ASYMPTOTIC BEHAVIOUR AND PROPAGATION PROPERTIES
OF THE ONE-DIMENSIONAL FLOW OF GAS
IN A POROUS MEDIUM

BY
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Abstract. The one-dimensional porous media equation \( u_t = (u^m)_{xx}, m > 1 \), is considered for \( x \in \mathbb{R}, t > 0 \) with initial conditions \( u(x,0) = u_0(x) \) integrable, nonnegative and with compact support. We study the behaviour of the solutions as \( t \to \infty \) proving that the expressions for the density, pressure, local velocity and interfaces converge to those of a model solution. In particular the first term in the asymptotic development of the free-boundary is obtained.

0. Introduction. Suppose we have a certain distribution of gas whose density at time \( t = 0 \) is given by a function \( u_0(x) \) of one spatial direction \( (x \in \mathbb{R}) \). If the gas flows through a homogeneous porous medium the density \( u = u(x, t) \) at time \( t > 0 \) is governed by the equation

\[
\frac{\partial u}{\partial t} = (u^m)_{xx}
\]

for \( x \in \mathbb{R} \) and \( t > 0 \); \( m \) is a physical constant, \( m > 1 \), and we have scaled out other physical constants (see [1] for a physical derivation). \( u \) satisfies the initial condition

\[
u(x,0) = u_0(x)
\]

where \( u_0 \) satisfies the following assumptions:

\[
u_0 \in L^1(\mathbb{R}), \quad u_0 \geq 0, u_0 \not\equiv 0,
\]

and \( u_0 \) is compactly supported, i.e. if \( \Omega_0 = \{ x \in \mathbb{R} : u_0(x) > 0 \} \) we have

\[
\inf \Omega_0 < -\infty, \quad \sup \Omega_0 < \infty
\]

Sticking to the above application we define the pressure by \( p = mM^{m-1}/(m - 1) \) on \( Q = \mathbb{R} \times (0, \infty) \) and the local velocity by \( V = -v_x \) on the domain of dependence

\[
\Omega = \Omega[u] = \{ (x, t) \in Q : u(x, t) > 0 \}.
\]

The total mass at time \( t > 0 \) is \( M(t) = \int u(x, t) \, dx \) and the center of mass is \( x(t) = M(t)^{-1} \int u(x, t) \, dx \). Set \( M_0 = \int u_0(x) \, dx \) and \( x_0 = M_0^{-1} \int u_0(x) x \, dx : M_0 > 0 \) and \( a_1 < x_0 < a_2 \). \( l_0 = a_2 - a_1 \) measures the dispersion of the initial data.

Much is already known for problem (0.1)–(0.4); see [19] for a survey of results up to 1980, where the \( n \)-dimensional case is considered, \( n \geq 1 \). In particular (0.1)–(0.4)
admits a unique continuous weak solution \( u(x, t) \geq 0 \) [18], [3], such that for \( t > 0 \), \( u(\cdot, t) \) has compact support [14]. Thus \( \Omega(t) = \{ x \in \mathbb{R} : u(x, t) > 0 \} \) is bounded for every \( t > 0 \) and two outer interfaces arise with equations \( x = \xi_{i}(t) \), \( i = 1, 2 \), \( t \geq 0 \), where

\[
\begin{align*}
\xi_1(t) &= \inf \Omega(t) \quad \text{if } t > 0, \quad \xi_1(0) = a_1, \\
\xi_2(t) &= \sup \Omega(t) \quad \text{if } t > 0, \quad \xi_2(0) = a_2.
\end{align*}
\]

As a consequence of the inequality [10]

\[
u_i \geq -\frac{u}{(m + 1)t},
\]

the set \( \{ \Omega(t) : t > 0 \} \) is ordered by inclusion: \( \Omega(t') \supset \Omega(t) \) if \( t' > t \) so that \( (-1)^{\xi_i} \) is a nondecreasing function. Moreover there exist \( t^*_i > 0 \) (called waiting-times) such that for \( 0 \leq t \leq t^*_i \), \( \xi_i(t) = a_i \) [16] and, for \( t > t^*_i \), \( \xi_i(t) \) is continuously differentiable and \( (-1)^{\xi_i(t)} > 0 \) (once the interface starts to move it never stops) [11], [16] proves that when \( t \to \infty \) \( (-1)^{\xi_i}(t) \) behaves like \( t^{1/(m + 1)} \). These results are proved under the simplifying hypotheses that \( u_0 \) is continuous, \( u_0(x) > 0 \) for every \( x \in I = (a_1, a_2) \) and vanishes outside \( I \), but the proofs apply under conditions (0.3), (0.4). On the contrary under only these two conditions the property that \( \Omega(t) = (\xi_1(t), \xi_2(t)) \) does not hold in general, i.e. inner interfaces can appear that make \( \Omega(t) \) disconnected for some time interval \( 0 < t < T, T > 0 \).

We shall be concerned in this paper with the following question: Give significant information about the behaviour of the solutions to (0.1)--(0.4) in terms of a simple information on \( u_0 \), specifically in terms of \( M_0, x_0, a_1, a_2 \). Our contribution deals with the asymptotic behaviour of density, pressure, velocity and free-boundaries and on the global properties of \( \Omega \).

To describe the large-time behaviour we take as model solutions the class of explicit self-similar solutions corresponding to an initial “instantaneous source” given by Barenblatt in 1952 [6], i.e. solutions of (0.1) with initial data

\[
u_0(x) = M\delta(x - a)
\]

where \( M > 0, a \in \mathbb{R} \) and \( \delta \) is Dirac’s delta function. The unique weak solution \( \widetilde{u}(x, t; M, a) \) is given in terms of its pressure by

\[
\tilde{\nu}(x, t; M, a) = \left[ r(t)^2 - (x - a)^2 \right]^{\frac{1}{2(m + 1)t}}
\]

where

\[
r(t) = c_m(M^{m-1}t)^{1/(m+1)}
\]

with

\[
c_m = \left\{ \frac{2m(m + 1)}{m - 1} B\left( \frac{m}{m - 1}, \frac{1}{2} \right) \left( \frac{m}{m - 1} \right)^{1-m} \right\}^{1/(m+1)},
\]

\(B(\cdot, \cdot)\) being Euler’s Beta function (\([s]_+\) means \( \max(s, 0) \)). Its interfaces are the strictly monotone \( C^\infty \)-curves given by \( \xi_i(t) = (-1)^{r(t)} + a \), \( i = 1, 2 \), and the velocity is defined on \( \Omega[\tilde{u}] = \{(x, t) \in Q : |x - a| < r(t)\} \) by \( \tilde{V}(x, t) = (x - a) \cdot ((m + 1)t)^{-1} \).

Let \( \tilde{u}(x, t; M) = \tilde{u}(x, t; M, 0) \); then \( \tilde{u}(x, t; M, a) = \tilde{u}(x - a, t; M) \).
Our main result shows to what extent the solution $u(x, t)$ to (0.1)–(0.4) resembles the self-similar $\bar{u} = \bar{u}(x, t; M_0, x_0)$ with same mass and center of mass.

**Theorem A.** Let $u(x, t)$ be the solution to (0.1)–(0.4) and let $\bar{u} = \bar{u}(x, t; M_0, x_0)$. Then:

(i) for every $t \geq T^* = (l_0/c_m)^{m+1}M_0^{1-m}$, $\Omega(t)$ is the open interval $(\zeta_i(t), \zeta_2(t))$ with strictly expanding borders;

(ii) as $t \to \infty$ we have

\[
\begin{align*}
(0.10) & \quad (-1)^i(\zeta_i(t) - \bar{\zeta}_i(t)) \to 0; \\
(0.11) & \quad \frac{\zeta_i'(t)}{\bar{\zeta}_i'(t)} \to 1 \quad \text{and} \quad t | \zeta_i'(t) - \bar{\zeta}_i'(t) | \to 0; \\
(0.12) & \quad t \left| \nabla(x, t) - \frac{x - x_0}{(m + 1)t} \right| \to 0 \quad \text{uniformly in} \ x \in \Omega(t), \\
(0.13) & \quad t^{m/(m+1)} | v(x, t) - \bar{v}(x, t) | \to 0 \quad \text{uniformly in} \ x \in \mathbb{R}
\end{align*}
\]

and for every $x \in \mathbb{R}$, $t > 0$

\[
(0.13)' \quad v(x, t) \leq \max_{x \in \mathbb{R}} \bar{v}(x, t) = c_m^2(2(m + 1))^{-1} (M_0^{2m+1}(m-1)/(m+1))
\]

We may write (0.10), (0.11) as giving the first term in the asymptotic development of $\zeta_i(t)$ and $\bar{\zeta}_i(t)$:

\[
\begin{align*}
(0.14) & \quad \zeta_i(t) = x_0 + (-1)^i c_m M_0^{(m-1)/(m+1)} t^{1/(m+1)} + o(1), \\
(0.15) & \quad \bar{\zeta}_i(t) = (-1)^i (c_m/(m + 1)) M_0^{(m-1)/(m+1)} t^{m/(m+1)} + o(1/t)
\end{align*}
\]

where $o(1)$ and $o(1/t)$ are the usual Landau $o$’s taken as $t \to \infty$. Theorem A shows that $M_0$ and $x_0$ are the only relevant initial data in the first approximation to the large-time behaviour of the solutions to (0.1)–(0.4). In particular (0.13) implies for $1 < m =< 2$ the estimate

\[
(0.16) \quad t^{2/(m+1)} | u(x, t) - \bar{u}(x, t) | \to 0
\]

uniformly in $x \in \mathbb{R}$. If $m > 2$, however, (0.16) holds uniformly in $x: |x - x_0| < \alpha r(t; M_0)$ for every $0 < \alpha < 1$ and we obtain $t^\alpha | u(x, t) - \bar{u}(x, t) | \to 0$ at $t \to \infty$ uniformly in $x \in \mathbb{R}$ for $\sigma = m/(m^2 - 1)$.

