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AXIOMS FOR STIEFEL-WHITNEY HOMOLOGY CLASSES OF SOME SINGULAR SPACES[1]

BY

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ABSTRACT. A system of axioms for the Stiefel-Whitney classes of certain type of singular spaces is established. The main examples of these singular spaces are Euler manifolds mod 2 and homology manifolds mod 2. As a consequence, it is shown that on homology manifolds mod 2 the generalized Stiefel conjecture holds.

The purpose of this paper is to set up a system of axioms which describe in a unique way the Stiefel-Whitney (S.W.) homology classes of some class of singular spaces, called in this paper allowable class. The main examples include Euler manifolds mod 2 and homology manifolds mod 2. This axiomatic characterization of S.W. homology classes then gives as corollaries affirmative answers to generalized Stiefel conjectures. The classical Stiefel conjecture says that on a smooth manifold M^n, the Poincaré dual of the ith cohomology S.W. class is represented by the cycle mod 2 which is equal to the sum (mod 2) of all $(n-i)$-simplices in the first barycentric subdivision of some triangulation of M **[21]**. In the generalized Stiefel conjecture we deal with Steenrod squares of Wu classes instead of ordinary cohomology S.W. classes.

The question about a possible axiomatic description of S.W. classes for Euler manifolds or for more general spaces was posed by Blanton and Schweitzer **[4]**. Our corollary on homology manifolds mod 2 was also obtained by Taylor **[19]**, but from a completely different viewpoint.

The main tools in proving that our axioms determine a unique class are block bundle transversality and a description of cohomology classes as morphisms on bordism groups. These techniques then provide the "transversality classes" $\tau(\varphi)$ of an embedding φ of our singular space X into the interior of a PL manifold M. $\tau(\varphi)$ lies in $H^*(M, \partial M; \mathbf{Z}_2)$ and determines characteristics (mod 2) of transversal intersections of X with singular manifolds in M. These ideas are based on the work of Latour **[13]**. In §1 we give some basic facts about Euler manifolds mod 2 and their S.W. homology classes, which then motivates §§2 and 3 where we introduce allowable classes of spaces and introduce axioms for their S.W. classes. In §4 we prove that the axioms determine unique classes and derive the above-mentioned consequences.

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1. Geometry of Euler manifolds. By a "polyhedron" we shall understand a compact polyhedron, but the whole theory makes sense for locally finite polyhedra (and then appropriate homology based upon infinite, but locally finite chains). First, we recall some basic definitions, facts and examples.

A *geometric cycle* of dimension n is an n-dimensional polyhedron X with a triangulation in which $(n-1)$-simplices have as links just two points and lower dimensional simplices have nonempty and connected links.

An n-dimensional *geometric cycle with boundary* $(X, \partial X)$ is an n-polyhedron X with a triangulation in which $(n-1)$-simplices have as links one or two points, lower dimensional simplices have nonempty and connected links and all the faces of $(n-1)$-simplices with one point links constitute an $(n-1)$-geometric cycle, called the boundary ∂X of X. We always assume ∂X is locally collared and hence collared in Y.

Note that an n-geometric cycle is an n-pseudomanifold (circuit), and is "purely n-dimensional".

If X is an n-dimensional geometric cycle, then its *singular set* $S(X) = \{x \in X \mid d_X(x) < n\}$ has dimension $\leqslant n-3$, where $d_X(x)$ is the intrinsic dimension of x in X. For every connected n-dimensional geometric cycle with boundary $(X, \partial X)$, we have a canonical isomorphism $H_n(X, \partial X; \mathbf{Z}_2) \approx \mathbf{Z}_2$. The generator of this group is denoted by $[X, \partial X]$ or simply by $[X]$.

1.1. DEFINITION. An n-dimensional geometric cycle X is called *n-Euler manifold* mod 2 (or *$\mathbf{Z}_2$-Euler manifold*) if it is an *Euler space*, i.e. $\chi(X, X \backslash \{x\}) \equiv 1$ for all $x \in X$, or equivalently if $\chi(\mathrm{Lk}(x, X)) \equiv 0$, for all $x \in X$, or equivalently if there is a triangulation K of X such that for all $\sigma \in K$, $\chi(\mathrm{Lk}(\sigma, K)) \equiv 0$. Here, $\mathrm{Lk}(\sigma, K)$ is the boundary of the simplicial neighborhood of σ in K, χ is the Euler characteristic, and $\equiv$ means $\equiv$ (mod 2) throughout this paper. More generally, *n-Euler manifold* mod 2 *with boundary* is an n-dimensional geometric cycle with boundary $(X, \partial X)$ such that $\chi(X, X \backslash \{x\}) \equiv 1$, for all $x \in X \backslash \partial X$ and

$$\chi(X, X \backslash \{x\}) \equiv 0, \quad \chi(\partial X, \partial X \backslash \{x\}) \equiv 1,$$

for all $x \in \partial X$. Equivalently, there is a triangulation (K, L) of $(X, \partial X)$ such that $\chi(\mathrm{Lk}(\sigma, K)) \equiv 0$ for all $\sigma \in K - L$ and $\chi(\mathrm{Lk}(\sigma, K)) \equiv 1$, $\chi(\mathrm{Lk}(\sigma, L)) \equiv 0$ for all $\sigma \in L$.

That all these definitions are equivalent follows from the following facts:

$$\tilde{H}_s(\mathrm{Lk}(x, X)) \approx H_{s+1}(X, X \backslash \{x\}),$$
$$\mathrm{Lk}(x, X) \approx_{\mathrm{PL}} \partial\sigma * \mathrm{Lk}(\sigma, K) \approx_{\mathrm{PL}} \Sigma^{k-1}\mathrm{Lk}(\sigma, K),$$

where $x \in \mathring{\sigma}$, $\sigma \in K$, $|K| = X$, and for any compact polyhedron P, $\chi(P) \equiv \chi(\Sigma P)$. Note that there is an appropriate notion of (integral) Euler manifolds ($\chi(X, X \backslash \{x\}) = (-1)^{\dim X}$, for all $x \in X$), but we will not discuss them in this paper.

Examples of $\mathbf{Z}_2$-Euler manifolds include all mod 2 (polyhedral) homology manifolds, and suspensions of connected mod 2 homology manifolds with even Euler characteristics, since the only two bad points have connected links with $\chi(\text{link}) \equiv 0$. Further, let M^n be a connected closed mod 2 homology manifold, and ξ^k a (disk or block) bundle over M whose sphere bundle is connected. Then the Thom space $T(\xi) = T$ is an $(n+k)$-Euler manifold mod 2. For, let $E = E(\xi)$ be the total space of ξ. Then $T = E/\partial E$ and let $*$ be the base point (the only "suspicious" point), and

$$\chi(T, T\backslash\{*\}) = \chi(\text{cone}(\partial E), \partial E) = 1 - \chi(\partial E) \equiv 1,$$

since $\chi(\text{boundary of mod 2 homology manifold}) \equiv 0$. Further, as Sullivan **[17]** pointed out, compact connected real analytic spaces as well as complex projective varieties are Euler spaces. Sometimes they are not Euler manifolds, i.e. links of "bad points" are not connected, as the "pinched torus" $x^3 + y^3 = xyz$ in homogeneous coordinates $[x, y, z]$ in $\mathbf{C}P^2$ shows, but sometimes they are. For example the quadratic cone X^4: $x^2 + y^2 + z^2 = 0$ in $\mathbf{C}P^3$ with homogeneous coordinates $[x, y, z, w]$. The only singular point is $p = [0, 0, 0, 1]$ and $\text{Lk}(p, X) \approx$ (tangent circle bundle of $S^2) \approx \mathbf{R}P^3$. Note that X is homeomorphic to the Thom space of the tangent bundle of S^2. Finally, King and Akbulut **[12]** showed recently that 2-dimensional real algebraic sets are topologically characterized as 2-dimensional Euler spaces.

Let us mention only that if X is a $\mathbf{Z}_2$-Euler manifold (or just a geometric cycle) with isolated singularities which is also a $\mathbf{Z}_2$-Euler-Poincaré complex, then X is, in fact, a $\mathbf{Z}_2$-homology manifold. This follows from McCrory **[15]**.

Note that if X is an n-Euler manifold mod 2, K a triangulation of X and $\sigma^k \in K$, then $\text{Lk}(\sigma^k, K)$ is an $(n-k-1)$-Euler manifold mod 2.

An appropriate class of maps relating mod 2 Euler manifolds are mod 2 *Euler resolutions*. Let f: $X \to Y$ be a map between two polyhedra. We say that f is a mod 2 Euler resolution if f is a PL map, and $f^{-1}(y)$ is a nonempty, connected set with $\chi(f^{-1}(y)) \equiv 1$, for all $y \in Y$. It is easy to see that if f: $X \to Y$ is a mod 2 Euler resolution between two n-polyhedra and X is a mod 2 Euler manifold, then Y is too, and $\chi(X) \equiv \chi(Y)$. In particular, being a mod 2 Euler manifold is a PL property. Goldstein and Turner **[9]** showed that being a mod 2 Euler space is a topological property.

1.2. PROPOSITION. *Let f: $X \to Y$ be a* mod 2 *Euler resolution between two n-dimensional* mod 2 *Euler manifolds*; *then C_f, the simplicial mapping cylinder, is a* mod 2 *Euler manifold of dimension $(n+1)$ with boundary $X \amalg Y$. The converse is also true.*

For a proof see **[20]**. □

Let us denote by $\mathbf{E}^n_\partial(2)$ the class of all n-dimensional Euler manifolds mod 2 with boundary and by $\mathbf{E}^n(2)$ the class of those with empty boundary. $\mathbf{E}_\partial(2) = \bigcup_{n \geq 0} \mathbf{E}^n_\partial(2)$, $\mathbf{E}(2) = \bigcup_{n \geq 0} \mathbf{E}^n(2)$. Note that $(X, \partial X) \in \mathbf{E}^n_\partial(2)$ implies $\partial X \in \mathbf{E}^{n-1}(2)$.

1.3. PROPOSITION. (a) GLUING AND CUTTING: *Let $(X_j, \partial X_j) \in \mathbf{E}^n_\partial(2)$, $Y_j \subset \partial X_j$, $j = 0, 1$, be components of the boundaries and f: $Y_0 \to Y_1$ be a* PL *homeomorphism. Let*

$A_j = \partial X_j \backslash Y_j$ and $X = X_0 \cup_f X_1$. Then $(X, A_0 \cup A_1) \in \mathbf{E}^n_\partial(2)$. If $(X, A_0 \cup A_1)$, $(X_j, \partial X_j) \in \mathbf{E}^n_\partial(2)$, then also $(X_{1-j}, \partial X_{1-j}) \in \mathbf{E}^n_\partial(2)$.

