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QUADRATIC SPACES OVER LAURENT EXTENSIONS OF DEDEKIND DOMAINS

BY

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ABSTRACT. Let R be a Dedekind domain in which 2 is invertible. We show in this paper that any isotropic quadratic space over $R[T, T^{-1}]$ is isometric to $q_1 \perp Tq_2$ where q_1, q_2 are quadratic spaces over R. We give an example to show that this result does not hold for anisotropic spaces.

Introduction. Let R be a Dedekind domain in which 2 is invertible. Karoubi, in [**3**], shows that if q is any quadratic space over $R[T, T^{-1}]$, then, the class $[q]$ of q in the Wittring of $R[T, T^{-1}]$ is $[q_1] + [Tq_2]$, where q_1 and q_2 are quadratic spaces over R. In this paper, we prove a nonstable version of this theorem.

In §1, we list a few results on quadratic spaces over $K[T, T^{-1}]$, K denoting a field of characteristic $\neq 2$ and also recall the stable results of Karoubi. In §2, we show (Theorem 2.4) that if R is a complete discrete valuation ring in which 2 is invertible, any quadratic space over $R[T, T^{-1}]$ is isometric to $q_1 \perp Tq_2$, q_1 and q_2 being quadratic spaces over R. This shows in particular that any quadratic space over $R[T, T^{-1}]$ is an orthogonal sum of rank one spaces. If R is a field, this result is due to Harder (see [**4**, 13.4.4]). In §3, using the technique of patching diagrams, we prove the main result (Theorem 3.5), namely, if R is a Dedekind domain in which 2 is invertible, every isotropic quadratic space over $R[T, T^{-1}]$ is isometric to $q_1 \perp Tq_2$ where q_1 and q_2 are quadratic spaces over R. We give an example to show that this result does not hold for anisotropic spaces.

In this paper, R denotes an integral domain in which 2 is invertible. A quadratic space (P, q) over R is denoted by q, suppressing the underlying module. We say that a quadratic space (P, q) is isotropic if there exists $v \in P$, $v \neq 0$ such that $q(v) = 0$. For $\lambda_i \in U(R)$, $1 \leqslant i \leqslant n$, $\langle \lambda_1, \ldots, \lambda_n \rangle$ denotes the diagonal form. We denote by h the standard hyperbolic plane over R whose matrix is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. *The Witt index of a quadratic space q over R is $\geqslant n$* if $q \simeq q' \perp H(P)$, with rank $P \geqslant n$, where $H(P)$ denotes the hyperbolic space.

1. Some preliminary results. In this section we prove some lemmas which will be used in the later sections. We assume throughout the paper that in all the rings considered, 2 is invertible. If q_1, q_2 are quadratic spaces over R, $q_1 \perp Tq_2$ denotes the quadratic space $(q_1 \otimes_R R[T, T^{-1}]) \perp T(q_2 \otimes_R R[T, T^{-1}])$ over $R[T, T^{-1}]$.

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LEMMA 1.1 *Let R be an integral domain. Let q_1 and q_2 be quadratic spaces over R. If the quadratic space $q_1 \perp Tq_2$ over $R[T, T^{-1}]$ is isotropic, then q_1 or q_2 is isotropic over R.*

PROOF. Clearly it suffices to prove the lemma when R is a field. Let $q_1 \simeq \langle\lambda_1,\ldots,\lambda_r\rangle$, $q_2 \simeq \langle\mu_1,\ldots,\mu_s\rangle$, λ_i, $\mu_i \in R - (0)$. If $q_1 \perp Tq_2$ is isotropic, there exist f_i, $g_j \in R[T]$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, such that $\Sigma\lambda_i f_i^2 + T\cdot\Sigma\mu_j g_j^2 = 0$. Let $n = \max_i\{\text{degree } f_i\}$ and $m = \max_j\{\text{degree } g_j\}$. Let θ_i (resp. φ_j) be the coefficients of T^n (resp. T^m) in f_i (resp. g_j). A comparison of degree shows that $\Sigma\lambda_i\theta_i^2 = 0$ or $\Sigma\mu_j\varphi_j^2 = 0$ i.e. q_1 or q_2 is isotropic over R.

LEMMA 1.2. *Let K be a field of characteristic $\neq 2$. Any quadratic space over $K[T, T^{-1}]$ is isometric to $q_1 \perp Tq_2 \perp h^n$ where q_1 and q_2 are quadratic spaces over K and $n =$ Witt index of q.*

PROOF. Since $K[T, T^{-1}]$ is Euclidean, $q \simeq q' \perp h^n$, where q' is anisotropic and $n =$ Witt index of q. Since $\operatorname{Spec} K[T, T^{-1}]$ is the complement in $\mathbf{P}_1(K)$ of two points, each of degree one, it follows from [**4**, 13.4.4] that q' is diagonalisable. Hence $q' \simeq q_1 \perp Tq_2$, q_1, q_2 quadratic spaces over K.