As a precedent to these results Kamin [15] proved the convergence of $u$ towards a self-similar $\bar{u}$ with equal mass with an estimate

\[
(0.17) \quad t^{1/(m+1)} | u(x, t) - \bar{u}(x, t) | \to 0
\]

that does not allow for the characterization of $x_0$. Friedman and Kamin [13] extend (0.17) to dimensions $n \geq 1$. Several terms of the asymptotic representation of $u$ were stated in [7] without proof.

We begin by reviewing in §1 several properties of the solutions. In particular we prove the time-invariance of the mass and the center of mass, i.e. for every $t > 0$, $M(t) = M_0$, $x(t) = x_0$ (Lemma 1.1).

In §2 we introduce a comparison principle, based on the evaluation of masses, that we name “Shifting-Comparison Principle” (Sh.C.P.) (Lemma 3.2). As immediate corollaries we derive the estimate for the free boundaries $(-1)^i(\zeta_i(t) r^{-1/(m+1)}) \to c_m M_0^{(m-1)/(m+1)}$, which improves Knerr’s result [16], and the estimate in Theorem
A(i), where we remark that \( T^* \) is optimal in terms of \( M_0 \) and \( l_0 \) as an upper bound for both the occurrence of waiting-times and that of an inner free-boundary.

§3 is devoted to proving Theorem A. As a main ingredient we use a sharp version of Caffarelli and Friedman’s [11] differential inequality for the interfaces that in fact gives the monotonicity of \( \zeta'(t) t^{m/(m+1)} \) (Lemma 3.1).

The case where \( u_0 \) is a symmetric function is considered in §4. Then \( u(x, t) \) is symmetric with respect to \( x \) and we prove optimal rates of convergence in the results of Theorem A by means of a new “Concentration-Comparison Principle” (Theorem B).

Finally §5 considers the right interface \( \xi(t) \) of a solution of (0.1), (0.2) with \( u_0 \) satisfying (0.3) and, instead of (0.4),
\[
\text{ess sup}_t \Omega_0 = 0.
\]
The behaviour of \( \xi \) as \( t \to \infty \) and \( t \to 0 \) is investigated as well as its dependence on the \( L^p \)-norm of the initial data \( u_0 \), \( 1 \leq p \leq \infty \).

Let us remark that the asymptotic behaviour of the porous medium equation in bounded domains of \( \mathbb{R}^n \) has been studied recently by Aronson and Peletier [5].

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1. Preliminaries.

1.1 Existence of solutions. We begin by reviewing the existence and properties of weak solutions to (0.1), (0.2). It is known [3] that for every \( u_0 \in L^1(\mathbb{R}), u_0 \geq 0 \), there exists a unique continuous function \( u = u(x, t) \) in \( Q = \mathbb{R} \times (0, \infty) \) with the following properties:
\[
\begin{align*}
(i) \quad & u \in C([0, \infty); L^1(\mathbb{R})) \cap L^\infty(\mathbb{R} \times [\delta, \infty)) \text{ for every } \delta > 0, \\
(ii) \quad & u_t = (u^m)_{xx} \text{ in the sense of distributions on } Q, \\
(iii) \quad & u(0) = u_0 \text{ in } L^1(\mathbb{R}).
\end{align*}
\]

Here \( u(t) \) denotes the element \( u(\cdot, t) \) in \( L^1(\mathbb{R}) \). Moreover \( u_t \) and \( (u^m)_{xx} \) exist a.e., \( u_t \in L^1_{\text{loc}}(Q) \) satisfies (0.7) and
\[
\|u_t(\cdot, t)\|_1 \leq \frac{2}{(m+1)t}\|u_0\|_1.
\]

To obtain the solution we may approximate \( u_0 \) by a decreasing sequence of strictly positive, smooth functions \( u^n_0 \), apply to \( u^n_0 \) the existence theorem of [18] and derive (1.1) in the limit.

In §2 we shall need an alternative approach: we discretize (0.1) in time and have recourse to Crandall and Liggett’s Generation Theorem (see [12]). In fact given a continuous increasing function \( \phi: \mathbb{R} \to \mathbb{R}, \phi(0) = 0 \), the operator \( A = A_\phi \) defined on
\[
D(A) = \{ u \in L^1(\mathbb{R}); \phi(u) \in W^{1,\infty}(\mathbb{R}) \text{ and } \phi(u)_{xx} \in L^1(\mathbb{R}) \}
\]
by
\[
Au = -\phi(u)_{xx} \text{ if } u \in D(A)
\]
is \( m\)-accretive in \( L^1(\mathbb{R}) \), i.e. the resolvent \( (I + \lambda A)^{-1} \) is a contraction on \( L^1(\mathbb{R}) \) for any \( \lambda > 0 \) [9], and the closure of \( D(A) \) in \( L^1(\mathbb{R}) \) is \( L^1(\mathbb{R}) \) [10]. Hence the formula

\[
S(t)u_0 = \lim_{n \to \infty} \left( I + \left( t/n \right) A \right)^{-n}u_0 \quad \text{for} \quad t \geq 0, \quad u_0 \in L^1(\mathbb{R}),
\]

defines a semigroup of contractions \( S(t) = S_0(t), \ t \geq 0, \) in \( L^1(\mathbb{R}) \). Bénilan has proved [8] that \( u(x, t) = (S(t)u_0)(x) \) solves in a generalized sense (called integral or mild sense) the evolution problem

\[
\begin{cases}
    u_t = \phi(u)_{xx}, & x \in \mathbb{R}, \ t > 0, \\
    u(x, 0) = u_0(x) \in L^1(\mathbb{R}),
\end{cases}
\]

and that these mild solutions are unique.

Setting \( \phi(t) = s|s|^{m-1} \) we recover problem (0.1), (0.2) and both constructions give the same unique solution satisfying (1.1).

Since for every \( f_i \in L^1(\mathbb{R}), \ i = 1, 2, \) and \( \lambda > 0 \) we have [9]

\[
\left\| \left[ (I + \lambda A)^{-1}f_1 - (I + \lambda A)^{-1}f_2 \right] \right\|_1 \leq \left\| \left[ f_1 - f_2 \right] \right\|_1,
\]

using (1.5) we obtain for every \( u_i \in L^1(\mathbb{R}), \ i = 1, 2,
\]

\[
\left\| \left[ S(t)u_1 - S(t)u_2 \right] \right\|_1 \leq \left\| \left[ u_1 - u_2 \right] \right\|_1.
\]

(1.7) implies obvious comparison results for the solutions of (0.1), (0.2).

**Remark.** Since the solutions \( \bar{u}(x, t; M, a) \) are limits of solutions of (0.1) with initial data \( u_0^\alpha \geq 0, u_0^\alpha \in L^1(\mathbb{R}) \), such that \( u_0^\alpha \to M\delta(x) \) in the \( *\)-topology of \( \mathcal{M}(\mathbb{R}) \), the space of bounded Radon measures (take \( u_0^\alpha(x) = \bar{u}(x, 1/n; M, a) \)), the comparison results valid for solutions with \( L^1 \)-data apply also to \( \bar{u} \).

1.2. Group of transformations. Equation (0.1) admits the biparametric group of transformations

\[
\begin{align*}
    \hat{u} &= ku, \\
    \hat{x} &= L^{-1}x, \\
    \hat{t} &= k^{1-m}L^{-2}t,
\end{align*}
\]

i.e. if \( u(x, t) \) is a solution of (0.1) with initial condition \( u_0(x) \) then for every \( k, L > 0, \) \( \hat{u}(x, t) \) defined by

\[
\hat{u}(x, t) = ku(Lx, L^2k^{m-1}t)
\]

is a solution with initial condition \( \hat{u}_0(x) = ku_0(Lx) \). We write \( \hat{u} = T_{K,L}u \). The transformation \( T_{K,L} \) preserves the (initial) mass iff \( k = L \).

We can use the group of transformations \( T_{K,L} \) to reduce a problem (0.1), (1.2) to a simple one. Thus if \( u(x, t) \) is such that \( \int u_0(x) \, dx = M \) and we put \( u = T_{M,1}\hat{u} \) we have \( \hat{M} = \int \hat{u}_0(x) \, dx = 1 \). By means of \( T_{M,1} \) our conclusions on \( \hat{u} \) apply to \( u \). For instance the respective free boundaries \( \xi(t) \) and \( \hat{\xi}(t) \) are related by

\[
\xi(t) = \hat{\xi}(M^{m-1}t).
\]

Notice that the self-similar solutions \( \bar{u}(x, t; M) \) centered at \( a = 0 \) are invariant under \( T_{k,k}, \ k > 0, \) i.e.

\[
\bar{u}(x, t; M) = k\bar{u}(kx, k^{m+1}t; M).
\]
1.3. Two invariants. We establish here the invariance of the total mass $M$ and the center of mass $x_c$ for the solutions of (0.1)–(0.4):

**Lemma 1.1.** For every $t > 0$ $M(t) = M_0$ and $x_c(t) = x_0$.

**Proof.** Assume first that $u_0$ is also continuous, positive on $I = (a_1, a_2)$ and such that $u_0(x) > |x - a_i|^{1/(m-1)}$ in a neighbourhood of both $a_1$ and $a_2$ (so that the waiting-times $t_1^*$ and $t_2^*$ vanish, see [16]). Then $u \in C^\infty(\Omega)$ and $v \in C^1(\Omega \cap Q)$, see [11]. It follows that $u^m \in C^1(\Omega \cap Q)$ and $(u^m)_x$ vanishes on both interfaces.

