$ (use subharmonicity in $z_2$). The function $f = (P + z_1^{N-1})/\|P + z_1^{N-1}\|_{t,\alpha}$ gives the nonopenness of the product map. Changing $\alpha_1$ to $-1$ and making other minor changes in the proof of Remark 5, we get Theorem 2 for $U^n$.

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**References**


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