LEMMA 1.3. *Let K be a field of characteristic $\neq 2$. Then cancellation holds for quadratic spaces over $K[T, T^{-1}]$.*

PROOF. Let q, q', q'' be quadratic spaces over $K[T, T^{-1}]$ such that $q \perp q'' \simeq q' \perp q''$. If q is isotropic, in view of cancellation for isotropic quadratic spaces over principal ideal domains [**14**, Theorem 3.1], $q \simeq q'$. Let q be anisotropic. Then q' is also anisotropic and by Lemmas 1.2 and 1.1, $q \simeq q_1 \perp Tq_2$, $q' \simeq q_1' \perp Tq_2'$ where q_i, q_i', $i = 1, 2$, are anisotropic quadratic spaces over K. Let $q_1 \simeq \langle\lambda_1,\ldots,\lambda_r\rangle$, $q_2 \simeq \langle\mu_1,\ldots,\mu_s\rangle$, $q_1' \simeq \langle\lambda_1',\ldots,\lambda_l'\rangle$, $q_2' \simeq \langle\mu_1',\ldots,\mu_k'\rangle$, λ_i, μ_i, λ_i', $\mu_i' \in K^*$. Over $K(T)$, $q_1 \perp Tq_2 \simeq q_1' \perp Tq_2'$ and hence $q_1' \perp Tq_2'$ represents λ_1 over $K(T)$. Thus, there exist f_i, g_i, $g \in K[T]$ such that

$$(*) \qquad \lambda_1 g^2 = \sum_{i=1}^{l} \lambda_i' f_i^2 + T \sum_{j=1}^{k} \mu_j' g_j^2.$$

Since the polynomials $\Sigma\lambda_i' f_i^2$ and $\Sigma\mu_j' g_j^2$ have even degrees, q_1' and q_2' being anisotropic, a comparison of degrees in $(*)$ shows that degree $\Sigma\lambda_i' f_i^2 >$ degree $T\cdot\Sigma\mu_j' g_j^2$. Comparing the leading coefficients in $(*)$, it follows that q_1' represents λ_1. By induction, one shows that $q_1 \simeq q_1'$ and $q_2 \simeq q_2'$.

Let now R be a Dedekind domain. The canonical map $R \hookrightarrow R[T, T^{-1}]$ induces an isomorphism $\operatorname{Pic} R \xrightarrow{\sim} \operatorname{Pic} R[T, T^{-1}]$. In what follows, we shall regard this as an identification.

LEMMA 1.4. *Let R be a Dedekind domain and q a quadratic space over $R[T, T^{-1}] = A$. Then there exist quadratic spaces q_1 and q_2 over R such that $q \perp h^m \simeq q_1 \perp Tq_2 \perp h^{m-1} \perp H(Q)$ for some $Q \in \operatorname{Pic} R$.*

PROOF. By a theorem of Karoubi [**3**, Theorem 3.11], the class $[q]$ of q in the Wittring of A is equal to $[q_1] + [Tq_2]$, q_1, q_2 being quadratic spaces over R. Hence

$q \perp h^m \simeq q_1 \perp Tq_2 \perp H(P)$ for some projective A-module P. We assume without loss of generality that q_1 and q_2 are anisotropic since over any Dedekind domain, any isotropic space splits off a hyperbolic summand. Hence, by Lemma 1.1, $q_1 \perp Tq_2$ is anisotropic. Comparing the Witt indices of $q \perp h^m$ and $q_1 \perp Tq_2 \perp H(P)$ over $K[T, T^{-1}]$, K denoting the quotient field of R, it follows that rank $P \geqslant m$. Hence $H(P) \simeq h^{m-1} \perp H(Q) \perp h^r$ where $r + m = \operatorname{rank} P$ and $Q = \bigwedge^{r+m} P \in \operatorname{Pic} R$. If $q_1' = q_1 \perp h^r$, then $q \perp h^m \simeq q_1' \perp Tq_2 \perp h^{m-1} \perp H(Q)$.