Take now two arbitrary times $t_2 > t_1 > 0$ and set $G = \{(x, t): t_1 < t < t_2\}$. Then

$$\int_{\Omega} u(x, t_2) dx - \int_{\Omega} u(x, t_1) dx = \int_{G} u_x dx dt
= \int_{t_1}^{t_2} dt \left[(u^m)_x (\xi_2(t), t) - (u^m)_x (\xi_1(t), t)\right] = 0,$$

$$\int_{\Omega} u(x, t_2) dx - \int_{\Omega} u(x, t_1) dx = \int_{G} u_x dx dt = \int_{t_1}^{t_2} dt \left[x(u^m)_x - u^m\right]_{(\xi_1(t), t)}^{(\xi_2(t), t)} = 0,$$

and the result follows. For general $u_0$ approximate by a decreasing sequence $\{u^n\}$ as above and pass to the limit using the $L^1$-continuity of the map $u_0 \mapsto u(t)$ (formula (1.7)).

**Remarks.** (1) The result is valid in a much more general context: for instance for the solutions of $(P_\phi)$, without the restriction of nonnegativity.

(2) The invariance of the total mass has been widely used in connection with this problem: [10, 15]. The invariance of the center of mass has been pointed out in [7].

1.4. Regularity up to the interfaces. We know that the solutions are classical in $\Omega$. [1] proves that $v(x, t)$ is Lipschitz-continuous in $x$ in $Q_\tau = \mathbb{R} \times (\tau, \infty)$ for every $\tau > 0$. But $v_x$ need not be continuous at the interfaces (check the self-similar solutions). However [16] proves that $v_x(\xi(t), t)$ exists for every $t > 0$ as the limit of $v_x(x, t), x \to \xi'(t), x \in \Omega(t)$ and

$$\xi'(t) \equiv -v_x(\xi(t), t) = \xi(t)$$

where $\xi(t)$ is the right derivative of $\xi$ at $t$. [11] proves that $v_x$ is continuously differentiable up to the boundary $x = \xi(t)$ if $t > t^*_x$.

2. Comparison by shifting.

2.1. We introduce in this section a “Shifting-Comparison Principle” that allows us to compare a solution with given initial condition with the one corresponding to a displaced initial condition. To measure the relative displacement we use the corresponding distribution functions defined by

$$U(x, t) = \int_{-\infty}^{x} u(x, t) dx = \text{amount of mass in } (-\infty, x].$$

The idea behind the principle is that it is more feasible to compare masses than to compare point densities. The principle is in fact a maximum principle for the “integrated” equation $U_t = (U_x | U_x |^{m-1})_x$.
We prove the principle in an elliptic version. Then (1.5) allows us to derive the
evolution version.

**Lemma 2.1 (Shifting Comparison Principle. Elliptic Version).** Let $\beta$ be a
continuous nondecreasing function such that $0 = \beta(0) \subset \text{Int} \beta(\mathbb{R})$ and let $f_i, i = 1, 2,$ be
integrable functions such that for every $x \in \mathbb{R},$

$$\int_{-\infty}^{x} f_i(x) \, dx \leq \int_{-\infty}^{x} f_i'(x) \, dx.$$

Let $u_i$ be the solution (E): $-u'' + \beta(u) = f_i,$ with $u_i \in W^{1,\infty}(\mathbb{R})$ and $w_i = \beta(u_i) \in
L^1(\mathbb{R})$ (see [9]). Then for every $x \in \mathbb{R},$

$$\int_{-\infty}^{x} w_i(x) \, dx \leq \int_{-\infty}^{x} w_i'(x) \, dx.$$

**Proof.** Set $F_i(x) = \int_{-\infty}^{x} f_i(x) \, dx$ and $W_i(x) = \int_{-\infty}^{x} w_i(x) \, dx.$ Assume that (2.3)
does not hold so that $G = \{ x \in \mathbb{R} : W_1(x) > W_2(x) \}$ is nonvoid. Let $I = (a, b),
-\infty < a < b \leq \infty,$ be a maximal interval in $G.$ For every $x \in I$ we have by
integration of (E):

$$u_1'(x) = W_1(x) - F_1(x) > W_2(x) - F_2(x) = u_2'(x)$$

so that $u_1 - u_2$ is strictly increasing on $I.$

Assume now that $a > -\infty.$ Then by continuity $W_1(a) = W_2(a)$ and $u_1(a) > u_2(a)$
(if $u_1(a) < u_2(a)$ we would have $u_1(x) < u_2(x)$ if $|x - a| < \epsilon$ for an $\epsilon > 0$ so that
$w_1(x) \leq w_2(x)$, hence $W_1 - W_2$ is nonincreasing in $a - \epsilon < x < a + \epsilon,$ contradicting
the definition of $a$). By (2.4) we have $u_1 > u_2$ on $I,$ so that $W_1 - W_2$ is
nondecreasing on $I.$ This implies that $b = \infty$ and $W_1(\infty) > W_2(\infty).$ But this
contradicts the fact that $F_1(\infty) \leq F_2(\infty)$ and that $W_i(\infty) = f_i(\infty)$ for the solutions
of (E) (see [9, formula (4.3)]).

If $a = -\infty,$ $W_i(-\infty) = 0$ and $u_1(-\infty) \geq u_2(-\infty)$ by the preceding argument.
Hence the same conclusion holds. #

**Remark.** It is clear that the proof of Lemma 2.1 applies to much more general
situations. In particular it is true for the solutions of

$$-\frac{d}{dx} A(x, u'(x)) + B(x, u(x)) = f(x)$$

where $A$ and $B$ are, say, increasing in $u$ and continuous in $x,$ and $f \in L^1(\mathbb{R}).$

Besides it holds for suitable Dirichlet or Neumann boundary conditions if (2.5) is
posed in a bounded interval or a half-line.

Similar remarks apply to Lemma 2.2 to follow. #

The change of variables $w = \beta(u), u = \phi(w)$ transforms $-u'' + \beta(u) = f$
into $-\phi'(w) w'' + w = f,$ that can be written as $w = (I + A_\phi)^{-1}(f).$ Therefore setting $\phi(s) = s |s|^{m-1},$ i.e. $\beta(s) = s |s|^{(1-m)/m},$ we derive via (1.5) the following evolution
version for the solutions of (0.1), (0.2):

**Lemma 2.2 (Shifting-Comparison Principle. Parabolic version).** Let $u^1(x, t),
u^2(x, t)$ be solutions of (0.1), (0.2) with initial data $u^1_0(x), u^2_0(x) \in L^1(\mathbb{R}).$ If for every
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we have

\[ \int_{-\infty}^{x} u_0'(x) \, dx \leq \int_{-\infty}^{x} u_0^2(x) \, dx, \quad \text{i.e.} \quad u_0'(x) \leq u_0^2(x), \]

then for every \( t > 0 \) and every \( x \in \mathbb{R} \),

\[ \int_{-\infty}^{x} u'(x, t) \, dx \leq \int_{-\infty}^{x} u^2(x, t) \, dx, \quad \text{i.e.} \quad u'(x, t) \leq u^2(x, t). \]

Remarks. (1) We say that the mass of \( u^1 \) is shifted to the right with respect to that of \( u^2 \) at time \( t = 0 \); this situation is preserved for every \( t > 0 \).

(2) As said in §1.1 we may consider initial data of the form \( M \delta(x - x_0) \).

2.2. First applications. An easy application of the Sh.C.P., comparing with self-similar solutions that concentrate all the mass \( M_0 \) of \( u_0 \) at the extreme points \( x = a_1 \) or \( x = a_2 \), allows us to bound the right and left interface of \( u \) from above and below, giving a first estimate of their asymptotic behaviour. Here and in the sequel we fix \( M = M_0 \) in \( r(t) : r(t) = c_m(M_0^{m-1})^{1/(m+1)} \).

Corollary 2.3. For every \( t > 0 \) we have

\[ \frac{a_1 - r(t)}{a_2 - r(t)}, \quad \frac{a_1 + r(t)}{a_2 + r(t)} > \frac{1}{a_2} - r(t), \]

so that

\[ \lim_{t \to \infty} \frac{\xi(t)}{t^{1/(m+1)}} = (-1)^n c_m M_0^{-(m-1)/(m+1)}. \]

Proof. We take \( u'(x, t) = \bar{u}(x, t; M_0, a_1) \) and \( u^2(x, t) = \bar{u}(x, t; M_0, a_2) \). Since we have \( U_0'(x) \geq U_0(x) \geq U_0^2(x) \), we conclude that \( U'(x, t) \geq U(x, t) \geq U^2(x, t) \).

But since the interfaces can be characterized in terms of \( U : \xi(t) = \inf\{x \in \mathbb{R} : U(x, t) > 0\}, \xi_2(t) = \sup\{x \in \mathbb{R} : U(x, t) < M_0\} \), it follows that \( \xi(t) = a_1 - r(t) \leq \xi_1(t) \leq \xi_2(t) = a_2 - r(t) \) and likewise for \( \xi_2(t) \).

We must prove that the inequalities (2.8) are strict: assume for instance that for a \( t_0 > 0, \xi(t_0) = a_2 + r(t_0) \). Since \( \xi(t) = a_2 + r(t) \) for every \( t > 0 \), we have \( \xi(t_0) = r(t_0) \). Take now \( \bar{u}(x, t) = \bar{u}(x - a_2, t; M_0) \). For \( t = t_0 \) we have \( \xi(t_0) = \xi_2(t_0) \).