(b) Product: *Let $(X, \partial X) \in \mathbf{E}^p_\partial(2)$, $(Y, \partial Y) \in \mathbf{E}^q_\partial(2)$. Then $(X, \partial X) \times (Y, \partial Y) = (X \times Y, \partial X \times Y \cup X \times \partial Y) \in \mathbf{E}^{p+q}_\partial(2)$. Conversely, if $(X, \partial X)$ and $(Y, \partial Y)$ are geometric cycles of dimension p (resp. q) such that $(X, \partial X) \times (Y, \partial Y) \in \mathbf{E}^{p+q}_\partial(2)$, then $(X, \partial X) \in \mathbf{E}^p_\partial(2)$, $(Y, \partial Y) \in \mathbf{E}^q_\partial(2)$.*

(c) Joins, suspensions, cones: *Let $X \in \mathbf{E}^p(2)$, $Y \in \mathbf{E}^q(2)$, X, Y connected and $\chi(X) \equiv \chi(Y) \equiv 0$. Then $X * Y \in \mathbf{E}^{p+q+1}(2)$. Furthermore, let $X \in \mathbf{E}^p(2)$, X connected and $\chi(X) \equiv 0$. Then $\Sigma X \in \mathbf{E}^{p+1}(2)$ and* (cone X, X) $\in \mathbf{E}^{p+1}_\partial(2)$.

(d) Bicollarity: *Let $X^n \in \mathbf{E}^n_\partial(2)$ and $Y^{n-1} \subset X$ be a subpolyhedron, and $Y \subset X \backslash \partial X$ and Y (locally) bicollared in $X \backslash \partial X$. Then $Y \in \mathbf{E}^{n-1}_\partial(2)$.*

(e) Regular neighborhoods and complements: *Let $X \in \mathbf{E}^n_\partial(2)$ and $P \subset X \backslash \partial X$ be a subpolyhedron of X. Let N be a regular neighborhood of P in X and ∂N its boundary. Then $(N, \partial N)$, $(X \backslash \mathring{N}, \partial N \cup \partial X) \in \mathbf{E}^n_\partial(2)$.*

(f) Block bundle: *Let E be the total space of a k-bundle ξ^k* (fibre: D^k) *over an n-polyhedron X, and ∂E the total space of the associated sphere bundle. Then $X \in \mathbf{E}^n_\partial(2) \Leftrightarrow (E, \partial E) \in \mathbf{E}^{n+k}_\partial(2)$.*

Proof. The analogous properties for geometric cycles are proved in **[13]**, so we only examine local Euler numbers.

(a) Since links are PL invariants, we may assume that $Y_0 = Y_1 = Y$ and $f = \text{id}$. Let $y \in Y$. Then $\text{Lk}(y, X) = \text{Lk}(y, X_0) \cup \text{Lk}(y, X_1)$ with intersection $\text{Lk}(y, Y) = \text{Lk}(y, \partial X_0) = \text{Lk}(y, \partial X_1)$. Hence $\chi(\text{Lk}(y, X)) \equiv 1 + 1 - 0 \equiv 0$. It is clear that other points in X have appropriate local Euler numbers.

The cutting property can be proved similarly.

(b) It follows at once, using the fact that for $x \in X$, $y \in Y$, $\text{Lk}((x, y), X \times Y) \approx_{\text{PL}} \text{Lk}(x, X) * \text{Lk}(y, Y)$ and $\chi(A * B) \equiv \chi(A) + \chi(B) - \chi(A)\chi(B)$. The converse is true because for $a, b \in \mathbf{Z}$, $a + b \equiv ab \Leftrightarrow a \equiv 0 \,\&\, b \equiv 0 \pmod 2$.

(c) One can use the following precise formula for the link of a point in the join: For $x \in X, y \in Y, t \in [0, 1]$,

$$\text{Lk}(tx + (1-t)y, X * Y) = \begin{cases} X * \text{Lk}(y, Y), & t = 0, \\ (\Sigma \text{Lk}(x, X)) * \text{Lk}(y, Y), & t \in (0, 1), \\ \text{Lk}(x, X) * Y, & t = 1. \end{cases}$$

Then one uses the formula for $\chi(A * B)$ and the fact $\chi(\Sigma Z) \equiv \chi(Z)$. The cone and suspension are clear. In fact, the join-case follows from (a), (b) and the cone-case, using the fact that $X * Y \approx_{\text{PL}} cX \times Y \cup_{X \times Y} X \times cY$.

(d) The point here is that if (K, L) triangulates (X, Y), then for $\sigma \in L$, $\dim \sigma \leqslant n - 2$, bicollarity implies $(\text{Lk}(\sigma, K), \text{Lk}(\sigma, L)) \approx_{\text{PL}} (\Sigma \text{Lk}(\sigma, L), \text{Lk}(\sigma, L))$.

(e) Follows from (d).

(f) It is not hard to show that all local homology is getting shifted in E for $k = \dim \xi$; see **[20]** for details. □

1.4. Definition. Let X be an n-polyhedron, K its triangulation, K' the barycentric subdivision of K. For every p, $0 \leqslant p \leqslant n$, let

$$s_{n-p}(K') = \sum_{\sigma_0 < \cdots < \sigma_p \in K} \langle \underline{\sigma}_0 \cdots \underline{\sigma}_p \rangle \in C_p(K'; \mathbf{Z}_2),$$

where $\underline{\sigma}_i$ is the barycenter of σ_i.

More generally, let (X^n, Y^{n-1}) be a polyhedral pair, (K, L) its triangulation, and (K', L') the barycentric subdivision. We define

$$s_{n-p}(K', L') = \sum_{\sigma_0 < \cdots < \sigma_p \in K;\ \sigma_p \notin L} \langle \underline{\sigma}_0 \cdots \underline{\sigma}_p \rangle \in C_p(K', L'; \mathbf{Z}_2).$$

These chains mod 2 are called *Stiefel chains* mod 2.

It is known that if K triangulates an n-Euler manifold mod 2 then $s_q(K')$ is a cycle mod 2 for all $q < n$, and in the relative case that $\partial s_q(K', L') = s_q(L') \in C_{n-q-1}(L'; \mathbf{Z}_2)$ for all $q < n$ (cf. **[10]**).

Note that instead of barycentric, we can take any derived subdivision K^* of K by starring the simplices at any interior point $\sigma^* \in \mathring{\sigma}$, and then the simplicial isomorphism $\underline{\sigma} \to \sigma^*$ carries $s_q(K')$ to $s_q(K^*)$, and since it is isotopic to the identity, if one of them is a cycle, the other one is too. So we can write $s_q(K)$ instead of $s_q(K')$, etc.

1.5. Definition. Let X^n be an Euler manifold mod 2. Then the homology class of $s_q(K)$ is denoted by $W_q(X) \in H_{n-q}(X; \mathbf{Z}_2)$, for every q, $0 \leqslant q \leqslant n-1$, and for $q = n$, let $W_n(X) \in H_0(X; \mathbf{Z}_2)$ be the class of $s_n(K) = \Sigma_{\sigma \in K} \langle \underline{\sigma} \rangle \in C_0(K'; \mathbf{Z}_2)$.

Similarly, if $(X, \partial X)$ is an n-Euler manifold mod 2 then the homology class of $s_q(K, L)$ is denoted by $W_q(X, \partial X) \in H_{n-q}(X, \partial X; \mathbf{Z}_2)$. The class $W_q(X, \partial X) = W_q(X)$ is called the *qth Stiefel-Whitney homology class* (S.W. class) of $(X, \partial X)$, and

$$W(X, \partial X) = W(X) = W_0(X) + W_1(X) + \cdots + W_n(X) \in \bigoplus_{i=0}^{n} H_i(X, \partial X; \mathbf{Z}_2)$$

is called the *total* S.W. class.

1.6. Remarks. (a) $W_q(\partial X) = \partial W_q(X, \partial X) \in H_{n-q-1}(\partial X; \mathbf{Z}_2)$, where $(X, \partial X) \in \mathbf{E}^n_\partial(2)$ and ∂ is the boundary operator. This follows from the above formula on the chain level.

(b) Let $(X, \partial X) \in \mathbf{E}^n_\partial(2)$ and let ε: $H_0(X, \partial X; \mathbf{Z}_2) \to \mathbf{Z}_2$ be the augmentation. Then $\varepsilon W_n(X, \partial X) \equiv \chi(X, \partial X)$. Further, since the diagram

$$\begin{array}{ccc} H_0(\partial X; \mathbf{Z}_2) & \xrightarrow{i_*} & H_0(X; \mathbf{Z}_2) \\ & \varepsilon \searrow \quad \swarrow \varepsilon & \\ & \mathbf{Z}_2 & \end{array}$$

commutes, it follows that

$$\chi(\partial X) \equiv \varepsilon\big(W_{n-1}(\partial X)\big) = \varepsilon i_* W_{n-1}(\partial X) = \varepsilon i_* \partial W_{n-1}(X) = 0,$$

since $i_* \partial = 0$.

(c) Note that $W_0(X) \in H_n(X, \partial X; \mathbf{Z}_2)$ is the mod 2 fundamental class $[X]$ for every $(X, \partial X) \in \mathbf{E}^n_\partial(2)$.

(d) Using 1.2 it follows easily that if $f\colon X \to Y$ is an Euler resolution mod 2 between two n-Euler manifolds mod 2, then $f_* W_q(X) = W_q(Y)$ for all q, $0 \leqslant q \leqslant n$.

In particular, S.W. homology classes are PL invariants. This justifies the notation $W_q(X)$ (instead of $W_q(K)$).

The definition of Euler cobordism mod 2 is the usual one. Denote by $\mathfrak{N}_n^{\mathbf{E},2}$ the nth cobordism group based on the class $\mathbf{E}_\partial(2)$. Denote by $\bar{\chi}$ the Euler characteristics reduced mod 2.

1.7. PROPOSITION. $\bar{\chi}\colon \mathfrak{N}_n^{\mathbf{E},2} \to \mathbf{Z}_2$ *is an isomorphism for* $n \neq 1$ *and* $\mathfrak{N}_1^{\mathbf{E},2} = 0$.