COROLLARY 1.5. *Let R be a Dedekind domain and q an isotropic quadratic space over $R[T, T^{-1}]$. Then there exist quadratic spaces q_1, q_2 over R and $Q \in \operatorname{Pic} R$ such that*

$$q \perp h^m \simeq \left(q_1 \perp Tq_2 \perp H(Q)\right) \perp h^m.$$

PROOF. By Lemma 1.4, $q \perp h^m \simeq q_1' \perp Tq_2' \perp h^{m-1} \perp H(P)$, $P \in \operatorname{Pic} R$, q_1', q_2' quadratic spaces over R. Since q is isotropic, Witt index $(q_1' \perp Tq_2' \perp h^{m-1} \perp H(P))$ $\geqslant m + 1$ over $K[T, T^{-1}]$. Thus $q_1' \perp Tq_2'$ and hence by Lemma 1.1, q_1' or q_2' is isotropic. Let $q_1' \perp Tq_2' \simeq q_1 \perp Tq_2 \perp H(Q)$, $Q \in \operatorname{Pic} R$. Then

$$q \perp h^m \simeq q_1 \perp Tq_2 \perp h^{m-1} \perp H(P \oplus Q) \simeq q_1 \perp Tq_2 \perp h^m \perp H(P \otimes_R Q).$$

COROLLARY 1.6. *Let R be a principal ideal domain. If q is any quadratic space over $R[T, T^{-1}]$, there exist quadratic spaces q_1 and q_2 over R such that $q \perp h^m \simeq q_1 \perp Tq_2 \perp h^m$.*

PROOF. Since $\operatorname{Pic} R$ is trivial, the corollary follows from Lemma 1.4.

2. Quadratic spaces over Laurent extensions of complete d.v.r. We begin with the following.

PROPOSITION 2.1. *Let R be a p.i.d. and q an isotropic quadratic space over $R[T, T^{-1}]$. If q is stably extended from R, then q is extended from R.*

PROOF. Let $A = R[T, T^{-1}]$. Since q is stably extended from R, there exists a quadratic space q_0 over R such that $q \perp h^m \simeq (q_0 \otimes_R A) \perp h^m$ for some integer $m \geqslant 0$. The ring $R(T)$ obtained from $R[T]$ by inverting all the monic polynomials in T is a p.i.d. In view of [**14**, Theorem 3.1], we have $q \otimes_A R(T) \simeq q_0 \otimes_R R(T)$. Hence there exists a monic $g \in R[T]$ such that $q \otimes_A A_g \simeq q_0 \otimes_R A_g$. Let $g = T^n f$, $n =$ degree g, $f \in R[T^{-1}]$. Then T^{-1} and f are coprime in $R[T^{-1}]$ and the quadratic spaces q over A, $q_0 \otimes_R R[T^{-1}]_f$ over $R[T^{-1}]_f$ and an isometry φ: $q \otimes_A A_f \simeq q_0 \otimes_R A_f$ define a quadratic space $\tilde{q}$ over $R[T^{-1}]$ which is isotropic. In view of [**9**, Theorem 3.2], $\tilde{q}$ is extended from R and hence q is extended from R.

The proof above is an adaptation of a proof of Swan [**12**, Theorem 1.1]. Using a theorem of Suslin and Kopeiko [**13**] and a cancellation theorem of Roy [**11**], one can prove the following theorem on the same lines as the above proposition.

THEOREM. *Let q be a quadratic space over $R[T, T^{-1}]$ which is stably extended from a quadratic space q_0 over R. If Witt index of $q_0 \geqslant \dim R + 1$, then q is extended from R.*

We note however that Proposition 2.1 is a sharper result for dimension one in view of the extendibility of isotropic quadratic spaces over $R[T]$, R a Dedekind domain.

LEMMA 2.2 *Let R be a local domain with maximal ideal $\mathfrak{m}$ and residue field k. Let q_1, q_2 be quadratic spaces over R and let $\psi \in O_{k(T)}(\bar{q}_1 \perp T\bar{q}_2 \perp h)$, bar denoting reduction modulo* $\mathfrak{m}$. (O denotes the orthogonal group.) *Then there exists $f \in R[T]$, with $f(0) = 1$ and $\varphi \in O_{R[T,T^{-1}]_f}(q_1 \perp Tq_2 \perp h)$ such that $\bar{\varphi} = \psi$.*

PROOF. Let $EO(\bar{q}_1 \perp T\bar{q}_2, h)$ denote the subgroup of $O(\bar{q}_1 \perp T\bar{q}_2 \perp h)$ generated by elementary orthogonal transformations [**11**]. Then

$$O_{k(T)}(\bar{q}_1 \perp T\bar{q}_2 \perp h) = EO_{k(T)}(\bar{q}_1 \perp T\bar{q}_2, h)\cdot O_{k(T)}(h)$$

and hence there exists $f \in k[T]$ with $f(0) = 1$ such that

$$\psi = \eta\tau, \qquad \eta \in EO_{k[T,T^{-1}]_f}(\bar{q}_1 \perp T\bar{q}_2, h), \tau \in O_{k[T,T^{-1}]_f}(h).$$

Then τ is of the form

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & u \\ u^{-1} & 0 \end{pmatrix}, \qquad u \in U\big(k[T, T^{-1}]_f\big).$$

Let $f = \prod f_i^{n_i}$, f_i irreducible over $k[T]$ and $f_i(0) = 1$. Let $g_i \in R[T]$ be lifts of f_i with $g_i(0) = 1$. Let $g = \prod g_i$. The map $R[T, T^{-1}]_g \to k[T, T^{-1}]_f$ is surjective and induces a surjection also on the corresponding group of units. Hence both η and τ can be lifted to $O_{R[T,T^{-1}]_g}(q_1 \perp Tq_2 \perp h)$ and this proves the lemma.