Using the fact that \( \varphi(t, t) = -\varphi(t, t) \) for any \( t > 0 \) [16], and that \( \varphi_{xx} = 0 - 1/(m + 1) = \varphi_{xx} (\text{in } \mathfrak{G}[\bar{u}]) [3] \), we conclude that, at \( t = t_0 \), we have \( \bar{u}(x, t_0) \geq \bar{u}(x, t_0, t_0) \). Since both have mass \( M_0 \) it follows that \( u = \bar{u} \) at \( t = t_0 \). But this is impossible since \( x_0 < a_2 \) and the center of mass is invariant. #

As a consequence of (2.8) we bound above the waiting-times \( t^*_e \) in terms of \( M_0 = \|u_0\|_1 \) and \( l_0 = a_2 - a_1 \):

Corollary 2.4. We have

\[ t^*_e < T^* \equiv (l_0/c_m)^{m+1} M_0^{1-m}. \]

Proof. For \( t > T^*, \xi_2(t) > a_1 + r(T^*) = a_1 + l_0 = a_2 \). Similarly for \( \xi_1 \). #

Remarks. (1) (2.10) is sharp in terms of \( M_0 \) and \( l_0 \). To see this choose an initial datum \( u_0 \) with two components: one, \( u_0^1(x) \), of mass \( M_0 - \varepsilon, \varepsilon > 0 \) small, supported in \([a_1, a_1 + \varepsilon]\), and the other, \( u_0^2(x) \), of mass obviously \( \leq \varepsilon \), supported in \([a_1 - \varepsilon, a_2]\), and such that the corresponding solution \( u^2(x, t) \) has vertical interfaces for at least a time \( T^* \). Up to the time where the interfaces of the solutions to both
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partial initial data, \( u^1(x, t) \) and \( u^2(x, t) \), meet we have \( u(x, t) = u^1(x, t) + u^2(x, t) \) so that \( \delta(t) = a_2 \). But as \( \varepsilon \to 0 \) this time is easily seen to approach \( T^* \) (use the Sh.C.P.).

(2) For recent work on the determination of the waiting-times see [4 and 17].

We now turn to the existence of an inner free-boundary, \( \Gamma_{in} \). Since we are mainly interested in the large-time behaviour we want to bound above the time at which the inner free-boundary ceases to exist. For that we define on \( \mathbb{R} \) the nonnegative function

\[
\tau(x) = \sup \{ t > 0 : u(x, t) = 0 \}.
\]

It is clear that \( \Omega[u] = \{(x, t) \in Q : t > \tau(x)\} \). We have as a consequence of Corollary 2.4:

**Corollary 2.5.** For every \( x \in [a_1, a_2] \), \( \tau(x) < T^* \) so that

\[
\Gamma_{in} \subset (a_1, a_2) \times (0, T^*).
\]

**Proof.** If \( x = a_1 \) or \( x = a_2 \), \( \tau(a_i) = \tau_i^* \) and we are reduced to (2.10). For any \( \bar{x} \) \( a_1 < \bar{x} < a_2 \) such that \( \tau(\bar{x}) > 0 \), we write \( u_0(x) \) as \( u_0 = u_0^{(1)} + u_0^{(2)} \) with \( u_0^{(1)} = u_0 \chi_{(-\infty, \bar{x}]} \) and \( u_0^{(2)} = u_0 \chi_{(\bar{x}, \infty)} \chi_E \) denoting the characteristic function of a set \( E \subset \mathbb{R} \). Let \( u^{(1)}, u^{(2)} \) be the respective solutions. By comparison \( u(x, t) \geq u^{(i)}(x, t) \) so that \( \tau(\bar{x}) \leq \tau^{(i)}(\bar{x}), i = 1, 2 \).

Now observe that \( \tau^{(1)}(\bar{x}) = \tau_1^*(\bar{x}) \) = the right waiting-time of \( u^{(1)} \) and \( \tau^{(2)}(\bar{x}) = \tau_1^*(\bar{x}) \) = the left waiting-time of \( u^{(2)} \), so that

\[
\tau(\bar{x}) < \left( \frac{\bar{x} - a_1}{c_m} \right)^{m+1} M_1^{1-m} \quad \text{with} \quad M_1 = \int_{a_1}^{\bar{x}} u_0(x) \, dx
\]

and

\[
\tau(\bar{x}) < \left( \frac{a_2 - \bar{x}}{c_m} \right)^{m+1} M_2^{1-m} \quad \text{with} \quad M_2 = M_0 - M_1.
\]

It follows from (2.13), (2.14) that

\[
l_0 = (a_2 - \bar{x}) + (\bar{x} - a_1) > c_m \left( M_1^{(m-1)/(m+1)} + M_2^{(m-1)/(m+1)} \right) \tau(\bar{x})
\]

\[
> c_m M_0^{(m-1)/(m+1)} \tau(\bar{x})
\]

and the result follows.

**Remarks.** (1) (2.12) is sharp: argue as in Remark (1) to Corollary 2.2.

(2) It is not difficult to see that

\[
\Gamma_{in} = \{(x, \tau(x)) : x \in \text{Int}(\Lambda)\} \cup \{(x, t) : x \in \partial \Lambda \cap \Lambda \text{ and } 0 < t < \tau(x)\}
\]

where \( \Lambda = \{(a_1, a_2) : \tau(x) > 0\} \) and that for every maximal open interval \( I \) in \( \Lambda \), \( \Gamma_{in} \cap (I \times (0, \infty)) \) consists of one or two monotone \( C^1 \)-arcs (pieces of interfaces to subsolutions as above).

3. The asymptotic behaviour. This section is devoted to proving Theorem A (ii): the fact that \( \max_{x \in \mathbb{R}} u(x, t) \leq \max_{x \in \mathbb{R}} \tilde{u}(x, t; M_0) \) for every \( t > 0 \) follows easily
from two properties: (i) $u$ and $\tilde{u}$ have the same mass, $M_0$; (ii) $\nu_{xx} \geq -((m + 1)t)^{-1}$ in $\Omega[u]$, $\tilde{\nu}_{xx} = -((m + 1)t)^{-1}$ in $\Omega[\tilde{u}]$.

The rest is based on a precise description of the outer interfaces. We begin by revisiting the second-order differential inequality for the $\xi_i$'s obtained by Caffarelli and Friedman [11]:

**Lemma 3.1.** There exist nonnegative measures $\mu_i$, $i = 1, 2$, on $(0, \infty)$ such that

$$
\xi_1''(t) + \left(\frac{m}{m + 1}t\right)\xi_1'(t) = \mu_1(t)(-1)^i
$$

in the sense of distributions. Hence the expression $(-1)^i\xi_i'(t)t^m/(m + 1)$ is nondecreasing in $(0, \infty)$.

**Remarks.** (1) The coefficient $m/(m + 1)$ is best possible: the self-similar solutions $\tilde{u}(x, t; M, a)$ satisfy (3.1) with $\mu_i = 0$.

(2) Caffarelli and Friedman’s result states (3.1) in the form $f''(t) + k f'(t) = \mu_i(t)(-1)^i$ with a constant $k > 0$. But the specification of $k$ as $m/(m + 1)t$ plays a fundamental role in the sequel for it gives the monotonicity of $\xi'(t)t^m/(m + 1)$.

**Proof.** We review the proof in [11] to point out how $k$ may be replaced by $m/(m + 1)t$.

Let us take the case $i = 2$ and drop the $i$ for simplicity. At a point of the interface $(\tilde{\xi}(t_0), t_0)$ with $t_0 > t^*$, we adapt a self-similar solution $\tilde{u} = \tilde{u}(x - x_1, t; M_1)$ with $x_1, M_1$ so chosen as to have (i) $\tilde{\xi}(t_0) = \tilde{\xi}(t)$ and (ii) $\tilde{\xi}'(t_0) = \tilde{\xi}'(t_0)$, i.e. $r_x(\tilde{\xi}(t_0), t_0) = \tilde{\nu}_{xx}(\tilde{\xi}(t_0), t_0)$. Since $\nu_{xx} \geq -((m + 1)t)^{-1}$ in $\Omega[\tilde{u}]$, it follows for $t > t_0$ that $u(x, t) \geq \tilde{u}(x, t)$. As in [11] we conclude that for $h > 0$,

$$
\xi(t_0 + h) - \xi(t_0) - \xi'(t_0)h \geq \tilde{\xi}(t_0 + h) - \tilde{\xi}(t_0) - \tilde{\xi}'(t_0)h.
$$

Using the fact that for the self-similar solution $\tilde{u}$,

$$
\tilde{\xi}''(t_0) + \left(\frac{m}{m + 1}t_0\right)\tilde{\xi}'(t_0) = 0,
$$

the second-member of (3.2) equals $-\left(h^2/2\right)\left(m/(m + 1)t_0\right)\tilde{\xi}'(t_0) + O(h^3)$.

Now define the function $\Phi_h$ for $h > 0$ fixed by

$$
\Phi_h(t_0) = \frac{\xi(t_0 + h) - \xi(t_0) - h\xi'(t_0)}{h^2/2} \geq -\frac{m}{(m + 1)t_0} \xi'(t_0) + O(h).
$$

One proves as in [11] that a subsequence of $\Phi_h$ converges weakly towards a signed measure so that in the limit (3.5) gives (dropping the zeros)

$$
\xi''(t) + \left(\frac{m}{m + 1}t\right)\xi'(t) = \mu
$$

in the distribution sense in $(t^*, \infty)$ and $\mu$ is nonnegative. Now divide (3.5) by $\xi'(t) > 0$ to get

$$
\log\left(\frac{\xi'(t)t^m/(m + 1)}{\xi'(t)}\right) = \mu(t)/\xi'(t) \geq 0.
$$

Therefore $\xi'(t)t^m/(m + 1)$ is nondecreasing in $(t^*, \infty)$. But since $\xi(t) = a_2$ in $0 \leq t \leq t^*$, the assertions of the lemma hold in $(0, \infty)$. #

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1 If $t = t^*$, $\xi_i(t^*_i)$ means the right derivative $\xi_i(t^*_i)$.  