PROOF. Since a circle bounds, $\mathfrak{N}_1^{\mathbf{E},2} = 0$. That $\bar{\chi}$ is a well-defined homomorphism follows from 1.6(b). Let $n > 1$, $X \in \mathbf{E}^n(2)$ connected with $\bar{\chi}(X) = 0$. Then by 1.3(c) $(\text{cone } X, X) \in \mathbf{E}_\partial^{n+1}(2)$ and X is a boundary. If X is not connected, take a cone over each component. So $\bar{\chi}$ is a monomorphism. To see that $\mathfrak{N}_n^{\mathbf{E},2} \neq 0$, for $n \geqslant 2$, we have in dimension 2, $\bar{\chi}(\mathbf{R}P^2) = 1$ and for $n \geqslant 3$,

$$X^n = S^1 \times D^{n-1} \cup \text{cone}(S^1 \times S^{n-2}) \in \mathbf{E}^n(2) \quad \text{with } \bar{\chi}(X^n) = 1. \quad \square$$

The Euler bordism mod 2 is also defined in a standard way. So, denote by $\mathfrak{N}_n^{\mathbf{E},2}(A, B)$ the nth Euler bordism group mod 2 of the polyhedral pair (A, B). $\mathfrak{N}_*^{\mathbf{E},2}(\ )$ is clearly a homology theory.

Next, denote by $H_n^{\mathbf{E},2}(A, B) = \bigoplus_{i \neq 1} H_{n-i}(A, B; \mathbf{Z}_2)$. Now, $H_*^{\mathbf{E},2}(\ )$ is also a homology theory and there is a natural transformation $\omega\colon \mathfrak{N}_*^{\mathbf{E},2}(\ ) \to H_*^{\mathbf{E},2}(\ )$ defined by

$$\omega_{A,B}([X, f]) = \sum_{i \neq 1} f_* W_i(X).$$

(Here one uses 1.6(a).)

1.8. PROPOSITION. *The natural transformation* $\omega\colon \mathfrak{N}_*^{\mathbf{E},2}(\ ) \to H_*^{\mathbf{E},2}(\ )$ *is an equivalence.*

This follows from the fact that $\omega_{\text{point}}\colon \mathfrak{N}_*^{\mathbf{E},2}(\text{point}) \to H_*^{\mathbf{E},2}(\text{point})$ is an isomorphism (cf. 1.7 and 1.6(b)). $\square$

1.9. COROLLARY. *The Hurewicz map* $\mu\colon \mathfrak{N}_n^{\mathbf{E},2}(A, B) \to H_n(A, B; \mathbf{Z}_2)$ *is an epimorphism, whose kernel is generated by decomposable elements* $[X \times Y, f \circ \text{pr}_1] = [X, f] \cdot [Y]$, *where Y is without boundary.*

PROOF. The epi part follows from Proposition 1.8 and 1.6(c), and the result about the kernel from degeneracy of the spectral sequence of the homology theory $\mathfrak{N}_*^{\mathbf{E},2}(\ )$ as in [7] (cf. [13]). $\square$

2. Allowable classes.

2.1. DEFINITION. For any $n \geqslant 0$, let L_∂^n be a subclass of n-dimensional geometric cycles with boundary, closed under PL isomorphisms, $(D^n, S^{n-1}) \in L_\partial^n$ and let L^n be the subclass of L_∂^n consisting of those with empty boundary. Then an L_∂^n*-manifold* is an n-dimensional geometric cycle with boundary $(X, \partial X)$, such that $\forall x \in X \backslash \partial X$ $|\text{Lk}(x, X)| \in L^{n-1}$; $\forall x \in \partial X$ $|\text{Lk}(x, X)| \in L_\partial^{n-1}$ and $|\text{Lk}(x, \partial X)| \in L^{n-2}$. If $\partial X = \varnothing$ we will call $(X, \varnothing) = X$ a (closed) L^n*-manifold*.

An *allowable class* $\mathbf{A}$ consists in each dimension $n \geqslant 0$ of a class $\mathbf{A}^n$, each member of which is an L^n_∂-manifold and such that the following global properties hold:

(a) *Gluing and cutting*. Let $(X_j, \partial X_j) \in \mathbf{A}^n$, $Y_j \subset \partial X_j$, $j = 0, 1$, be a component of the boundary (or several components) and f: $Y_0 \to Y_1$ a PL homeomorphism. Denote $A_j = \partial X_j \backslash Y_j$ and $X = X_0 \cup_f X_1$. Then $(X, A_0 \cup A_1) \in \mathbf{A}^n$. Conversely, if $(X, A_0 \cup A_1), (X_j, \partial X_j) \in \mathbf{A}^n$, then $(X_{1-j}, \partial X_{1-j}) \in \mathbf{A}^n$.

(b) *Boundary*. If $(X, \partial X) \in \mathbf{A}^n$, then $\partial X \in \mathbf{A}^{n-1}$ and $\chi(\partial X) \equiv 0 \pmod 2$.

(c) *Product*. Let $(X, \partial X) \in \mathbf{A}^p$, $(Y, \partial Y) \in \mathbf{A}^q$. Then $(X, \partial X) \times (Y, \partial Y) \in \mathbf{A}^{p+q}$.

(d) *Block bundle*. Let $E = E(\xi)$ be the total space of a k-block bundle ξ^k(fibre : D^k) over an n-geometric cycle X. Then $(X, \partial X) \in \mathbf{A}^n \Leftrightarrow (E, \partial E) \in \mathbf{A}^{n+k}$, where $\partial E = E(\xi/\partial X) \cup$ (the total space of the associated sphere bundle).

(e) *Steenrod representability*. Denote by $\mathfrak{N}^{\mathbf{A}}_n(A, B)$ the appropriate notion of unoriented bordism groups based on the class $\mathbf{A}$, $n \geqslant 0$. Then for any pair (A, B) of polyhedra, the "Hurewicz map" μ_n: $\mathfrak{N}^{\mathbf{A}}_n(A, B) \to H_n(A, B; \mathbf{Z}_2)$, $\mu_n([X, f]) = f_*[X]$ is an epimorphism and the kernel of μ_n is generated by the *decomposable elements* in the image of the natural pairing $\mathfrak{N}^{\mathbf{A}}_*(A, B) \otimes \mathfrak{N}^{\mathbf{A}}_*(\mathrm{pt}) \to \mathfrak{N}^{\mathbf{A}}_*(A, B)$; that is by elements $[X, f]\cdot[Y] = [X \times Y, f \circ \mathrm{pr}_1]$, where f: $(X, \partial X) \to (A, B), \partial Y = \varnothing$, $\dim Y > 0$.

2.2. Remarks. (1) If $(X, \partial X) \in \mathbf{A}^n$ and $Y \subset X \backslash \partial X$ is a subpolyhedron and $(N, \partial N)$ is a regular neighborhood of Y in $X, C = X \backslash \mathring{N}, \partial C = \partial N \cup \partial X$, then clearly from local properties it follows that $(N, \partial N) \in \mathbf{A}^n$ and by (a), $(C, \partial C) \in \mathbf{A}^n$ (cf. [**6**]).

(2) Examples of allowable classes include smooth and PL manifolds.

Now, by 1.3, 1.6(b) and 1.9 it follows:

2.3. Proposition. *The class* $\mathbf{E}_\partial(2)$ *of Euler manifolds* mod 2 *is an allowable class.* □

2.4. Proposition. *The class* $\mathbf{H}_\partial(2)$ *of (polyhedral)* $\mathbf{Z}_2$*-homology manifolds is an allowable class.*

Proof. Local properties and the gluing-cutting property follow easily using Mayer-Vietoris arguments; the boundary property follows by $\mathbf{Z}_2$-Poincaré duality. The product and the block bundle property can be easily checked; cf. [**6**]. The Steenrod representability follows as in 1.9. □

3. Stiefel-Whitney homology classes of allowable pairs.

3.1. Definition. Let $\mathbf{A}$ be an allowable class. For every $n \geqslant 0$, we assign to every $(X, \partial X) \in \mathbf{A}^n$ a homology class

$$W(X, \partial X) = W(X) = W_0(X) + W_1(X) + \cdots + W_n(X) \in H_*(X, \partial X; \mathbf{Z}_2)$$

such that the following axioms hold:

(1) For each integer q, $0 \leqslant q \leqslant n$, $W_q(X) \in H_{n-q}(X, \partial X; \mathbf{Z}_2)$, and $W_0(X) = [X]$, the fundamental class of X, and $\varepsilon(W_n(X)) \equiv \chi(X, \partial X)$, where ε: $H_0(X, \partial X; \mathbf{Z}_2) \to \mathbf{Z}_2$ is the augmentation.

(2) *Restriction*. Let $(X_j, \partial X_j) \in \mathbf{A}^n$ and $Y_j \subset \partial X_j$ be one (or several) component of the boundary, $j = 0, 1$, $f\colon Y_0 \to Y_1$ a PL homeomorphism. Let $X = X_0 \cup_f X_1$ and $\partial X = (\partial X_0 \backslash Y_0) \cup (\partial X_1 \backslash Y_1)$. Then $(X, \partial X) \in \mathbf{A}^n$ and $r_j W_q(X) = W_q(X_j)$ for every q, $0 \leqslant q \leqslant n$, where

$$\begin{array}{ccc} H_{n-q}(X, \partial X) & \xrightarrow{\quad r_j \quad} & H_{n-q}(X_j, \partial X_j) \\ & {\scriptstyle i_{j*}} \searrow \quad \swarrow {\scriptstyle \approx\ \mathrm{exc}} & \\ & H_{n-q}(X, \partial X \cup X_{1-j}) & \end{array}$$

(i_j = inclusion).

(3) *Boundary*. $\partial W_q(X, \partial X) = W_q(\partial X)$, for $(X, \partial X) \in \mathbf{A}^n$, $0 \leqslant q \leqslant n-1$.

(4) *Product*. Let $(X, \partial X) \in \mathbf{A}^n$, $(Y, \partial Y) \in \mathbf{A}^m$. Then $W(X \times Y) = W(X) \times W(Y)$, i.e. $W_q(X \times Y) = \sum_{i+j=q} W_i(X) \times W_j(Y)$.

(5) *Normalization*. $W_q(\mathbf{R}P^n) = \binom{n+1}{q} x_{n-q}$, where x_{n-q} is the unique nonzero element in $H_{n-q}(\mathbf{R}P^n; \mathbf{Z}_2)$.

$W_q(X)$ is called the qth, and $W(X)$ the total *Stiefel-Whitney homology class* of the allowable pair $(X, \partial X)$.

Now we prove existence of these elements for $\mathbf{A} = \mathbf{E}_\partial(2)$ and $\mathbf{A} = \mathbf{H}_\partial(2)$.