LEMMA 2.3. *Let R, $\mathfrak{m}$, k be as in Lemma 2.2. Let $\mathfrak{F}$ be a vector bundle over $\mathbf{P}_1(R)$ such that $\bar{\mathfrak{F}} = \mathfrak{F} \otimes k \tilde{\to} \oplus_{r\ copies}\, \mathfrak{o}_{\mathbf{P}_1(k)} \oplus_{s\ copies}\, \mathfrak{o}_{\mathbf{P}_1(k)}(1)$. Let $\mathcal{G}$ be a trivial subbundle of $\bar{\mathfrak{F}}$, which is a direct summand. Then there exists a trivial subbundle $\mathfrak{F}_1$ of $\mathfrak{F}$ which is a direct summand and such that $\bar{\mathfrak{F}}_1 = \mathcal{G}$.*

PROOF. Let $i: \mathcal{G} \to \bar{\mathfrak{F}}$ be the inclusion and $p: \bar{\mathfrak{F}} \to \mathcal{G}$ a projection. Since $\mathrm{Hom}(\mathcal{G}, \bar{\mathfrak{F}})$ and $\mathrm{Hom}(\bar{\mathfrak{F}}, \mathcal{G})$ are direct sums of copies of $\mathfrak{o}$, $\mathfrak{o}(1)$ or $\mathfrak{o}(-1)$, it follows that $H^1(\mathbf{P}_1(k), \mathrm{Hom}(\mathcal{G}, \bar{\mathfrak{F}})) = H^1(\mathbf{P}_1(k), \mathrm{Hom}(\bar{\mathfrak{F}}, \mathcal{G})) = 0$. Let $\mathfrak{F}_1$ be a trivial bundle over $\mathbf{P}_1(R)$ with $\bar{\mathfrak{F}}_1 = \mathcal{G}$. Then the maps i and p can be lifted to $\tilde{i}: \mathfrak{F}_1 \to \mathfrak{F}$ and $\tilde{p}: \mathfrak{F} \to \mathfrak{F}_1$ [**2**, 4.6.2]. The map $\psi = \tilde{p}\tilde{i}$ is an automorphism of $\mathfrak{F}_1$, since its reduction modulo $\mathfrak{m}$ is identity. Then $j = \psi^{-1}\tilde{i}: \mathfrak{F}_1 \to \mathfrak{F}$ is a direct injection and $\overline{j(\mathfrak{F}_1)} = \mathcal{G}$.

THEOREM 2.4. *Let R be a complete discrete valuation ring. Any quadratic space over $R[T, T^{-1}]$ is isometric to $q_1 \perp Tq_2$, q_1, q_2 being quadratic spaces over R. In particular, any quadratic space over $R[T, T^{-1}]$ is an orthogonal sum of rank one quadratic spaces.*

PROOF. Let π be a parameter in R and $k = R/(\pi)$, bar denoting reduction modulo π, $K = R_\pi$ = quotient field of R, and let $A = R[T, T^{-1}]$. Let q be a quadratic space over A. Then by Corollary 1.6, q is stably isometric to $q_1 \perp Tq_2$, q_1, q_2 quadratic spaces over R. In view of Lemma 1.3, over $K[T, T^{-1}]$ and $k[T, T^{-1}]$, q and $q_1 \perp Tq_2$ are isometric.

Let q be anisotropic. Then q_1 and q_2 are anisotropic over R and since R is complete, the reductions $\bar{q}_1$ and $\bar{q}_2$ modulo (π) are anisotropic. Hence, by Lemma

1.1, $\bar{q}_1 \perp T\bar{q}_2$ is anisotropic. Since $\bar{q} \xrightarrow{\sim} \bar{q}_1 \perp T\bar{q}_2$, $\bar{q}$ is also anisotropic. In view of [**5**, Proposition 1.1] any isometry φ: $q \otimes_A K[T, T^{-1}] \simeq (q_1 \perp Tq_2) \otimes_A K[T, T^{-1}]$ is defined over A and hence $q \simeq q_1 \perp Tq_2$ over A.