---

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We recall that for the self-similar solution $u = \bar{u}(x, t; M_0)$, the interfaces are given by $(-1)^{-1} \xi'(t) = r(t) \equiv c_m(M_0^{m-1})^{1/(m+1)}$. Combining Corollary 2.3 and Lemma 3.1 we obtain

**Lemma 3.2.** For $t > t^*$ we have

\[(3.7) \quad (-1)^{-1} \xi'(t) \leq r(t) \quad \text{and} \quad (-1)^{-1} \xi'(t)/r'(t) \uparrow 1 \quad \text{as} \quad t \to \infty.
\]

There exist $b_i$, $i = 1, 2$, such that $a_1 \leq b_i \leq a_2$, and as $t \to \infty$,

\[(3.8) \quad \xi_2(t) - r(t) \downarrow b_2, \quad \xi_1(t) + r(t) \uparrow b_1.
\]

Finally

\[(3.9) \quad \lim_{t \to \infty} |\xi'(t) - (-1)^{1} r'(t)| = 0.
\]

**Proof.** We may consider only the case $M_0 = 1$ and $i = 2$. Also we drop the $i$'s.

Since $\xi'(t) t^{m/(m+1)}$ is nondecreasing (Lemma 3.1) there exists the limit

\[\lim_{t \to \infty} \xi'(t) t^{m/(m+1)} = K \leq \infty.
\]

Since $\lim_{t \to \infty} \xi'(t) t^{-1/(m+1)} = c_m$ (Corollary 2.3) we conclude that $K = c_m/(m + 1)$ and (3.7) is proved.

In particular we have $\xi'(t) \leq r'(t)$ so that $\eta(t) = \xi(t) - r(t)$ is nonincreasing in $t$.

Since $a_1 < \eta(t) < a_2$, (3.8) follows.

To prove (3.9) write (3.1) in the form

\[(3.10) \quad (t \xi')' = \xi'/(m + 1) + t \mu(t).
\]

Integrating in $t$ gives

\[(3.11) \quad t \xi'(t) = \frac{\xi(t) - a}{m + 1} + \int_{0}^{t} t \mu(t) dt.
\]

Let $\xi(t) = \int_{0}^{t} t \mu(t) dt$. (3.11) can be written as $t \eta'(t) = (\eta(t) - a_2)/(m + 1) + \xi(t)$ so that (3.9) is equivalent to $\xi(\infty) = (a_2 - b_2)/(m + 1)$. If this is not true and, say, $\xi(\infty) \geq (a_2 - b_2)/(m + 1) + \varepsilon$ for an $\varepsilon > 0$, then we would have $\lim_{t \to \infty} t \eta'(t) \geq \varepsilon$ as $t \to \infty$ so that $\lim_{t \to \infty} \eta(t)/\log t \geq \varepsilon$. But since $|\eta(t)| \leq \max(|a_1|, |a_2|)$, this is not possible. The same argument holds if $\xi(\infty) < (a_2 - b_2)/(m + 1)$.

**Remark.** (3.11) implies that $(-1)^{-1} (\xi_2(t) - a_2) t^{-1/(m+1)}$ is monotone nondecreasing. Hence we can formulate (2.9) more precisely:

\[(3.12) \quad (-1)^{1} (\xi_2(t) - a_2) t^{-1/(m+1)} \uparrow c_m M_0^{m-1}/(m+1) \quad \#.
\]

To obtain the asymptotic expression (0.10) for $\xi(t)$ we need yet to show that $b_1 = b_2 = x_0$. We introduce the following expression: $d(t) = l(t) - 2 r(t)$, where $l(t) = \xi(t) = \xi_2(t) + \xi_3(t)$ is the dispersion of $u$ at time $t$. (3.7) says that $d'(t) \leq 0$ and (3.8) that $d(t) \downarrow b_2 - b_1$ as $t \to \infty$. We show next that $b_1 = b_2$.

**Lemma 3.3.** $b_1 = b_2$, i.e., there exists $b \in (a_1, a_2)$ such that as $t \to \infty$,

\[(3.13) \quad \xi_1(t) = (-1)^{1} r(t) + b + o(1).
\]

**Proof.** We divide the proof in two parts.

(1) We prove first that $b_2 \geq b_1$, i.e. that $\lim_{t \to \infty} d(t) \geq 0$. For that we evaluate $\nu$ at a fixed $t > 0$. Since $\nu(\xi_1(t), t) = 0$,

\[\nu_\xi(\xi_1(t), t) = -\xi_1(t) = (r(t) - \varepsilon(t))/(m + 1) t
\]
where \( \rho(t) \geq 0 \) as \( t \to \infty \), and \( \nu_{xx} \geq ((m + 1)t)^{-1} \), we have

\[
(3.14) \quad \nu(x, t) \geq -\frac{\dot{x}^2}{2(m + 1)t} + \frac{(r(t) - \epsilon(t))\dot{x}}{(m + 1)t} \quad \text{if } \dot{x} \geq 0,
\]

where \( \dot{x} = x - \xi(t) \). Hence \( \nu(x, t) = 0 \), \( x > \xi(t) \) implies \( \dot{x} \geq 2(r(t) - \epsilon(t)) \). But since \( \nu(x, t) = 0 \) for \( x = \xi(t) \), i.e., for \( \dot{x} = l(t) = 2r(t) + d(t) \), we conclude that \( d(t) \geq -2\epsilon(t) \). Let \( t \to \infty \) to conclude.

(II) We now prove that \( b_1 > b_2 \) cannot occur.

The idea is to compare \( u \) at fixed times \( t \to 0 \) with the self-similar solutions \( \tilde{u} = \tilde{u}(x - x^*(t), t; M_0) \), \( x^*(t) = \frac{1}{2}(\xi_1(t) + \xi_2(t)) \) (i.e., the one centered in \( \Omega(t) \)) and estimate the integrals at time \( t \):

\[
I_1(t) = \int_{\{u < \tilde{u}\}} (\tilde{u}(x, t) - u(x, t)) \, dx,
\]

\[
I_2(t) = \int_{\{u > \tilde{u}\}} (u(x, t) - \tilde{u}(x, t)) \, dx.
\]

Since \( \int u(x, t) \, dx = \int \tilde{u}(x, t) \, dx = M_0 \), \( I_1(t) = I_2(t) \) for every \( t \). Nevertheless we shall show that if \( b_2 > b_1 \), \( I_2 \) is asymptotically larger than \( I_1 \). We begin by defining in \( \{(x, t): t > \gamma^*, \xi_1(t) \leq x \leq \xi_2(t)\} \) the function

\[
f(x, t) = \nu_x(x, t) + (x - x^*(t))/((m + 1)t)
\]

(cf. §1.4). We have

\[
(3.15) \quad \frac{\partial f(x, t)}{\partial x} = \nu_{xx}(x, t) + ((m + 1)t)^{-1} \geq 0
\]

so that \( f \) is nondecreasing in \( x \). Also

\[
(3.16) \quad f(\xi_1(t), t) = -\xi_1(t) - \frac{l(t)}{2(m + 1)t} = \frac{d(t) - 2\epsilon(t)}{2(m + 1)t}.
\]

Since \( 0 \leq d(t) \leq l_0 \) we conclude that \( f(\xi_1(t), t) = O(1/t) \) as \( t \to \infty \). Similarly \( f(\xi_2(t), t) = O(1/t) \). This and (3.17) give

\[
(3.17) \quad f(x, t) = O(1/t) \quad \text{as } t \to \infty \text{ uniformly in } x \in \Omega(t).
\]

Next we estimate \( \nu(x, t) - \tilde{\nu}(x, t) \) for large \( t \): for every \( t \geq T^* \) and \( x \): \( x - x^*(t) \leq r(t) \) we have \( \partial(\nu - \tilde{\nu})/\partial x = f \) so that

\[
(3.18) \quad |\nu(x, t) - \tilde{\nu}(x, t)| \leq |\nu(x^*(t) - r(t), t)| + \int_{x^*(t) - r}^x |f(x, t)| \, dx
\]

\[
\leq \sup |\nu_x| \cdot d(t)/2 + O(1/t) \cdot r(t) = O(r(t)/t)
\]

and the same estimate clearly holds for \( \xi_1(t) \leq x \leq \xi_2(t) \) and \( \tilde{\xi}_2(t) \leq x \leq \xi_2(t) \). (The estimate for \( |\nu_x| \) comes from (3.16).) Hence we conclude that

\[
(3.19) \quad |\nu(x, t) - \tilde{\nu}(x, t)| = O(t^{-m/(m + 1)}) \quad \text{as } t \to \infty \text{ uniformly in } x \in \mathbb{R}.
\]

From this it follows that uniformly in \( x \) such that \( |x - x^*(t)| \leq \alpha r(t), 0 < \alpha < 1 \), if \( m > 2 \) or in \( x \in \mathbb{R} \) if \( 1 < m \leq 2 \):

\[
(3.20) \quad |u(x, t) - \tilde{u}(x, t)| = O(t^{-2/(m + 1)})
\]

(a simple application of the Mean Value Theorem since \( u = ((m - 1)\nu/m)^{(m-1)/(m-1)} \)).
We are now in a position to estimate $I_1$ and $I_2$ under the hypothesis that $b_2 > b_1$; since $\int u(x, t) \, dx = \int \tilde{u}(x, t) \, dx$ and $l(t) - \tilde{l}(t) = 2d(t) > 0$, the set $G(t) = \{ x \in \mathbb{R} : v(x, t) < \tilde{v}(x, t) \}$ is nonvoid. Let $x_1(t) = \inf G(t)$, $x_2(t) = \sup G(t)$. We have $\bar{x}(t) \leq x_1(t) \leq x_2(t) \leq \bar{x}(t)$. Since

\[
\tilde{v}(x, t) = \left( r(t)^2 - |x - x^*(t)|^2 \right) / 2(m + 1)t
\]

for every $x$: $|x - x^*(t)| \leq r(t)$, recalling (3.14) we obtain, if $\lim d(t) = d > 0$, the estimate

\[
x^*(t) - x_1(t) = o(r(t)),
\]

and the same applies to $x_2(t)$. Thus as $t \to \infty$,

\[
I_1(t) = \int_{\{ u \leq \tilde{u} \}} |\tilde{u} - u| \, dx
\leq |x_2(t) - x_1(t)| \sup_{x_1(t) \leq x \leq x_2(t)} |u(x, t) - \tilde{u}(x, t)|
\leq o(r(t)) \cdot O\left( r(t)^2 \right) = o\left( t^{-1/(m+1)} \right).
\]