3.2. THEOREM. *On the class* $\mathbf{E}_\partial(2)$ *of Euler manifolds* mod 2 (*with boundary*), *there exist Stiefel-Whitney homology classes. They are given as homology classes of* mod 2 *Stiefel chains.*

To prove this theorem, we have to check that combinatorially defined homology classes $W(\ )$ of Stiefel chains mod 2 satisfy properties (1)–(5) in 3.1. We do it in the next lemmas.

First note that axioms (1) and (3) are satisfied by 1.6.

3.3. LEMMA. $W(\ )$ *satisfies axiom* (2) *on* $\mathbf{E}_\partial(2)$.

PROOF. To prove the restriction property, let us assume first that $(X_j, \partial X_j) \in \mathbf{E}^n_\partial(2)$, $j = 0, 1$, $Y \subset \partial X_j$ a boundary component, $A_j = \partial X_j \backslash Y$, $X = X_0 \cup_Y X_1$, i.e. $Y_0 = Y_1 = Y$, $f = \mathrm{id}$. Then $(X, A_0 \cup A_1) \in \mathbf{E}^n_\partial(2)$. We want to prove that in the diagram

$$\begin{array}{ccc} H_{n-q}(X, \partial X) & \xrightarrow{\quad r_j \quad} & H_{n-q}(X_j, \partial X_j) \\ & {\scriptstyle i_{j*}} \searrow \quad \swarrow {\scriptstyle \approx\ \mathrm{exc}} & \\ & H_{n-q}(X, \partial X \cup X_{1-j}) \big(\approx H_{n-q}(X, X \backslash \mathring{X}_j)\big) & \end{array}$$

we get $i_{0*} W_q(X) = e_* W_q(X_0) \in H_{n-q}(X, \partial X \cup X_1)$, where e is the inclusion which gives excision. Triangulate X by K, such that K_0, K_1, L_0, L_1, L are subcomplexes which triangulate X_0, X_1, A_0, A_1, Y, respectively. Consider the difference

$$s_{n-q}(K', L'_0 \cup L'_1) - s_{n-q}(K'_0, L'_0 \cup L').$$

It is carried by $X_1 \subset \partial X \cup X_1$. Indeed, a simplex $\langle \underline{\sigma}_0 \cdots \underline{\sigma}_q \rangle$ in this difference which actually occurs is in $[K' - (L'_0 \cup L')] - [K'_0 - (L'_0 \cup L')] = K'_1 - L'_1$. But $\langle \underline{\sigma}_0 \cdots \underline{\sigma}_q \rangle \in K'_1 - L'_1$ implies $\sigma_q \in K_1 - L_1$, and hence $\langle \underline{\sigma}_0 \cdots \underline{\sigma}_q \rangle$ is in $X_1 \backslash A_1$. Thus on the homology level, this difference is zero in $H_*(X, \partial X \cup X_1)$. The general case is similar. □

It is worth mentioning a few corollaries.

3.4. Corollary (Additivity). *Let* $(X_j, \partial X_j) \in \mathbf{E}^n_\partial(2), j = 0, 1$. *Then*

$$W(X_0 \amalg X_1) = W(X_0) + W(X_1).$$

Proof. Here $Y = Y_0 = Y_1 = \varnothing$. □

3.5. Corollary. *Let* $(X_j, \partial X_j) \in \mathbf{E}^n_\partial(2), j = 0, 1, X_0 \cap X_1 = \partial X_0 = \partial X_1 = Y$ *and* $X = X_0 \cup_Y X_1$. *Then* $\Delta W_q(X) = W_q(Y)$, *where* Δ *is the Mayer-Vietoris boundary given by*:

$$\begin{array}{ccc} H_{n-q}(X) & \overset{i_{0*}}{\rightarrow} & H_{n-q}(X, X_1) \\ \Delta\downarrow & & \uparrow\approx \\ H_{n-q-1}(Y) & \underset{\partial}{\leftarrow} & H_{n-q}(X_0, Y) \end{array}$$

Proof. By Axiom (3) and 3.3:

$$\Delta W_q(X) = \partial r_0 W_q(X) = \partial W_q(X_0, Y) = W_q(Y). \quad □$$

3.6. Corollary. *Let* $(X, \partial X) \in \mathbf{E}^n_\partial(2)$, $A \subset X \backslash \partial X$ *a subpolyhedron of* X *and* N *a regular neighborhood of* A *in* X *with boundary* ∂N. *Let* q: $(X, \partial X) \to (N/\partial N, *)$ *be the natural collapsing map. Then* $q_* W(X) = W(N)$.

Proof. Let $C = X \backslash \mathring{N}$. Then $(C, \partial N \amalg \partial X) \in \mathbf{E}^n_\partial(2)$ (cf. 1.3(e)). Then X is the union of $(N, \partial N)$ and $(C, \partial N \amalg \partial X)$ and we apply 3.3 using the fact that $H_i(N, \partial N) \approx H_i(N/\partial N, *)$ (cf. 4.6). □

3.7. Corollary (Suspension). *Let* $X \in \mathbf{E}^n(2)$, $\chi(X) \equiv 0$, X *connected. Then* $s_* W_q(X) = W_q(\Sigma X)$, *where* s_*: $\tilde{H}_{n-q}(X) \to \tilde{H}_{n-q+1}(\Sigma X)$ *is the suspension isomorphism.*

Proof. $\Sigma X \in \mathbf{E}^{n+1}(2)$ by 1.3(c). Write $\Sigma X = c_0 X \cup c_1 X$ with intersection X. By 3.5 $\Delta W_q(\Sigma X) = W_q(X)$, where:

$$\begin{array}{ccc} \tilde{H}_*(\Sigma X) & \overset{i_{0*}}{\rightarrow} & \tilde{H}_*(\Sigma X, c_1 X) \\ \Delta\downarrow & & \uparrow\approx \\ \tilde{H}_{*-1}(X) & \leftarrow & H_*(c_0 X, X) \end{array}$$

Since $\Delta^{-1} = s_*$, the claim follows. □

As a corollary of 1.2, Axiom (3) and 3.4 we get

3.8. Corollary. *Let* f: $X \to Y$ *be an Euler resolution* (mod 2) *between* $\mathbf{Z}_2$*-Euler manifolds* X, Y. *Then* $f_* W_q(X) = W_q(Y)$. *In particular, the classes* W *are* PL *invariants.*

3.9. Lemma. $W(\ )$ *satisfies the product axiom* (4) *on* $\mathbf{E}_\partial(2)$.

Proof of this fact is given in [11] and [13].

3.10. LEMMA. *$W_{n-p}(\mathbf{R}P^n) = \binom{n+1}{p+1}x_p$, for all p, $0 \leqslant p \leqslant n$, where $0 \neq x_p \in H_p(\mathbf{R}P^n; \mathbf{Z}_2)$.*

PROOF. Let $\Sigma^n = \{(x_1,\ldots,x_{n+1}) \in \mathbf{R}^{n+1} \mid \Sigma_{i=1}^{n+1} |x_i| = 1\}$. Consider the standard triangulation on Σ^n with vertices $v_i^{\pm 1}$ given by

$$v_i^{\pm 1} = (0,\ldots,0,\pm 1,0,\ldots,0) \quad (i\text{th spot}), \qquad i = 1,\ldots,n+1.$$

We order these vertices by $v_i^{\pm 1} < v_j^{\pm 1} \Leftrightarrow i < j$.

Let $\Pi^n = \Sigma^n/\sim$, where $x \sim -x$. Then an (ordered) p-simplex of the regular cell complex $\Pi = \Pi^n$ is of the form $\sigma^p = v_{i_0}^{\varepsilon_0} \cdots v_{i_p}^{\varepsilon_p}$, $1 \leqslant i_0 < \cdots < i_p \leqslant n+1$, $\varepsilon_i \in \{-1, 1\}$ together with the identifications

$$v_{i_0}^{\varepsilon_0} \cdots v_{i_p}^{\varepsilon_p} = v_{i_0}^{-\varepsilon_0} \cdots v_{i_p}^{-\varepsilon_p}.$$

Let $K = \Pi'$ be the first barycentric subdivision of Π. Although $W_{n-p}(\mathbf{R}P^n)$ is represented by the sum of all p-simplices of K', it is not hard to prove that it is, in fact, represented by the sum of all p-simplices of K. (E.g., define a map $f\colon K' \to K$ on a vertex $\underline{s}$ of K' as $f(\underline{s}) = \underline{\sigma}_p$, where $s = \langle \underline{\sigma}_0 \cdots \underline{\sigma}_p \rangle \in K$ and extend it linearly. Then f is simplicial, induces identity on homology, $\chi(f^{-1}y) \equiv 1$ for all $y \in |K|$, and hence $W_{n-p}(K') = W_{n-p}(K)$.)

Now let $x \in H^1(\mathbf{R}P^n; \mathbf{Z}_2)$ be the generator. We prove that

$$(*) \qquad \left\langle x^p, W_{n-p}(K) \right\rangle = \binom{n+1}{p+1}.$$

This formula clearly implies Lemma 3.10. We shall prove $(*)$ on the cochain level, constructing a cochain c^p on K representing x^p, such that $(*)$ holds. First let d^p be a cochain on Π defined by

$$\left\langle d^p, v_{i_0}^{\varepsilon_0} \cdots v_{i_p}^{\varepsilon_p} \right\rangle \equiv \begin{cases} 1, & \text{if } \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_p \text{ alternate,} \\ 0, & \text{otherwise.} \end{cases}$$

It follows easily that d^p represents the generator in cohomology and $d^p \cup d^q = d^{p+q}$. Now a simplex in K is of the form $\langle \underline{\sigma}_0 \cdots \underline{\sigma}_p \rangle$, where $\sigma_0 < \sigma_1 < \cdots < \sigma_p \in \Pi$. For $k = 0, 1, \ldots, p$ let

$$\sigma_k = v_{k(0)}^{\varepsilon_k(0)} v_{k(1)}^{\varepsilon_k(1)} \cdots v_{k(|\sigma_k|)}^{\varepsilon_k(|\sigma_k|)}, \qquad |\sigma_k| = \dim \sigma_k.$$

Define

$$\left\langle c^p, \left\langle \underline{\sigma}_0 \cdots \underline{\sigma}_p \right\rangle \right\rangle = \left\langle d^p, v_{0(|\sigma_0|)}^{\varepsilon_0(|\sigma_0|)}, \ldots, v_{p(|\sigma_p|)}^{\varepsilon_p(|\sigma_p|)} \right\rangle = \begin{cases} 1, & \text{if } \varepsilon_{0(|\sigma_0|)}, \ldots, \varepsilon_{p(|\sigma_p|)} \text{ alternate,} \\ 0, & \text{otherwise.} \end{cases}$$

Since $W_{n-p}(K)$ is represented by $\Sigma_{\sigma_0 < \cdots < \sigma_p \in \Pi} \langle \underline{\sigma}_0 \cdots \underline{\sigma}_p \rangle$, we will get as many units in the sum $\Sigma_{\sigma_0 < \cdots < \sigma_p \in \Pi} \langle c^p, \langle \underline{\sigma}_0 \cdots \underline{\sigma}_p \rangle \rangle$ as there are increasing sequences $i_0 < i_1 < \cdots < i_p$ between 1 and $n+1$, i.e. $\binom{n+1}{p+1}$. This proves $(*)$ and hence Lemma 3.10. □

For a different proof of this lemma see [**3**] or [**9**]. So, 3.2 is proved.