Let now q be isotropic. Then by Lemma 1.1, q_1 or q_2 is isotropic and hence splits off an h. We assume that q is stably isometric to $q_1 \perp Tq_2 \perp h$, where q_1 and q_2 are quadratic spaces over R. Since all projective modules over $R[T, T^{-1}]$ are free, we identify isometry classes of quadratic forms with equivalence classes of symmetric matrices. Let β_1 be the matrix representing q_1, and β_2 the matrix representing q_2 (over R). Then the matrix

$$\alpha_0 = \begin{pmatrix} \beta_1 & & \\ & T\beta_2 & \\ & & h \end{pmatrix}$$

represents $q_1 \perp Tq_2 \perp h$, where $h = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let α be the matrix of q. We show that $\alpha \sim \alpha_0$, under the orthogonal equivalence of matrices. By Lemma 1.3, there exists $V \in \mathrm{GL}_N(k[T, T^{-1}])$ such that $V\bar{\alpha}V^t = \bar{\alpha}_0$, where $N = n + m + 2$, $n = \operatorname{rank} q_1$, $m = \operatorname{rank} q_2$. Since the map $\mathrm{GL}_N(R[T, T^{-1}]) \to \mathrm{GL}_N(k[T, T^{-1}])$ is surjective, $k[T, T^{-1}]$ being Euclidean, there exists $U \in \mathrm{GL}_N(A)$ such that $\bar{U} = V$. Replacing α by $U\alpha U^t$, we assume that $\bar{\alpha} = \bar{\alpha}_0$. Since $R(T^{-1})$ (the ring obtained by inverting monics in $R[T^{-1}]$) is a p.i.d., by [**14**, Theorem 3.1], α and α_0 are equivalent over $R(T^{-1})$. Hence there exists $f \in R[T]$ with $f(0) = 1$ such that $\alpha \sim \alpha_0$ over A_f. Let $W\alpha W^t = \alpha_0$, $W \in \mathrm{GL}_N(A_f)$. Then $\bar{W} \in O_{k[T,T^{-1}]_{\bar{f}}}(\bar{\alpha}_0)$ and by Lemma 2.2, there exists $f' \in R[T]$ with $f'(0) = 1$ and $X \in O_{R[T,T^{-1}]_{f'}}(\alpha_0)$ such that $\bar{X} = \bar{W}$. Replacing f by ff' and denoting it again by f, we have $(X^{-1}W)\alpha \cdot (X^{-1}W)^t = \alpha_0$ with $X^{-1}W \in \mathrm{GL}_N(R[T, T^{-1}]_f)$ and $\overline{X^{-1}W} =$ Identity. Let $\tilde{\alpha}$ be the symmetric matrix over $R[T]$ obtained by taking α over $R[T, T^{-1}]$, α_0 over $R[T]_f$ and patching them over $R[T, T^{-1}]_f$ by $Z = X^{-1}W$ (noting that T and f are coprime in $R[T]$). Then $\bar{\tilde{\alpha}}$ is obtained by patching $\bar{\alpha}_0$ over $k[T, T^{-1}]$ and $k[T]_{\bar{f}}$ by $\bar{Z} =$ Identity over $k[T, T^{-1}]_{\bar{f}}$. Hence there exists $\bar{Y} \in \mathrm{GL}_N(k[T])$ such that $\bar{Y}\bar{\tilde{\alpha}}\bar{Y}^t = \bar{\alpha}_0$. Let $Y \in \mathrm{GL}_N(R[T])$ be a lift of $\bar{Y}$. We replace $\tilde{\alpha}$ by $Y\tilde{\alpha}Y^t$ and assume that $\bar{\tilde{\alpha}} = \bar{\alpha}_0$.

Since $\tilde{\alpha} \sim \alpha$ over $R[T, T^{-1}]$, over $R(T)$, which is a p.i.d., we have $\tilde{\alpha} \sim \alpha_0$. Hence there exists $f \in R[T^{-1}]$ with $f(0) = 1$ and $B \in \mathrm{GL}_n(A_f)$ with $B\tilde{\alpha}B^t = \alpha_0$. Using similar arguments as above, by changing f suitably, we assume that $\bar{B} =$ Identity. Let

$$\alpha_0' = \begin{pmatrix} \beta_1 & & \\ & T^{-1}\beta_2 & \\ & & h \end{pmatrix}.$$

We define a vector bundle $\mathcal{F}$ over $\mathbf{P}_1(R)$ with a quadratic structure (not necessarily nonsingular) as follows: $\tilde{\alpha}$ over Spec $R[T]$, α_0' over Spec $R[T^{-1}]_f$ and the matrix

$$\begin{pmatrix} I_n & & \\ & T^{-1}I_m & \\ & & I_2 \end{pmatrix} \cdot B$$

over $\operatorname{Spec} A_f$ as a patching isometry. The bundle $\bar{\mathfrak{F}}$ is given by the patching matrix (over $k[T, T^{-1}]_{\bar{f}}$)