In the following $C$ will stand for any positive constant depending only on $m$. For large $t$ it follows from (3.14), (3.22) that for $r(t)/3 < |x - x^*(t)| < r(t)/2$,

\[
v(x, t) - \tilde{v}(x, t) \geq C t^{-m/(m+1)},
\]

so that we estimate $I_+$ from below:

\[
I_+(t) \geq 2 \cdot (r(t)/6) \cdot C t^{2/(m+1)} = Ct^{-1/(m+1)}.
\]

(3.24), (3.25) contradict the fact that $I_+(t) = I_-(t)$ for every $t$. Hence the assumption $b_2 > b_1$ was false. #

**Proof of Theorem A.** Part (i) was proved in Corollaries 2.4 and 2.5. To prove part (ii) we repeat the calculations of the preceding lemma, taking into account the fact that $d(t) \to 0$. Thus $x^*(t) = b + o(1)$ and (3.18), (3.20), (3.21) give, respectively:

\[
\nu(x, t) + (x - b) / (m + 1)t = o(1/t) \quad \text{uniformly in } x \in \Omega(t),
\]

\[
\nu(x, t) - \tilde{v}(x, t; M_0, b) = o\left( t^{-m/(m+1)} \right) \quad \text{uniformly in } x \in \mathbb{R},
\]

\[
u(x, t) - \bar{u}(x, t; M_0, b) = o\left( t^{-2/(m+1)} \right),
\]

uniformly in $x$: $|x - b| < \alpha r(t)$, $0 < \alpha < 1$, if $m > 2$ or in $x \in \mathbb{R}$ if $1 < m \leq 2$.

It only remains to prove that $b = x_0$.

If $1 < m \leq 2$ the proof is immediate from (3.28) and the invariance of the center of mass:

\[
M_0 | x_0 - b | = \left| \int x(u(x, t) - \bar{u}(x, t; M_0, b)) \, dx \right|
= O(r(t)^2) \cdot o\left( t^{-2/(m+1)} \right) = o(1).
\]

Now let $t \to \infty$ to obtain $x_0 = b$. 

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Case $m > 2$. To simplify the calculation we may assume that $b = 0$ (if $b \neq 0$ shift the $x$-axis). Then

\begin{equation}
M_0 \mid x_0 \mid = \left| \int x(u - \bar{u}) \, dx \right|
\end{equation}

\begin{align*}
&\leq \int_{-r(t)}^{r(t)} xu(x, t) \, dt + \int_{r(t)}^{r(t)} |xu(x, t)| \, dt \\
&\quad + \int_{-r(t)}^{r(t)} x |u(x, t) - \bar{u}(x, t)| \, dx \\
&= I_1 + I_2 + I_3.
\end{align*}

Using (3.27) we estimate $u$ in the region $|x| \geq r(t)$ as $o(t^{-m/(m-1)})$ so that

\begin{equation}
I_1 + I_2 = o(t^{-m/(m-1)}) \cdot r(t) \cdot d(t) = o(t^{-1/(m+1)}) \to 0.
\end{equation}

Next we estimate $I_3$. By (3.27) there is a function $\epsilon(t) \geq 0$, $\epsilon(t) \to 0$ such that for $|x| \leq r(t)$,

\begin{equation}
\bar{v}(x, t) - \epsilon(t) t^{-m/(m+1)} \leq v \leq \bar{v} + \epsilon(t) t^{-m/(m+1)}.
\end{equation}

Let $F(t) = \{|x| \leq r(t): u(x, t) \geq \bar{u}(x, t)\}$. We have ($C$ represents any constant $> 0$):

\begin{equation}
\left| \int_{F(t)} x(u - \bar{u}) \, dx \right| \leq 2 \int_{0}^{r(t)} x C \mid v - \bar{v} \mid^{1/(m-1)} \, dx
\end{equation}

\begin{align*}
&\leq C \int_{0}^{r(t)} x \, dx \left\{ \frac{r(t)^2 - x^2 + Ce(t)r(t)}{2(m+1)t} \right\}^{1/(m-1)} - \left\{ \frac{r(t)^2 - x^2}{2(m+1)t} \right\}^{1/(m-1)} \right\} \\
&\leq Cr(t) \left\{ -Ce(t)r(t)^{-m/(m+1)} + (1 + Ce(t)/r(t))^{m/(m-1)} - 1 \right\} \\
&= o(r(t)^{-1/(m-1)}) + o(1) \to 0.
\end{align*}

A similar computation applies on the set $G(t) = \{|x| \leq r(t): u(x, t) \leq \bar{u}(x, t)\}$ (by replacing $\epsilon(t)$ by $-\epsilon(t), \ldots$) so that finally $I_3(t) \to 0$ and $x_0 = 0$ (i.e. $x_0 = b$) follows from (3.30) letting $t \to \infty$.

4. Symmetric solutions. In this section we assume that $u_0(x)$ is symmetric, i.e.

\begin{equation}
u_0(x) = u_0(-x).
\end{equation}

Since (0.1) is symmetry-invariant the corresponding solution $u(x, t)$ satisfies

\begin{equation}
u(x, t) = u(-x, t)
\end{equation}

for every $x, t > 0$. We set $\xi(t) = \xi_2(t) = -\xi_1(t)$.

We introduce a comparison principle based on the estimate of the concentration of mass around the origin by means of which we prove optimal rates of convergence.
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in the results of Theorem A:

**THEOREM B.** Let \( u(x, t) \) be a solution of (0.1), (0.2) with \( u \) satisfying (0.3), (0.4), (4.1). Then there exist \( \tau > 0 \) and \( t_1 \geq 0 \) such that

\[
\begin{align*}
(4.3) &= (0.10') \quad 0 \leq (\xi(t) - r(t)) t^{m/(m+1)} \leq k_1 \cdot M_0^{(m-1)/(m+1)} \tau \quad & \text{if } t \geq t_1, \\
(4.4) &= (0.11') \quad 0 \leq (\xi'(t) - r'(t)) t^{(2m+1)/(m+1)} \leq k_2 \cdot M_0^{(m-1)/(m+1)} \tau \quad & \text{if } t \geq t_1, \\
(4.5) &= (0.12') \quad V(x, t) - \frac{x}{(m+1)t} t^{(2m+1)/(m+1)} \leq k_3 \cdot M_0^{(m-1)/(m+1)} \tau \quad & \text{if } |x| \leq \xi(t), t \geq t_1, \\
(4.6) &= (0.13') \quad v(x, t) - \frac{\left(r(t)^2 - x^2\right)}{2(m+1)t} t^{2m /(m+1)} \leq k_3 \cdot M_0^{(m-1)/(m+1)} \tau \quad & \text{if } x \in \mathbb{R}, t \geq t_1,
\end{align*}
\]

for some positive constants \( k_1 = c_m/(m+1), k_2 \leq k_1 ((m+1)/m)^m, k_3 \leq c_m (k_1/(m+1) + k_2) \). Moreover there exists \( \tau_1 = \tau_1(m) > 0 \) such that \( \tau \leq \tau_1 M_0^{1-n_0^{m+1}} \).

**REMARKS.** (1) (4.6) and \( u = ((m-1)\nu/m)^{1/(m-1)} \) give the estimate for \( u \),

\[
(4.7) = (0.16') \quad u(x, t) = \bar{u}(x, t; M_0) + O(t^{-(m+2)/(m+1)}),
\]

if \( t \geq t_1 \) uniformly in \( x \): \( |x| \leq a \tau(t), 0 < \alpha < 1 \) (if \( 1 < m \leq 2 \) uniformly in \( x \in \mathbb{R} \)).

(2) The exponents in (0.10')–(0.14') are best possible. To see this apply the theorem to the explicit solution \( u(x, t) = \bar{u}(x, t + \tau; M_0) \) for some \( \tau > 0 \), that serves as a model.

(3) We do not treat the problem of determining the best \( \tau \) in Theorem B and its consequences. In this respect see the indications in [21].

(4) The proof of Theorem B can be adapted to treat radially-symmetric solutions in spatial dimension \( n \geq 1 \).

The proof of Theorem B proceeds by comparing \( u \) with the self-similar solutions \( \bar{u}(x, t; M_0) \) and \( \bar{u}(x, t + \tau; M_0) \) for some suitable \( \tau > 0 \). As in §2 we first prove an elliptic version of the comparison principle.

**LEMMA 4.1 (CONCENTRATION-COMPARISON PRINCIPLE. ELLIPTIC VERSION).** Let \( \beta \) be a continuous nondecreasing function such that \( O = \beta(0) \subset \text{Int} \beta(\mathbb{R}) \), let \( f_i, i = 1, 2 \), be symmetric and integrable functions in \( \mathbb{R} \) and define for \( r > 0 \),

\[
(4.8) \quad F_i(r) = \int_{|x| \leq r} f_i(x) \, dx.
\]

Let \( u_i, i = 1, 2 \), be the (symmetric) solutions of \(-u_i'' + \beta(u_i) = f_i[9], \) set \( w_i = \beta(u_i) \) and for \( r > 0 \),

\[
(4.9) \quad W_i(r) = \int_{|x| \leq r} w_i(x) \, dx.
\]

Then if \( F_i(r) \leq F_2(r) \) for every \( r > 0 \), then \( W_i(r) \leq W_2(r) \) for every \( r > 0 \).
Remark. We say that \( f_2 \) is more concentrated than \( f_1 \): \( f_2 \succ f_1 \). The lemma implies then that \( w_2 \succ w_1 \).