Now we turn to the subclass $\mathbf{H}_\partial(2)$ of $\mathbf{E}_\partial(2)$ of homology manifolds mod 2. We prove the existence of S.W. classes on a bit more general class $P\mathbf{E}_\partial(2)$ of Euler manifolds mod 2 which are also Poincaré duality spaces with $\mathbf{Z}_2$ coefficients. Note that $P\mathbf{E}_\partial(2)$ is not an allowable class. On this class of spaces, the Wu class v_i is defined by

$$\langle \mathrm{Sq}^i x, [X]\rangle = \langle v_i \cup x, [X]\rangle, \quad \text{for all } x \in H^{n-i}(X, \partial X; \mathbf{Z}_2),$$

so that $v(X) = v = 1 + v_1 + \cdots$, $v_i \in H^i(X; \mathbf{Z}_2)$, $n = \dim X$.

3.11. Theorem. *On the class* $P\mathbf{E}_\partial(2)$, *and hence on the class* $\mathbf{H}_\partial(2)$, *there exist Stiefel-Whitney homology classes. They are given by* $W(X) = \mathrm{Sq}\, v(X) \cap [X]$.

Proof. Let us check that $\mathrm{Sq}\, v(X) \cap [X]$ satisfies axioms (1)–(5) in 3.1.

Axiom (1). Clearly

$$W_k(X) = \sum_{i+j=k} \left(\mathrm{Sq}^i v_j(X) \cap [X]\right) \in H_{n-k}(X, \partial X; \mathbf{Z}_2),$$

$W_0(X) = \mathrm{Sq}^0 1 \cap [X] = [X]$, for $(X, \partial X) \in P\mathbf{E}^n_\partial(2)$. Let us prove that $\varepsilon W_n(X) \equiv \chi(X)$. If n is odd, then $\chi(X) \equiv 0$, and since $\mathrm{Sq}^i x = 0$ for $i < p$, $x \in H^p(X)$ and $v_i = v_i(X) = 0$ for $i > n/2$, it follows that $\varepsilon W_n(X) = 0$. Now let n be even, say $n = 2p$. Then

$$W_n(X) = \varepsilon\left(\left(\underset{\substack{\parallel\\0}}{\mathrm{Sq}^{2p}1} + \cdots + \underset{\substack{\parallel\\0}}{\mathrm{Sq}^{p+1}v_{p-1}} + \mathrm{Sq}^p v_p + \underset{\substack{\parallel\\0}}{\mathrm{Sq}^{p-1}v_{p+1}} + \cdots + \underset{\substack{\parallel\\0}}{\mathrm{Sq}^0 v_{2p}}\right) \cap [X]\right)$$
$$= \varepsilon\left(v_p^2 \cap [X]\right) = \left\langle v_p^2, [X]\right\rangle.$$

Now recall that a $\mathbf{Z}_2$-Poincaré complex X^n has a unique "diagonal" cohomology class $U \in H^n(X \times X)$ s.t. $U \cap ([X] \times [X]) = d_*[X]$, where $d\colon X \to X \times X$ is the diagonal mapping.

Claim 1. Let $\{e_1, \ldots, e_r\}$ be a basis for the $\mathbf{Z}_2$-vector space $H_*(X)$ and $\{e_1^*, \ldots, e_r^*\}$ the dual basis, i.e., $\langle e_i^*, e_j\rangle = \delta_{ij}$. Then for $a \in H_n(X)$,

$$d_*(a) = \sum_{i,j} \langle e_i^* \cup e_j^*, a\rangle \cdot e_i \times e_j = \sum_j (e_j^* \cap a) \times e_j.$$

Proof of Claim 1. Let $d_*(a) = \Sigma c_{ij} e_i \times e_j$. Then

$$c_{ij} = \left\langle e_i^* \times e_j^*, d_*(a)\right\rangle = \left\langle d^*(e_i^* \times e_j^*), a\right\rangle = \left\langle e_i^* \cup e_j^*, a\right\rangle.$$

Also, $\langle e_i^* \cup e_j^*, a\rangle = \langle e_i^*, e_j^* \cap a\rangle$ and hence $d_*(a) = \Sigma_j (e_j^* \cap a) \times e_j$.

Claim 2. $\langle U \cup U, [X] \times [X]\rangle = \langle U, d_*[X]\rangle \equiv \chi(X)$.

Proof of Claim 2. Let $\{e_i\}_{i=1,\ldots,s}$ be a homogeneous basis for the graded vector space $H_*(X)$ and $\{e_i^*\}$ its dual basis. For $i \leq s$, let $u_i \in H^*(X)$ be the homogeneous element with $u_i \cap [X] = e_i$. Then $U = \Sigma e_i^* \times u_i$ and

$$U \cap ([X] \times [X]) = \sum (e_i^* \cap [X]) \times (u_i \cap [X])$$
$$= \sum_i (e_i^* \cap [X]) \times e_i = d_*[X].$$

Hence

$$\langle U \cup U, [X] \times [X]\rangle = \langle U, d_*[X]\rangle = \text{Euler characteristic of } (H^*(X) \cap [X])$$
$$= \text{Euler characteristic of } X.$$

Now recall that $v_p \cap [X] = \overline{\mathrm{Sq}}_p X$, where $\overline{\mathrm{Sq}}$ is the adjoint operator of Sq, i.e., $\langle \beta, \overline{\mathrm{Sq}}\, \alpha \rangle = \langle \mathrm{Sq}\, \beta, \alpha \rangle$. So we have

$$\begin{aligned}\left\langle v_p^2, [X] \right\rangle &= \left\langle d^*(v_p \times v_p), [X] \right\rangle = \left\langle v_p \times v_p, d_*[X] \right\rangle \\ &= \left\langle v_p \times v_p, U \cap ([X] \times [X]) \right\rangle \\ &= \left\langle U, (v_p \cap [X]) \times (v_p \cap [X]) \right\rangle \\ &= \left\langle U, \overline{\mathrm{Sq}}\, [X \times X] \right\rangle = \langle \mathrm{Sq}\, U, [X] \times [X] \rangle \\ &= \langle U \cup U, [X] \times [X] \rangle = \left\langle U, d_*[X] \right\rangle \equiv \chi(X).\end{aligned}$$

Axiom (2) (*Restriction*). Let $X = (X_0, \partial X_0) \cup_Y (X_1, \partial X_1)$, where $Y \subset \partial X_j$ is a component of the boundaries, $A_j = \partial X_j \backslash Y$, $\partial X = A_0 \cup A_1$. Let $e\colon (X_0, \partial X_0) \hookrightarrow (X, X_1 \cup A_0)$, $i\colon (X, \partial X) \hookrightarrow (X, X_1 \cup A_0)$ be inclusions. Clearly, $(X_j, \partial X_j) \in P\mathbf{E}^n_\partial(2) \Rightarrow (X, \partial X) \in P\mathbf{E}^n_\partial(2)$ and $\partial[X]/A_j = \partial[X_j]/A_j$ (see 2.3). We want to prove that in the diagram

$$\begin{array}{ccc} H_*(X, \partial X) & \xrightarrow{r_0} & H_*(X_0, \partial X_0) \\ & {}_{i_*}\searrow \quad \swarrow{}_{e_*}^{\approx} & \\ & H_*(X, X_1 \cup A_0) & \end{array}$$

$i_*(\mathrm{Sq}\, v(X) \cap [X]) = e_*(\mathrm{Sq}\, v(X_0) \cap [X_0])$. To do this, we need a different description of Stiefel-Whitney cohomology classes w of $\mathbf{Z}_2$-Poincaré space. Namely, if $(Z, \partial Z)$ is such a space, $w(Z) = w(\nu_Z^{-1})$, where ν_Z is a normal fibre space over Z, and ν_Z^{-1} its stable inverse (a fibre of ν_Z is a $\mathbf{Z}_2$-homology sphere; $w(\xi)$ for such a "spherical fibration" ξ^k is defined as $\Phi^{-1}\mathrm{Sq}\, U_\xi$, where $U_\xi \in H^k(T(\xi); \mathbf{Z}_2)$ is the Thom class of ξ). As Adams proved in **[1]**, the Wu theorem still holds for $\mathbf{Z}_2$-Poincaré spaces (in fact, for every $\mathbf{Z}_2$-Poincaré duality algebra). Therefore in our situation (assuming X_0, X_1 connected),

$$w(\nu_{X_j}) \cup \mathrm{Sq}\, v(X_j) = 1, \qquad w(\nu_X) \cup \mathrm{Sq}\, v(X) = 1.$$

Now starting with ν_X, we may assume that e pulls back ν_X to ν_{X_0} and that $i^*\nu_X = \nu_X$. So, $e^*w(\nu_X) \cup \mathrm{Sq}\, v(X_0) = 1$. Taking Poincaré duals ($\cap[X_0]$) on both sides and applying e_*, by naturality of $\cap$-product, we get

$$(1) \qquad w(\nu_X) \cap e_*(\mathrm{Sq}\, v(X_0) \cap [X_0]) = e_*[X_0].$$

Similarly, we get

$$(2) \qquad w(\nu_X) \cap i_*(\mathrm{Sq}\, v(X) \cap [X]) = i_*[X].$$

Now, $e_*[X_0] = i_*[X]$ so that the left-hand sides of (1) and (2) are equal, and then multiplying by $w(\nu_X^{-1})$ and using $w(\nu_X) \cup w(\nu_X^{-1}) = 1$, we get the desired equality.