$$\begin{pmatrix} I_n & & \\ & T^{-1}I_m & \\ & & I_2 \end{pmatrix}$$

and hence $\bar{\mathfrak{F}}$ is isometric to $\bar{\mathfrak{F}}_1 \perp \bar{\mathfrak{F}}_2 \perp \bar{\mathfrak{F}}_3$ where $\bar{\mathfrak{F}}_1 = \oplus_{n \text{ copies}}\, \mathfrak{o}$, $\bar{\mathfrak{F}}_2 = \oplus_{m \text{ copies}}\, \mathfrak{o}(1)$, $\bar{\mathfrak{F}}_3 = \oplus_{2 \text{ copies}}\, \mathfrak{o}$ and the quadratic form on $\bar{\mathfrak{F}}$ restricted to $\bar{\mathfrak{F}}_1$ is given by the matrix $\bar{\beta}_1$, the quadratic structure on $\bar{\mathfrak{F}}_2$ being given by $(T\bar{\beta}_2, T^{-1}, T^{-1}\bar{\beta}_2)$ and $\bar{\mathfrak{F}}_3 \xrightarrow{\sim} H(\mathfrak{o})$. By Lemma 2.3, $\mathfrak{F}$ contains a trivial subbundle $\mathfrak{F}_1$ of rank n which is a direct summand such that its reduction modulo π is $\bar{\mathfrak{F}}_1$. Since the bundle $\mathfrak{F}_1$ is trivial, the quadratic form on $\mathfrak{F}$, restricted to $\mathfrak{F}_1$, is extended from a form β_1' over R and its reduction modulo π is isometric to $\bar{\beta}_1$. Since R is complete, and $\bar{\beta}_1' \simeq \bar{\beta}_1$, we have $\beta_1' \simeq \beta_1$. Hence $(\mathfrak{F}_1, \beta_1)$ splits off an *orthogonal* summand of $\mathfrak{F}$. Restricting to $\operatorname{Spec} R[T]$, we see that $\tilde{\alpha}$ contains an orthogonal summand isometric to β_1. Since $\tilde{\alpha} \sim \alpha$ over $R[T, T^{-1}]$, it follows that α splits off an orthogonal summand isometric to β_1. Let

$$\alpha = \begin{pmatrix} \beta_1 & 0 \\ 0 & \alpha' \end{pmatrix}.$$

Then α' is stably isometric to

$$\begin{pmatrix} T\beta_2 & \\ & h \end{pmatrix}.$$

Hence $T^{-1}\alpha'$ is stably extended from

$$\begin{pmatrix} \beta_2 & \\ & h \end{pmatrix}$$

and by Proposition 2.1, extended from

$$\begin{pmatrix} \beta_2 & \\ & h \end{pmatrix}.$$

Thus

$$\alpha \sim \begin{pmatrix} \beta_1 & & \\ & T\beta_2 & \\ & & h \end{pmatrix}$$

and this completes the proof of the theorem.

3. Quadratic spaces over Laurent extensions of Dedekind domains. Let $R \hookrightarrow S$ be integral domains and $h \in R$ be a nonzero element of R. Let the natural map $R/hR \to S/hS$ be an isomorphism. We call the following cartesian square a *patching diagram*:

$$\begin{array}{ccc} R & \hookrightarrow & S \\ \downarrow & & \downarrow \\ R_h & \hookrightarrow & S_h \end{array}$$

Let $\mathfrak{M}$ denote the category of quadratic spaces over R and $\mathfrak{N}$ the category whose objects are triples (q_1, φ, q_2) where q_1 is a quadratic space over S, q_2 a quadratic space over R_h and φ: $q_1 \otimes_S S_h \xrightarrow{\sim} q_2 \otimes_{R_h} S_h$ an isometry, with obvious morphisms of triples. We have a natural functor T: $\mathfrak{M} \to \mathfrak{N}$ with $T(q) = (q \otimes_R S, \mathrm{Id}, q \otimes_R R_h)$. In view of [**6**, Theorem 1], T is an equivalence of categories.

LEMMA 3.1. *Let P be a finitely-generated projective R-module. For $\alpha(T) \in \mathrm{End}_{S_h[T]}(P \otimes_R S_h[T])$ with $\alpha(0) =$ Identity, there exists an integer $N \geqslant 0$ such that for $n \geqslant N$, $\alpha(h^nT) \in \mathrm{End}_{S[T]}(P \otimes_R S[T])$.*

PROOF. Let

$$\alpha(T) = 1 + \alpha_1 T + \cdots + \alpha_m T^m, \qquad \alpha_i \in \mathrm{End}_{S_h}(P \otimes_R S_h).$$

Since P is finitely generated, there exists an integer N such that for $n \geqslant N$, $h^n\alpha_i \in \mathrm{End}_S(P \otimes_R S)$ for $1 \leqslant i \leqslant m$. Then $\alpha(h^nT) \in \mathrm{End}_{S[T]}(P \otimes_R S[T])$.