Proof. As in Lemma 2.1 let \( G = \{ r > 0 : W_1(r) > W_2(r) \} \). If \( G \) is nonvoid, let \( I = (a, b) \) be a maximal interval in \( G \), \( 0 \leq a < b \leq \infty \). As in Lemma 2.1 \( u_1 - u_2 \) is strictly increasing on \( I \). Next if \( b < \infty \) we have \( W_1(b) = W_2(b) \), \( u_1(b) = u_2(b) \) arguing as there and we conclude that \( a = 0 \) and \( W_1(0) > W_2(0) \), impossible. The case \( b = \infty \) is also similar.

Via Semigroup Theory we pass to

Lemma 4.2 (Concentration-Comparison Principle. Parabolic Version). Let \( u^1(x, t) \), \( u^2(x, t) \) be solutions of (0.1), (0.2) with symmetric initial data \( u^1_0(x), u^2_0(x) \in L^1(\mathbb{R}) \). If for every \( r > 0 \),
\[
\int_{|x| < r} u^1_0(x) \, dx \leq \int_{|x| < r} u^2_0(x) \, dx,
\]
then for every \( t, r > 0 \) we have
\[
\int_{|x| < r} u^1(x, t) \, dx \leq \int_{|x| < r} u^2(x, t) \, dx.
\]

Remark. Both lemmas admit obvious \( n \)-dimensional counterparts valid for radially-symmetric solutions. Hence Theorem B admits an \( n \)-dimensional version.

Corollary 4.3. Let \( u(x, t) \) be a solution of (0.1)–(0.4), symmetric with respect to \( x \) and assume that there exist \( \tau > 0, t_1 \geq 0 \) such that \( u(x, t_1) \succ \bar{u}(x, t_1 + \tau; M_0) \). Then for every \( t \geq t_1 \), \( u(x, t) \succ \bar{u}(x, t + \tau; M_0) \) so that
\[
r(t) \leq \xi(t) \leq r(t + \tau) \leq r(t) + r(t) \tau / ((m + 1)t).
\]

Proof. The inequality \( r(t) \leq \xi(t) \) comes from Theorem A. To obtain the inequality \( \xi(t) \leq r(t + \tau) \) we use Lemma 4.2 and the fact that \( \xi(t) = \sup \{ r > 0 : \int_{|x| < r} u(x, t) \, dx < M_0 \} \).

We now show that for every \( u_0 \in L^1(\mathbb{R}) \) satisfying (0.3), (0.4), (4.2) there exist \( \tau > 0 \) and \( t_1 \geq 0 \) such that (4.12) holds. Moreover we bound above \( \tau \) in terms of \( M_0 \) and \( l_0 \):

Lemma 4.4. There exists \( \tau_1 > 0 \) such that (4.12) holds with \( \tau = \tau_1 \cdot M_0^{1-m} l_0^{m+1} \) for all large \( t \), and \( \tau_1 \geq (2c_m)^{-(m+1)} \).

Proof. By means of the group of transformations (1.9) we can reduce the proof to the case \( M_0 = l_0 = 1 \): if the lemma is true in this case and \( u(x, t) \) is a general solution of (0.1)–(0.4), (4.2), we define \( \hat{u} \) by
\[
u(x, t) = \frac{M_0}{l_0^{m+1}} \hat{u} x \frac{M_0^{m-1}}{l_0^{m+1}} + \tau_1; 1 \right).
\]
We have \( \hat{M}_0 = \hat{l}_0 = 1 \) so that \( \hat{u}(\cdot, t) \succ \bar{u}(\cdot, t + \tau_1; 1) \) for all \( t \geq \hat{t}_0 \geq 0 \). Then
\[
u(x, t) > \frac{M_0}{l_0^{m+1}} \hat{u} x \frac{M_0^{m-1}}{l_0^{m+1}} + \tau_1; 1 \right).
\]
Therefore we assume that $M_0 = l_0 = 1$. Now note that there is a worst situation with respect to the relation “$>$”, namely the one with initial condition

\[(4.15)\quad u_0(x) = \frac{1}{2} \delta(x - \frac{1}{2}) + \frac{1}{2} \delta(x + \frac{1}{2}).\]

We only have to prove that there exists a $\tau_1$ for this particular $u_0$ (the fact that $u_0$ is a measure causes no inconvenience, see remark in §1.1). Since $u_0(x) \geq \frac{1}{2} \delta(x - \frac{1}{2})$, $u(x, t) \geq \bar{u}(x - \frac{1}{2}, t; \frac{1}{2})$ so that for every $t > t_0 = (4c_m^{-1/2})^{-1}$ we have $u(x, t) \geq \bar{u}(\frac{1}{2}, t; \frac{1}{2}) > 0$ for $0 \leq x \leq 1$. Also the free-boundary $\bar{z}_2$ of $\bar{u}$ passes through $(1, t_0)$. By the Sh.C.P. we derive the estimate $1 < \bar{z}(t) \leq 1 + c_m(t - t_0)^{1/(m+1)}$. To obtain $u(x, t) > \bar{u}(x, t + \tau; 1)$ at a time $t > t_0$ we only have to take $\tau$ large enough, for instance such that

\[(4.16)\quad \bar{u}(0, t + \tau; 1) \cdot \bar{z}(t) \leq \bar{u}(\frac{1}{2}, t; \frac{1}{2}).\]

This $\tau = \tau(t, m)$ is to be minimized in $t > t_0$ to obtain $\tau_1$.

Consider now the explicit solution $u(x, t) = \bar{u}(x, t + \tau; 1)$ such that $l_0 = 1$; then $\tau = (2c_m)^{(m+1)}$, so that $\tau_1 > (2c_m)^{(m+1)}$.

**Proof of Theorem B.** Corollary 4.3 and Lemma 4.4 imply (4.3).

To prove (4.4) we restrict ourselves as above to the case $M_0 = l_0 = 1$. Let us estimate the derivative of $\eta(t) = \bar{z}(t) - r(t)$ using the fact that $r^{m/(m+1)} \eta'(t) \uparrow 0$ (Lemma 3.2). For $t > 0, \lambda > 1$ we have

\[
\eta'(t) \leq \int_t^{\infty} \eta'(s) \, ds \geq -\eta'(\lambda t)(\lambda t)^{m/(m+1)} \int_t^{\lambda t} s^{-m/(m+1)} \, ds
\]

Using (4.3) we obtain

\[(4.17)\quad -\eta'(t) \leq k_1 \tau \lambda \left\{ (m + 1)t^{2(m+1)/(m+1)}(\lambda^{1/(m+1)} - 1) \right\}^{-1}.
\]

The right-hand expression is minimized setting $\lambda = ((m + 1)/\lambda)^{m+1}$; then (4.18) gives (4.4) with $k_1 \leq k_2((m + 1)/\lambda)^m$.

Estimates (4.5), (4.6) are obtained by inserting the information (4.3), (4.4) into formulas (3.16)—(3.18) in Lemma 3.3.

**5. Other results on the growth of the interfaces.** In this section we consider solutions $u(x, t)$ of the Cauchy problem (0.1), (0.2), where $u_0$ satisfies (0.3), and instead of (0.4) the half-condition

\[(0.18)\quad \text{ess sup}(\text{support}(u_0)) = 0.
\]

Then a right free-boundary $x = \bar{z}(t)$, appears where $\bar{z}(t) = \{ \text{sup} x: u(x, t) > 0 \}$ for $t > 0, \bar{z}(0) = 0$. By the Sh.C.P. $\bar{z}(t)$ is finite and in fact

\[(5.1)\quad 0 \leq \bar{z}(t) \leq c_m \| u_0 \|_1^{(m-1)/(m+1)}t^{1/(m+1)}.
\]

What was said in the Introduction applies and thus there exists a time $t^* > 0$ such that $\bar{z}(t) = 0$ for $0 \leq t < t^*$ and $\bar{z} \in C^1([t^*, \infty))$ and $\bar{z}'(t) > 0$ if $t > t^*$. Also $\bar{z}(t)^{-1/(m+1)}$ and $\bar{z}'(t) t^{m/(m+1)}$ are nondecreasing in $(0, \infty)$. Furthermore we prove

**Theorem C.** As $t \to \infty$ we have (with $r(t) = c_n(M_0^{-1}t)^{1/(m+1)}$)

\[(5.2)\quad \frac{\bar{z}(t)}{r(t)} \uparrow 1, \quad \frac{\bar{z}'(t)}{r'(t)} \uparrow 1.
\]
As \( t \to 0 \) we have

\[
(5.3) \quad \xi(t)/r(t) \downarrow 0, \quad \xi'(t)/r'(t) \downarrow 0.
\]

Moreover if \( x_0 = M_0^{-1} \int u_0(x) x \, dx, -\infty \leq x_0 < 0, \)

\[
(5.4) \quad \xi(t) - r(t) \downarrow x_0.
\]

And if \( x_0 \) is finite, then

\[
(5.5) \quad t(\xi'(t) - r'(t)) \uparrow 0.
\]

**Proof.** Take the sequence of approximations to \( u(x, t), \{u^n(x, t)\} \) such that

\( u^n(x, t) \) is the solution of (0.1) with initial condition \( u^n_0 = u_0 \cdot x_{[-n,0]} \). If \( \xi^n(t) \) is the corresponding right-interface it follows from \( u(x, t) \geq u^n(x, t) \) that \( \xi(t) \geq \xi^n(t) \). But by (2.9) \( \xi^n(t) t^{-1/(m+1)} \to c_m(M_n)^{(m-1)/(m+1)} \), where \( M_n = \int_{-n}^0 u_0(x) \, dx \). Since \( M_n \to M_0 \) as \( n \to \infty \), this and (5.1) give (5.2). To obtain \( \xi'(t)/r'(t) \to 1 \) argue as in Lemma 3.2.