Axiom (3) (*Boundary*). Let $j\colon \partial X \hookrightarrow X$ be inclusion. Let $v = v(X, \partial X) \in H^*(X; \mathbf{Z}_2)$ be the Wu class of X.

$$\begin{aligned}\partial W_p(X, \partial X) &= \partial\Big(\sum_{k+l=p} \mathrm{Sq}^k v_l \cap [X]\Big) \quad \text{by stability of } \cap \\ &= \big(j^* \textstyle\sum \mathrm{Sq}^k v_l\big) \cap [X] \\ &= \textstyle\sum \mathrm{Sq}^k j^* v_l \cap [\partial X] \quad \text{by } (*) \text{ below} \\ &= \sum_{k+l=p} \mathrm{Sq}^k v_{l-1}(\partial X) \cap [\partial X] \quad \text{by definition} \\ &= W_{p-1}(\partial X).\end{aligned}$$

$$(*) \qquad j^* v_l(X) = v_{l-1}(\partial X).$$

To prove $(*)$, by definition of v's, it is enough to prove that for any $x \in H^{n-l-1}(\partial X)$, the following holds:

$$(**) \qquad \langle \mathrm{Sq}^l x, [\partial X]\rangle = \langle x \cup j^* v_l, [\partial X]\rangle.$$

But the right-hand side is

$$\begin{aligned}\langle x \cup j^* v_l, \partial[X]\rangle &= \langle \delta(x \cup j^* v_l), [X]\rangle \quad \text{by stability of } \cup\text{-product} \\ &= \langle \delta x \cup v_l, [X]\rangle \quad \text{by definition of } v \\ &= \langle \mathrm{Sq}^l(\delta x), [X]\rangle = \langle \delta \mathrm{Sq}^l x, [X]\rangle = \langle \mathrm{Sq}^l x, [\partial X]\rangle.\end{aligned}$$

This proves $(**)$, so $(*)$ and hence Axiom (3).

Axiom (4) (*Product*). This follows by the Cartan formula, multiplicativity of the $\cap$-product and the fact that $v(X \times Y) = v(X) \times v(Y)$. The last fact can be proved as follows. Since X, Y are compact and the coefficients are $\mathbf{Z}_2$, by Künneth $H^*(X \times Y) \approx H^*X \otimes H^*Y$ via $x \otimes y \mapsto x \times y$. Write $z \in H^*(X \times Y)$ as $z = x \times y$, $x \in H^*X$, $y \in H^*Y$ and (assuming X, Y connected)

$$\mathrm{Sq}\, x = v(X) \cup x, \quad \mathrm{Sq}\, y = v(Y) \cup y.$$

Multiplying these relations we get

$$\mathrm{Sq}(x \times y) = (v(X) \times v(Y)) \cup (x \times y),$$

or

$$\mathrm{Sq}\, z = (v(X) \times v(Y)) \cup z \quad \text{for all } z \in H^*(X \times Y).$$

Axiom (5) (*Normalization*). On smooth manifolds X, $\mathrm{Sq}\, v(X)$ is the S.W. cohomology class $w(X)$ (Wu theorem), and since we know $w(\mathbf{R}P^n) = (1 + x)^{n+1}$, it follows that

$$W_p(\mathbf{R}P^n) = \text{Poincaré dual of } w_p(\mathbf{R}P^n) = \binom{n+1}{p} x_{n-p},$$

where $0 \neq x_{n-p} \in H_{n-p}(\mathbf{R}P^n)$.

Theorem 3.11 is proved. □

4. Uniqueness of Stiefel-Whitney homology classes on allowable classes. In this section we prove the axiomatic characterization of Stiefel-Whitney homology classes on any allowable class.

4.1. THEOREM. *On any allowable class* $\mathbf{A}$ *there is at most one class* $W(\ )$ *satisfying axioms* (1)–(5) *in* 3.1.

In proving the uniqueness, the main ingredients are block bundle transversality and a Thom-Sullivan interpretation of cohomology as certain morphisms from PL-bordism into $\mathbf{Z}_2$, associated to a given cohomology characteristic class of PL manifolds, a construction often used in surgery. We use some ideas from [13].

If $\xi = (E \supset B)$ is a q-block bundle and $f\colon (P, \partial P) \to (E, \partial E)$ a map transverse to ξ, we write $f \perp \xi$. For a treatment of transversality see [13] or [14].

Denote by $\mathfrak{N}_n^{\mathrm{PL}}(A, B)$ the nth nonoriented bordism group based on PL manifolds of the polyhedral pair (A, B), and let $w(\ )$ denote Stiefel-Whitney cohomology classes of PL manifolds. Then there is a bijection (Thom-Sullivan)

$$\{\alpha \in \mathrm{Hom}\big(\mathfrak{N}_*^{\mathrm{PL}}(A, B), \mathbf{Z}_2\big) \mid \alpha[V_0 \times V_1, f_1 \circ \mathrm{pr}_2] \equiv \chi(V_0)\alpha[V_1, f_1],\ \partial V_0 = \varnothing,\ \dim V_0 > 0\} \leftrightarrow \{z \in H^*(A, B; \mathbf{Z}_2)\}.$$

The correspondence is given by $\alpha[V, f] = \langle f^*z \cup w(V), [V]\rangle$; cf. [13] or [5, Appendix].

4.2. LEMMA. *Let* $\mathbf{A}$ *be an allowable class,* $X \in \mathbf{A}^n$, $(M, \partial M)^{n+p}$ *a* PL *manifold, and* $\varphi\colon X \hookrightarrow \mathrm{Int}\, M$ *a* PL *embedding. Then there is a unique homomorphism* $\chi_X\colon \mathfrak{N}_{p+i}^{\mathrm{PL}}(M, \partial M) \to \mathbf{Z}_2$ *with the following property: if* $f\colon (V, \partial V)^{p+i} \to (M, \partial M)$ *is a map from a* PL *manifold* $(V, \partial V)$ *with* $f \perp X$ *and transverse intersection* T, *then* $\chi_X[V, f] \equiv \chi(T)$ (mod 2).

PROOF. Every element in $\mathfrak{N}_{p+i}^{\mathrm{PL}}(M, \partial M)$ can be represented by $f\colon (V, \partial V)^{p+i} \to (M, \partial M)$ transverse on X. Then the transverse intersection $T \in \mathbf{A}^i$, by 2.1(c), (d). Now, let $f'\colon (V', \partial V') \to (M, \partial M)$ be bordant to (V, f), i.e. determines the same element in $\mathfrak{N}_{p+i}^{\mathrm{PL}}(M, \partial M)$, $f' \perp X$. Then the associated transverse intersection T' is equal to T in $\mathfrak{N}_i^{\mathbf{A}}$. It is easy to see that 2.1(b) implies $\chi(T) \equiv \chi(T')$. So χ_X is well defined, clearly unique, and it is easy to check that it is a homomorphism of groups. □

Furthermore, by the transversality theorem it follows that χ_X has the property

$$\chi_X[V_0 \times V_1, f_1 \circ \mathrm{pr}_2] \equiv \chi(V_0)\chi_X[V_1, f_1], \quad \text{where } \partial V_0 = \varnothing$$

and

$$V_0 \times (V_1, \partial V_1) \xrightarrow{\mathrm{pr}_2} (V_1, \partial V_1) \xrightarrow{f_1} (M, \partial M).$$

Now, applying the above 1-1 correspondence on $\alpha = \chi_X$, the following definition is justified.

4.3. DEFINITION. Let $X \in \mathbf{A}^n$, $(M^{n+p}, \partial M)$ a PL manifold and $\varphi\colon X \hookrightarrow M$ a fixed PL embedding of X into the interior of M. Denote by $\tau_i(\varphi) \in H^{p+i}(M, \partial M; \mathbf{Z}_2)$ the *unique* class such that for any map $f\colon (V, \partial V) \to (M, \partial M)$ from a PL manifold $(V, \partial V)$ we have

$$\chi_X[V, f] \equiv \langle f^*\tau(\varphi) \cup w(V), [V]\rangle,$$

where $\tau(\varphi) = \tau_0(\varphi) + \tau_1(\varphi) + \cdots \in H^*(M, \partial M; \mathbf{Z}_2)$ is called (total) *transversality class* of the embedding φ, and $w(\ )$ is the Stiefel-Whitney cohomology class on PL manifolds.

Now we begin with the proof of 4.1—the uniqueness of the classes $W(\ )$ on the fixed allowable class **A**. The idea is to show axiomatically that S.W. classes determine and are determined by the transversality classes. Then by the uniqueness of the transversality classes, the uniqueness of the S.W. classes follows.

More precisely, we shall show (axiomatically) that whenever $W(\ )$ satisfies axioms (1)–(5) on **A**, the following relationship between S.W. classes and transversality classes hold: Let $X \in \mathbf{A}^n$ and embed X in some $\mathbf{R}^{n+p}$, and let N be a regular neighborhood with boundary ∂N, φ: $X \hookrightarrow N$ an embedding into the interior of N. Let $D_N = \cap[N]$: $H^{p+*}(N, \partial N) \to H_*(N)$ be the Poincaré duality for N ($\mathbf{Z}_2$ coefficients). Then

$$\chi_X[V, f] \equiv \left\langle f^*\left(D_N^{-1}\varphi_* W(X)\right) \cup w(N), [N]\right\rangle$$

for all singular manifolds f: $(V, \partial V) \to (N, \partial N)$.

We now start proving this relationship. Let $\xi^k = (E \supset B)$ be a PL block bundle, and g: $Y \to B$ a map from $Y \in \mathbf{A}^p$. Let $V = E(g^*\xi)$. Then by 2.1(d), $(V, \partial V) \in \mathbf{A}^{p+k}$, and by definition of pull-back this construction gives rise to a homomorphism ψ_ξ: $\mathfrak{N}_p^{\mathbf{A}}(B) \to \mathfrak{N}_{p+k}^{\mathbf{A}}(E, \partial E)$ defined by $\psi_\xi[Y, g] = [V, f]$, where f is a natural extension of g. By ξ-transversality theorem **[13]**, it follows that ψ_ξ is an isomorphism.

4.4. Lemma. *Suppose W satisfies axioms in* 3.1. *Let ξ^k be a block bundle over B. Then there is a unique class $\bar{\tau}(\xi) = \bar{\tau}_0(\xi) + \bar{\tau}_1(\xi) + \cdots$, $\bar{\tau}_i(\xi) \in H^{k+i}(E, \partial E)$, such that for any map f: $(V, \partial V) \to (E, \partial E)$, f transverse on ξ, $(V, \partial V) \in \mathbf{A}$, $\chi(f^{-1}B) \equiv \langle \bar{\tau}(\xi), f_* W(V)\rangle$ holds.*

Proof. We shall construct $\bar{\tau}_i(\xi)$'s inductively on i. First note that $\bar{\chi} \circ \psi_\xi^{-1}$: $\mathfrak{N}_k^{\mathbf{A}}(E, \partial E) \to \mathbf{Z}_2$ is zero on decomposable elements, and this is because ψ_ξ is an isomorphism of $\mathfrak{N}_*^{\mathbf{A}}$-modules, and so is ψ_ξ^{-1}. Now one applies 2.1(e), ξ-transversality theorem and Axiom (1) to get $\bar{\tau}_0(\xi)$.