LEMMA 3.2. *Let (P, q) be a quadratic space over R. Then, for*

$$\alpha(T) \in O_{S_h[T]}(q \otimes_R S_h[T]) \quad \textit{with } \alpha(0) = \textit{Identity},$$

there exists $N \geqslant 0$ such that $\alpha(h^nT) \in O_{S[T]}(q \otimes S[T])$, $n \geqslant N$.

PROOF. We have $O_{S_h[T]}(q \otimes_R S_h[T]) \cap \mathrm{End}_{S[T]}(P \otimes_R S[T]) = O_{S[T]}(q \otimes_R S[T])$.

LEMMA 3.3. *Let (P, q) be a quadratic space over R and $Q \in \mathrm{Pic}\, R$. Let $EO(q, H(Q))$ denote the elementary orthogonal subgroup of $O(q \perp H(Q))$. Then given $\sigma \in EO_{S_h}(q, H(Q))$ and $\tau \in O_{S_h}(H(Q))$, there exists $\sigma_1 \in O_S(q \perp H(Q))$ and $\sigma_2 \in O_{R_h}(q \perp H(Q))$ such that $\tau\sigma = \sigma_1\tau\sigma_2$.*

PROOF. Let α_i, $1 \leqslant i \leqslant l$ (resp. β_j, $l+1 \leqslant j \leqslant l+s$), be a set of generators of $\mathrm{Hom}(P, Q)$ (resp. $\mathrm{Hom}(P, Q^*)$). Let $e_\lambda^k = E_{\lambda\alpha_k}$ for $1 \leqslant k \leqslant l$ and $e_\lambda^k = E^*_{\lambda\beta_k}$, for $l+1 \leqslant k \leqslant l+s$, $\lambda \in S_h$, defined in [**11**, pp. 292, 293]. Then $EO_{S_h}(q, H(Q))$ is the subgroup generated by e_λ^k, $\lambda \in S_h$, $1 \leqslant k \leqslant l+m$. Let $\sigma = \prod_{k=1}^m e_{\lambda_k}^{i_k}$ and let $\sigma_p = \prod_{k=1}^{p-1} e_{\lambda_k}^{i_k}$, $1 \leqslant p \leqslant m$. Then by Lemma 3.2, there exists an integer N such that $(\tau\sigma_p) \cdot e_{h^N T}^{i_p} \cdot (\tau\sigma_p)^{-1} \in O_{S[T]}(q \perp H(Q))$ for $1 \leqslant p \leqslant m$. Since $Sh^N + R = S$, there exist $\mu_k \in S$, $\nu_k \in R_h$ such that $\lambda_k = h^N\mu_k + \nu_k$, $1 \leqslant k \leqslant m$. Then specialising $T = \mu_p \in S$, we have,

$$(\tau\sigma_p) \cdot e_{h^N \cdot \mu_p}^{i_p} \cdot (\tau\sigma_p)^{-1} \in O_S(q \perp H(Q)) \quad \text{for } 1 \leqslant p \leqslant m.$$

We have

$$\tau\sigma = \tau \cdot \prod_{k=1}^{m} e_{\lambda_k}^{i_k} = \tau \cdot \prod_{k=1}^{m} e_{h^N \cdot \mu_k}^{i_k} \cdot e_{\nu_k}^{i_k} = \tau \cdot \prod_{k=m}^{1} \left(\sigma_k \cdot e_{h^N \mu_k}^{i_k} \sigma_k^{-1}\right) \cdot \prod_{k=1}^{m} e_{\nu_k}^{i_k}$$

$$= \prod_{k=m}^{1} \tau\sigma_k \cdot e_{h^N \mu_k}^{i_k} (\tau\sigma_k)^{-1} \cdot \tau \cdot \prod_{k=1}^{m} e_{\nu_k}^{i_k} = \sigma_1 \cdot \tau \cdot \sigma_2$$

where $\sigma_1 \in O_S(q \perp H(Q))$, $\sigma_2 \in O_{R_h}(q \perp H(Q))$.

REMARK. The idea of the proof of the above lemma is due to Suslin.

PROPOSITION 3.4. *Let R be a principal ideal domain. Then every isotropic quadratic space over $R[T, T^{-1}]$ is isometric to $q_1 \perp Tq_2$, q_1, q_2 quadratic spaces over R.*

PROOF. By Corollary 1.5, q is stably isometric to $q_1 \perp Tq_2 \perp h$, q_1, q_2 quadratic spaces over R. Let K be the quotient field of R. Over $K[T, T^{-1}]$, by Lemma 1.3, $q \simeq q_1 \perp Tq_2 \perp h$. We assume, without loss of generality, that by inverting a prime $p \in R$, $q \xrightarrow{\sim} q_1 \perp Tq_2 \perp h$. Let $\hat{R}$ denote the completion of R at the prime ideal (p). We have a patching diagram:

$$\begin{array}{ccc} R[T, T^{-1}] & \hookrightarrow & \hat{R}[T, T^{-1}] \\ \downarrow & & \downarrow \\ R_p[T, T^{-1}] & \hookrightarrow & \hat{R}_p[T, T^{-1}] \end{array}$$