To prove (5.3) compare \( u \) with the solution \( u_\epsilon \) resulting from shifting the initial mass lying in \([-\epsilon, 0]\), for small \( \epsilon > 0 \), to 0 as a point mass \( M_\delta(x) \). In some time interval \([0, \tau_\epsilon]\), \( \tau_\epsilon > 0 \), the right-interface for \( u_\epsilon \) coincides with the one for this point mass and the Sh.C.P. gives us

\[
(5.6) \quad \xi(t) \leq c_m(M_{\epsilon}^{-1} t)^{1/(m+1)} \quad \text{for } 0 \leq t < \tau_\epsilon.
\]

Now let \( \epsilon \to 0 \): then \( M_\epsilon \to 0 \) and (5.4) implies that \( \xi(t)/r(t) \to 0 \). Since \( \xi'(t)t^{-m/(m+1)} \) is monotone the limit \( \xi'(t)/r'(t) \) exists and is zero.

To prove (5.4) notice that there exists \( b, -\infty \leq b \leq 0 \), such that \( \xi(t) - r(t) \downarrow b \) since \( \xi'(t) \leq r'(t) \) for every \( t > 0 \). We shall prove that \( b = x_0 \). For that we call \( \hat{u}^n(x, t) \) the solution resulting from shifting the mass of \( u_0 \) in \((-\infty, n]\) as a point mass to \( x = -n \), keeping \( \hat{u}^n_0(x) = u_0(x) \) for \( x > -n \). Let \( \hat{\xi}^n(t) \) and \( \hat{x}_n \) be the corresponding right-interface and center of mass. The sequence \( \{\hat{\xi}^n(t)\}_{n \in \mathbb{N}} \) is nonincreasing in \( n \) by the Sh.C.P. and Theorem A says that \( \hat{\xi}^n(t) = \hat{x}_n + r(t) + o(1) \). Since \( \hat{x}_n \downarrow x_0 \) as \( n \to \infty \), \( b = \lim_{t \to \infty} (\hat{\xi}(t) - r(t)) \leq x_0 \), so that in case \( x_0 = -\infty \) we are done.

It remains to prove that \( b \geq x_0 \) in case \( x_0 > -\infty \). Take an \( \epsilon > 0 \). It is clear that there exist \( n_\epsilon \) and \( t_\epsilon \) such that for \( n \geq n_\epsilon \) and \( t \geq t_\epsilon \), \( \hat{\xi}^n(t) < x_0 + r(t) + \epsilon \). Since \( \hat{\xi}'(t)/r'(t) \) is nondecreasing (we drop the index \( n \) in this calculation),

\[
\epsilon > \hat{\xi}(t) - r(t) - \hat{x}_0 = \int_t^\infty \left( r'(s) - \hat{\xi}'(s) \right) ds = \int_t^{2t} r'(s) \left[ 1 - \frac{\hat{\xi}'(s)}{r'(s)} \right] ds
\]

\[
\geq (r(2t) - r(t)) \cdot \left( 1 - \frac{\hat{\xi}'(2t)}{r'(2t)} \right) = r(2t)(1 - 2^{-1/(m+1)}) \left( 1 - \frac{\hat{\xi}'(2t)}{r'(2t)} \right).
\]
So that for $t \geq 2t_\epsilon$, $1 - \frac{\bar{\xi}'(t)/r'(t)}{r(t)} \leq k_m \epsilon/r(t)$ with $k_m > 0$ depending only on $m$. Since $-\bar{\varphi}(\bar{\xi}(t), t) = \bar{\xi}'(t)$ [16], and $\dot{\varphi}_{xx} \geq -((m + 1)t)^{-1}$, we have at $\bar{x}(t) = x_0 + r(t) - \epsilon$, for $t$ large enough ($t \geq 2t_\epsilon$, $2k_m \epsilon \leq r(t)$):

$$\dot{\varphi}(\bar{x}(t), t) \equiv \left( r'(t) - \frac{k_m \epsilon r'(t)}{r(t)} \right) \left( \bar{\xi}'(t) - \bar{x}(t) \right) = \frac{(\bar{\xi}'(t) - \bar{x}(t))^2}{2(m + 1)t}$$

so that $\dot{\varphi}(\bar{x}(t), t) \geq \epsilon r'(t)/2$ for $t$ large enough uniformly in $n$. Thus in the limit $\nu(\bar{x}(t), t) > 0$, so that $\bar{\xi}(t) \geq \bar{x}(t)$ for every large $t$, $\lim_{t \to \infty} (\bar{\xi}(t) - r(t) - x_0) > -\epsilon$ and the result follows.

(5.5) follows from (5.4) arguing as in Lemma 3.2. 

(5.2) makes clear that for an $u_0$ satisfying (0.3), (0.4'), $M_0$ and $x_0$ allow us to describe $\bar{\xi}(t)$ as $t \to \infty$ in the first approximation. As $t \to 0$ (5.3) shows that this is not the case: The description of $\bar{\xi}(t)$ requires further information: thus Knerr [16] proves that if $u_0 \in L^\infty(R)$ and $\|u_0\|_{L^\infty} \leq L (v_0 = mu_0^{m-1}/(m - 1))$, $\bar{\xi}(t) \leq 2(Lt)^{1/2}$ for every small $t$ and the exponent $1/2$ is sharp for this class of initial data.

We extend the result to cover the dependence of $\bar{\xi}(t)$ on the $L^p$-norm of $u_0$ for every $1 < p < \infty$: we consider the class of solutions

$$\mathcal{C}_{p,N} = \{u(x, t): u \text{ is solution of (0.1), (0.2), (0.3), (0.18)}$$
$$\text{with } u_0 \in L^p(R) \text{ and } \|u_0\|_p \leq N \}$$

where $1 \leq p \leq \infty$, $N > 0$ and $\| \cdot \|_p$ denotes the $L^p$-norm. If $\bar{\xi}_a(t)$ is the right-interface of $u \in \mathcal{C}_{p,N}$, we set for $t > 0$,

$$\mathcal{G}_{p,N}(t) = \sup \{\bar{\xi}_a(t): u \in \mathcal{C}_{p,N}\}.$$ 

We have

**Theorem D.** For every $p: 1 < p \leq \infty$ there exists a constant $C_{p,m} > 0$ such that

$$\mathcal{G}_{p,N}(t) = C_{p,m} (N^{m-1}t)^{\alpha}$$

with $\alpha = p/(2p + m - 1)$ if $1 \leq p < \infty$, $\alpha = \frac{1}{2}$ if $p = \infty$. We have $C_{1,m} = c_m$ ($c_m$ defined in (0.9)).

**Proof.** By means of the group of transformations $T_{k,L}$ ($\S 1.2$) we can reduce the proof to the case $N = t = 1$. In fact let $u$ be a solution with $L^p$-norm $N > 0$ and fix a certain $t > 0$. If we define $\hat{u}$ by

$$\hat{u}(x, t) = (T_{k,L}u)(x, t) = Ku(Lx, K^{m-1}L^2t)$$

with $L = (N^{m-1}t)^{\alpha}$ and $K = (N^{-2p+1})^{\beta}$ with $\beta = \alpha/p$ if $p < \infty$, $\beta = 0$ if $p = \infty$, then $\hat{u} \in \mathcal{C}_{p,1}$ and

$$\bar{\xi}(\hat{t}) = L^{\frac{\alpha}{p}} (k^{m-1}L)^{-\frac{\alpha}{p}} \Rightarrow \bar{\xi}(t) = (N^{m-1}t)^{\alpha} \cdot \bar{\xi}(1).$$

Hence we must prove that $C_{p,m} = \sup \{\bar{\xi}_a(1): u \in \mathcal{C}_{p,1}\}$ is finite for every $m > 1$, $1 \leq p \leq \infty$. 

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For \( p = 1 \) it follows from Corollary 2.3 that \( C_{1,m} = c_m \). On the other hand for 
\( p = \infty \) [16] proves that \( C_{\infty,m} \leq 2(m/(m - 1))^{1/2} \). For \( 1 < p < \infty \) we estimate \( C_{p,m} \) in terms of \( C_{\infty,m} \) by means of the following "\( L^p - L^\infty \) smoothing effect":

**Lemma 5.1** [20]. There exists a positive constant \( K = K(m, p) \) such that for every solution of (0.1), (0.2) with \( u_0 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R}) \), \( 1 \leq p < \infty \),

\[
\| u(t) \|_\infty \leq K \| u_0 \|_p^\alpha t^{-\delta}
\]

with \( \alpha = 2p/(2p + m - 1) \), \( \delta = 1/(2p + m - 1) \).

Hence for every \( t, h > 0 \) and every \( u \in C^1_{p,1} \) we have

\[
\xi(t + h)/\xi(t) \leq C_{\infty,m} \left( \| u(t) \|_\infty^{-1} t \right)^{1/2}.
\]

Fix \( t > 0 \) and set \( t_n = 2^{-n}t \), \( n \geq 1 \). (5.12) and (5.13) give

\[
\xi(t_{n-1}) \leq \xi(t_n) + C_{\infty,m} K^{(m-1)/2}(2^{-n}t)^{p/(2p+m-1)},
\]

and since \( \lim_{n \to \infty} \xi(t_n) = \xi(0) = 0 \), we conclude from (5.14) that

\[
\xi(t) \leq C_{\infty,m} K^{(m-1)/2} (2^\alpha - 1) t^\alpha
\]

with \( \alpha = p(2p + m - 1)^{-1} \). Therefore

\[
C_{p,m} \leq C_{\infty,m} K^{(m-1)/2} (2^\alpha - 1)^{-1}.
\]

**References**


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