Now suppose we have constructed $\bar{\tau}_j(\xi)$ for all $j < p$. Consider the homomorphism α: $\mathfrak{N}_{p+k}^{\mathbf{A}}(E, \partial E) \to \mathbf{Z}_2$ defined by

$$\alpha[V, f] = \bar{\chi} \circ \psi_\xi^{-1}[V, f] + \sum_{j<p} \left\langle \bar{\tau}_j(\xi), f_* W_{k+j}(V)\right\rangle.$$

(It is, indeed, a homomorphism by additivity—special case of Axiom (2).) By Axioms (4) and (1) and by inductive hypothesis it follows that α is zero on decomposable elements and hence it defines a homomorphism $H_{p+k}(E, \partial E) \to \mathbf{Z}_2$ which determines $\bar{\tau}_p(\xi)$. Applying the above construction to the family (V_i, f_i) of "singular **A**-manifolds", such that $\{f_{i_*}[V_i]\}$ generates $H_*(E, \partial E)$ (which exist by 2.1(e)), we get the uniqueness of $\bar{\tau}$. □

Let $\xi^k = (E \supset B)$ be a block bundle and denote by $\phi_\xi^*\colon H^*(B) \to H^{k+*}(E, \partial E)$ the Thom isomorphism for ξ. Define $\bar{w}(\xi) \in H^*(B)$ by $\bar{\tau}(\xi) = \phi_\xi^*(\bar{w}(\xi))$, or on components $\bar{w}_i(\xi) \in H^i(B)$ by $\bar{\tau}_i(\xi) = \phi_\xi^*(\bar{w}_i(\xi)) \in H^{k+i}(E, \partial E)$.

Then it is a matter of checking that $\bar{w}(\ )$ satisfies the following properties:

(a) If $g\colon B' \to B$, then $\bar{w}(\xi') = g^*\bar{w}(\xi)$, where $\xi' = g^*(\xi)$.

(b) $\bar{w}(\xi \oplus \eta) = \bar{w}(\xi) \cup \bar{w}(\eta)$, where $\xi \oplus \eta$ is the Whitney sum of block bundles ξ, η over B.

(c) If γ_n^1 is the canonical line bundle over $\mathbf{R}P^n$, then $\bar{w}(\gamma_n^1) = 1 + x + \cdots + x^n$, where $0 \neq x \in H^1(\mathbf{R}P^n)$.

To prove (a), let $E' = E(\xi')$, and let $\bar{g}\colon (E', \partial E') \to (E, \partial E)$ be the canonical extension of g. If $f\colon (V, \partial V) \to (E', \partial E')$ is transverse to ξ', then $\bar{g} \circ f$ is transverse to ξ. Now two classes $\bar{g}^*\bar{\tau}(\xi)$ and $\bar{\tau}(\xi')$ both satisfy Lemma 4.4, and then by the uniqueness of $\bar{\tau}$ it follows that $\bar{g}^*\bar{\tau}(\xi) = \bar{\tau}(\xi')$ and hence (by naturality of Thom isomorphism) we get (a).

To prove (b) it suffices to show the following: if ξ and ξ' are block bundles over B and B', respectively, and $\xi \times \xi'$ their (external) product over $B \times B'$, then $\bar{w}(\xi \times \xi') = \bar{w}(\xi) \times \bar{w}(\xi')$. To show this, it is enough to check that $\bar{\tau}(\xi) \times \bar{\tau}(\xi')$ satisfies the formula in Lemma 4.4 for "singular A-manifolds" of the form $(V_i, f_i) \times (V_j', f_j')$ where $\{f_{i*}[V_i]\}$ generates $H_*(B)$ and $\{f'_{j*}[V_j']\}$ generates $H_*(B')$. But this follows at once by multiplicativity of χ and Axiom (4).

To prove (c) consider the Thom space $T(\gamma_n^1) \approx \mathbf{R}P^{n+1}$. Then $\{j_*[\mathbf{R}P^q]\}_q$ generates $H_*(\mathbf{R}P^{n+1})$, where $j\colon \mathbf{R}P^q \hookrightarrow \mathbf{R}P^{n+1}$ are canonical inclusions. Note that $j \simeq Tj'$, where Tj' is an induced map on the Thom space-level of $j'\colon \mathbf{R}P^{q-1} \hookrightarrow \mathbf{R}P^{n+1}$ and $Tj' \perp \gamma_n^1$. Now (c) follows from the uniqueness of $\bar{\tau}$, the fact that $x = U_{\gamma_n^1}$ (= Thom class of γ_n^1) $\in H^1(\mathbf{R}P^{n+1})$ and

$$(*) \qquad \chi(\mathbf{R}P^{q-1}) = \left\langle x + x^2 + \cdots + x^{n+1}, j_*W(\mathbf{R}P^q)\right\rangle.$$

To prove $(*)$, observe that $\chi(\mathbf{R}P^{q-1}) \equiv q \pmod 2$. The right-hand side of $(*)$ is

$$\left\langle j^*x + \cdots + j^*x^q + j^*x^{q+1} + \cdots + j^*x^{n+1},\ W(\mathbf{R}P^q)\right\rangle$$
$$= \sum_{i=0}^{q-1} \left\langle x^{q-i}, W_i(\mathbf{R}P^q)\right\rangle = \sum_{i=0}^{q-1} \binom{q+1}{i} = 2^{q+1} - 2 - q \equiv q \pmod 2$$

by Axiom (5). So (a)–(c) are proved.

Note that if ξ is a vector bundle, then $\bar{w}(\xi) = (w(\xi))^{-1}$ = dual cohomology Stiefel-Whitney classes of ξ, because (a)–(c) characterize $(w(\xi))^{-1}$ for vector bundles ξ (cf. [**16**, p. 86]). So we have proved

4.5. LEMMA. *If ξ is a vector bundle over B, then $\bar{w}(\xi)$ agrees with $(w(\xi))^{-1}$ in $H^*(B)$.* □

4.6. LEMMA. *Let $(X, \partial X)$, $(Y, \partial Y) \in \mathbf{A}^n$ such that $Y \subset X \backslash \partial X$. Let Z be the complement of the interior of Y, $\partial Z = \partial X \amalg \partial Y$ and $q\colon (X, \partial X) \to (Y/\partial Y, *)$ the collapsing map. Then $(Z, \partial Z) \in \mathbf{A}^n$ and $q_*W(X) = W(Y)$.*

Proof. By the cutting property 2.1(a) it follows that $(Z, \partial Z) \in \mathbf{A}^n$. Since $(Y, \partial Y)$ is a polyhedral pair we have the natural isomorphism $H_*(Y, \partial Y) \approx H_*(Y/\partial Y, *)$, so that by the Restriction Axiom 3.1(1), and by commutativity of the diagram

$$\begin{array}{ccc} H_*(X,\partial X) & \xrightarrow{\;r\;} & H_*(Y,\partial Y) \approx H_*(Y/\partial Y, *) \\ & \searrow \qquad\qquad \swarrow_{\approx}^{\text{exc}} & \\ & H_*(X, \partial X \cup Y) & \end{array} \qquad q_* : H_*(X,\partial X) \to H_*(Y/\partial Y,*)$$

Lemma 4.6 follows. □

4.7. **Lemma.** *Let $W(\ )$ satisfy Axioms 3.1 on $\mathbf{A}$. Let ξ^k be a vector bundle on a polyhedron B, $f\colon (X, \partial X) \to (E(\xi), \partial E(\xi))$ transverse on ξ, where $(X, \partial X) \in \mathbf{A}^n$. Let $Y^{n-k} = f^{-1}(B)$, $g = f/Y$ and N a regular neighborhood of Y in X such that N is the total space of $\eta = g^*\xi$. Then $(N, \partial N) \in \mathbf{A}^n$ and*

$$\chi(Y) \equiv \left\langle \phi_\eta^*\left(w(\eta)^{-1}\right), W(N) \right\rangle.$$

Proof. $(N, \partial N) \in \mathbf{A}$ follows from the transversality theorem and 2.2(1). By Lemma 4.4

$$\chi(Y) \equiv \langle f^*\bar{\tau}(\xi), W(X) \rangle.$$

Now let $q\colon (X, \partial X) \to (N/\partial N, *)$ be the natural collapsing map. Since ξ is a vector bundle, it follows by Lemma 4.5 that $\bar{\tau}(\xi) = \phi_\xi^*(w(\xi)^{-1})$ and hence

$$\begin{aligned} f^*\bar{\tau}(\xi) &= f^*\phi_\xi^*\left(w(\xi)^{-1}\right) && \text{by the commutative diagram below} \\ &= q^*\phi_\eta^*\left(g^*\left(w(\xi)^{-1}\right)\right) && \text{by naturality of } w(\ )^{-1} \\ &= q^*\phi_\eta^*\left(w(\eta)^{-1}\right). \end{aligned}$$

Now Lemma 4.7 follows from Lemma 4.6: $q_* W(X) = W(N)$. The diagram

$$\begin{array}{ccc} & H^*(X, \partial X) & \\ {\scriptstyle q^*}\nearrow & & \nwarrow{\scriptstyle f^*} \\ H^*(N, \partial N) & \longleftarrow & H^*(E, \partial E) \\ \phi_\eta^* \uparrow & & \uparrow \phi_\xi^* \\ H^*(Y) & \xleftarrow[\;g^*\;]{} & H^*(B) \end{array}$$

commutes. □

4.8. **Proposition.** *Let $W(\ )$ satisfy Axioms* 3.1 *on an allowable class* $\mathbf{A}$. *Let* $X \in \mathbf{A}^n$ *be embedded in a euclidean space* $\mathbf{R}^{n+p}$ (p *big enough*). *Let* N *be its regular neighborhood and* φ: $X \hookrightarrow N$ *a* PL *embedding. Then*

$$\tau(\varphi) = D_N^{-1}\varphi_* W(X) \in H^*(N, \partial N),$$

where D_N: $H^{p+*}(N, \partial N) \to H_*(N)$ *is the Poincaré duality and* $\tau(\varphi)$ *the transversality class.*

Proof. We prove first $\tau_i(\varphi) = D_N^{-1}\varphi_* W_i(X)$, for $i < n/2$. It suffices to prove

$$\chi_X[V, f] \equiv \left\langle f^*\left(D_N^{-1}\varphi_* W(X)\right) \cup w(V), [V]\right\rangle$$

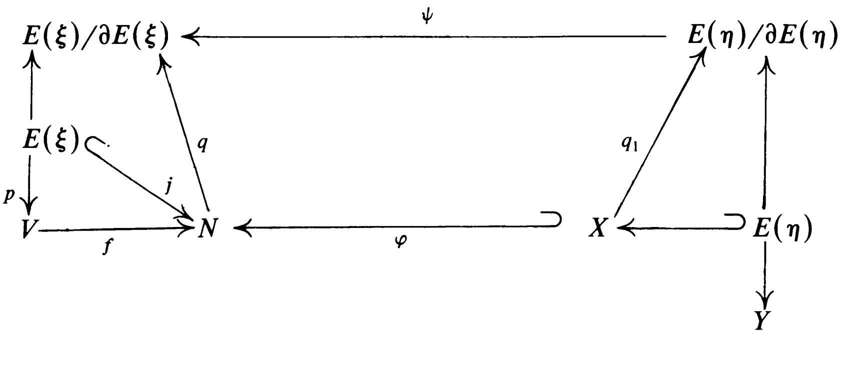
for all (V, f) for which $\{f_*[V]\}$ generates $\oplus_{j<n/2} H_{p+j}(N, \partial N)$. Let $x \in H_{p+i}(N, \partial N) \approx H^{n-i}(N) \approx H^{n-i}(X)$, $i < n/2$ and denote $k = n - i$. Since $n < 2k$, x can be represented by f_1: $X \to T\gamma_k$, where γ_k is the universal vector bundle over G_k and $f_1 \perp \gamma_k$. Let $Y^i = f_1^{-1}(G_k)$ and let η be a normal bundle on Y in X, $E(\eta)$ its total space. Let q_1: $X \to E(\eta)/\partial E(\eta)$ be the collapsing map and choose a deformation retraction r: $N \to X$ such that $q_1 \circ r$: $N \to E(\eta)/\partial E(\eta)$ is transverse to η on a neighborhood of Y. This gives rise to a PL manifold $(V, \partial V)^{p+i}$ and an embedding f: $(V, \partial V) \to (N, \partial N)$, $f \perp X$, with the transverse intersection Y and $f_*[V] = x \in H_{p+i}(N, \partial N)$.

Let ξ be a normal (disk) bundle of f, $E(\xi)$ its total space, and $\partial E(\xi)$ the total space of the associated sphere bundle. Let j: $(E(\xi), \partial E(\xi) \cap N) \hookrightarrow (N, \partial N)$ be inclusion and q: $N \to E(\xi)/\partial E(\xi)$ Thom's map for ξ. Note that on homology q_* acts as $D_{E(\xi)} j^* D_N^{-1}$ (i.e. like "Umkehrung"). Since N is parallelizible, for the tangent bundles we have $T_V \oplus \xi = T_N/V$ and so $w(V) = w(\xi)^{-1}$; hence

$$\left\langle f^*\left(D_N^{-1}\varphi_* W(X)\right) \cup w(V), [V]\right\rangle$$
$$= \left\langle f^*\left(D_N^{-1}\varphi_* W(X)\right) \cup w(\xi)^{-1}, p_*\left(U_\xi \cap [E(\xi)]\right)\right\rangle,$$

where U_ξ is the Thom class of ξ, p its projection.

Now the following diagram commutes (up to homotopy):



where unnamed arrows are natural projections, and ψ is induced by inclusion $(E(\eta), \partial E(\eta)) \hookrightarrow (E(\xi), \partial E(\xi))$. Let us continue our computations:

$$\begin{aligned}
&\left\langle f^*\left(D_N^{-1}\varphi_* W(X)\right) \cup w(\xi)^{-1}, p_*\left(U_\xi \cap [E(\xi)]\right)\right\rangle \\
&\quad= \left\langle p^*\left[f^*\left(D_N^{-1}\varphi_* W(X)\right) \cup w(\xi)^{-1}\right] \cup U_\xi, [E(\xi)]\right\rangle \\
&\quad= \left\langle j^* D_N^{-1}\varphi_* W(X) \cup p^* w(\xi)^{-1} \cup U_\xi, [E(\xi)]\right\rangle \\
&\quad= \left\langle \phi_\xi^*\left(w(\xi)^{-1}\right), j^*\left(D_N^{-1}\varphi_* W(X)\right) \cap [E(\xi)]\right\rangle \\
&\quad= \left\langle \phi_\xi^*\left(w(\xi)^{-1}\right), q_*\varphi_* W(X)\right\rangle \quad \text{by the diagram above} \\
&\quad= \left\langle \phi_\xi^*\left(w(\xi)^{-1}\right), \psi_* q_{1*} W(X)\right\rangle \quad \text{by Lemma 4.6} \\
&\quad= \left\langle \psi^*\phi_\xi^*\left(w(\xi)^{-1}\right), W(E(\eta))\right\rangle \quad \text{by naturality of Thom isomorphism} \\
&\quad= \left\langle \phi_\eta^*\left(w(\eta)^{-1}\right), W(E(\eta))\right\rangle \quad \text{by 4.7 } (\eta \approx (f_1/Y)^*\gamma_k \text{ is a vector bundle}) \\
&\quad\equiv \chi(Y) \equiv \chi_X[V, f].
\end{aligned}$$

So by the uniqueness of the classes $\tau(\varphi)$, it follows that

$$\tau_i(\varphi) = D_N^{-1}\varphi_* W_i(X) \quad \text{for } i < n/2.$$

Now the general case follows replacing $X^n \times S^K$ (K big enough) and using Axioms (1) and (4): $W(X^n \times S^K) = W(X) \times [S^K]$. Now we prove uniqueness. Let X be a closed **A**-space, i.e. $\partial X = \varnothing$. If $W(\ )$ and $W'(\ )$ satisfy our axioms then by the above construction,

$$D_N^{-1}\varphi_* W(X) = D_N^{-1}\varphi_* W'(X) \Rightarrow W(X) = W'(X).$$

If $(X, \partial X) \in \mathbf{A}^n$, we form the double $2X \in \mathbf{A}^n$ and then by (iterated) Axiom (2): $rW(2X) = W(X, \partial X)$, $rW'(2X) = W'(X, \partial X)$ and since for closed **A**-spaces, $W(2X) = W'(2X)$, we get finally $W(X, \partial X) = W'(X, \partial X)$. So the uniqueness, and hence Theorem 4.1, is proved. □

4.9. Remarks. (1) Instead of Axiom (3) as stated, we only need the following weaker

Axiom (3′). Let $(X, \partial X) \in \mathbf{A}^n$ and $P \subset \partial X$ be a subpolyhedron. Let

$$\partial X \underset{j}{\hookrightarrow} (\partial X, P) \underset{k}{\hookrightarrow} (X, P)$$

be inclusions. Then $(kj)_* W(\partial X) = 0 \in H_*(X, P)$. Clearly, Axiom (3) $\Rightarrow$ Axiom (3′).

(2) Let $\overline{H}_*$ denote simplicial homology based on locally finite, possibly infinite chains. Then all the results can be carried over to the corresponding theory using $\overline{H}_*$ instead of H_*.

Now there are several Stiefel conjecture-type consequences of our axioms. On smooth manifolds the first detailed proof of it appears in **[10]**, considering smooth vector fields on a manifold. Since then some other proofs are known; e.g. **[3]** and **[13]**. From our considerations the proof of the Stiefel conjecture for allowable classes of smooth or PL manifolds goes as follows: Let M^n be such a manifold, $\varphi: M \hookrightarrow N$ the zero section of the normal disk bundle

$$\nu = (N^{n+p}, r, M), \quad \phi_\nu^*: H^*(M) \to H^{p+*}(N, \partial N)$$

the Thom isomorphism. Then by the uniqueness of $\tau(\varphi)$, $\tau(\varphi) = \phi_\nu^*(w(M))$ and hence

$$W(M) = r_* D_N \tau(\varphi) = r_* D_N \phi_\nu^*(w(M)) = D_M(w(M)).$$

Another proof of the Stiefel conjecture for smooth (or PL) manifolds is to show first that our axioms imply Blanton and Schweitzer's axioms **[4]** on smooth manifolds. Then by their uniqueness and by our existence part (i.e. that combinatorially defined classes satisfy our axiom, see §3) the Stiefel conjecture follows.

To prove that our axioms imply Blanton and Schweitzer's, we just have to check Axiom (2) of Blanton and Schweitzer. But this follows as Theorem 1 in **[3]** from Propositions 1 and 2 and Lemma 1. The role of Lemma 1 plays our Axiom (3).

Taking for allowable class the class $\mathbf{H}_\partial(2)$, we finally get the following

4.10. COROLLARY. *On the class of* $\mathbf{Z}_2$*-homology manifolds, combinatorially defined homology Stiefel-Whitney classes agree with Poincaré duals of Stiefel-Wu classes* (*i.e.* $W(X) = \mathrm{Sq}\, v(X) \cap [X]$). *In particular*, $W(X)$ *is a homotopy invariant.*

It can be shown (McCrory, unpublished) that Halperin's conjecture (cf. **[8**, p. 112]) implies also our Corollary 4.10. The same result was obtained by L. Taylor **[19]** by using some techniques of Quinn.

We conjecture that the analogue of Corollary 4.10 is true for the class $P\mathbf{E}(2)$ of Euler manifolds mod 2 which are Poincaré duality spaces with $\mathbf{Z}_2$-coefficients. (One problem is that we do not know anything about regular neighborhoods of subpolyhedra in these spaces.)

At the end we mention only that by analogy with "tangential" S.W. classes, we can define normal (or dual) S.W. homology classes $\overline{W}(X) \in H_*(X; \mathbf{Z}_2)$ for $X \in \mathbf{A}^n$, where $\mathbf{A}$ is an allowable class as follows.

Let $\varphi: X^n \hookrightarrow N^{n+p}$ be an embedding into a euclidean regular neighborhood (p big enough) and $r: N \to X$ a deformation retraction. Define $\overline{W}_i(X) = r_* D_N \mathrm{Sq}^i \tau_0(\varphi) \in H_{n-i}(X; \mathbf{Z}_2)$. Since $\tau_0(\varphi) = D_N^{-1} \varphi_* W_0(X)$, it follows that

$$\overline{W}_i(X) = r_* D_N \mathrm{Sq}^i D_N^{-1} \varphi_* [X].$$

In case X is a smooth manifold, $\overline{W}_i(X) = D_X \overline{w}_i(X)$, i.e. Poincaré dual of the normal Steifel-Whitney cohomology class. A combinatorial formula for these normal classes is given in **[2]**.

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