Since $\hat{R}$ is a complete d.v.r., we have an isometry φ: $q \xrightarrow{\sim} q_1 \perp Tq_2 \perp h$ over $\hat{R}[T, T^{-1}]$. By assumption, there exists an isometry ψ: $q \xrightarrow{\sim} q_1 \perp Tq_2 \perp h$ over $R_p[T, T^{-1}]$. If φ and ψ coincide over $\hat{R}_p[T, T^{-1}]$, then they define an isometry

$$q \simeq q_1 \perp Tq_2 \perp h$$

over $R[T, T^{-1}]$. Otherwise, $\varphi\psi^{-1} \in O_{\hat{R}_p[T,T^{-1}]}(q_1 \perp Tq_2 \perp h)$. Since $\hat{R}_p$ is a field, in view of [**10**, Lemma 1.3], there exist $\eta \in EO_{\hat{R}_p[T,T^{-1}]}(q_1 \perp Tq_2, h)$ and $\tau \in O_{\hat{R}_p[T, T^{-1}]}(h)$ such that $\varphi\psi^{-1} = \tau\eta$. By Lemma 3.3,

$$\tau\eta = \eta_1\tau\eta_2, \qquad \eta_1 \in O_{\hat{R}[T,T^{-1}]}(q_1 \perp Tq_2 \perp h),\ \eta_2 \in O_{R_p[T,T^{-1}]}(q_1 \perp Tq_2 \perp h).$$

The element τ is of the form

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \text{ or } \begin{pmatrix} 0 & u \\ u^{-1} & 0 \end{pmatrix}, \qquad u \in U\big(\hat{R}_p[T, T^{-1}]\big).$$

Since $u = u_1u_2$, $u_1 \in U(\hat{R}[T, T^{-1}])$, $u_2 \in U(R_p[T, T^{-1}])$, $\tau = \tau_1\tau_2$, $\tau_1 \in O_{\hat{R}[T,T^{-1}]}(h)$, $\tau_2 \in O_{R_p[T,T^{-1}]}(h)$. Hence $\tau\eta = \eta_1\tau_1 \cdot \tau_2\eta_2 = \sigma_1 \cdot \sigma_2$, where $\sigma_1 = \eta_1\tau_1$, $\sigma_2 = \tau_2\eta_2$. Replacing φ and ψ by $\varphi' = (\eta_1\tau_1)^{-1}\varphi$, $\psi' = \tau_2\eta_2 \cdot \psi$, we have, $\varphi' = \psi'$ over $\hat{R}_p[T, T^{-1}]$ and hence define an isometry $q \simeq q_1 \perp Tq_2 \perp h$ over $R[T, T^{-1}]$.

THEOREM 3.5. *Let R be a Dedekind domain and q an isotropic quadratic space over $R[T, T^{-1}]$. Then $q \simeq q_1 \perp Tq_2$, q_1 and q_2 being quadratic spaces over R.*

PROOF. By Corollary 1.5, q is stably isometric to $q_1 \perp Tq_2 \perp H(Q)$, q_1, q_2 quadratic spaces over R and $Q \in \operatorname{Pic} R$. Let K denote the quotient field of R. Then by Lemma 1.3, $q \simeq q_1 \perp Tq_2 \perp H(Q)$ over $K[T, T^{-1}]$. Thus, there exists $\lambda \in R$, $\lambda \neq 0$, such that there is an isometry ψ: $q \otimes R_\lambda[T, T^{-1}] \simeq (q_1 \perp Tq_2 \perp H(Q)) \otimes R_\lambda[T, T^{-1}]$. Let $(\lambda) = \prod_{i=1}^r \wp_i^{n_i}$, $\wp_i \in \operatorname{Spec} R$ and let S denote the semilocalisation of R at the set $\{\wp_i\}_{i=1}^r$ of prime ideals of R. Then we have a patching diagram:

$$\begin{array}{ccc} R[T, T^{-1}] & \hookrightarrow & S[T, T^{-1}] \\ \downarrow & & \downarrow \\ R_\lambda[T, T^{-1}] & \hookrightarrow & S_\lambda[T, T^{-1}] \end{array}$$

Since S is a semilocal domain of dimension one, S is p.i.d. and by Proposition 3.4, we have an isometry φ: $q \otimes_R S \xrightarrow{\sim} (q_1 \perp Tq_2 \perp H(Q)) \otimes_R S$. If φ and ψ coincide over

$S_\lambda[T, T^{-1}]$, they define an isometry $q \xrightarrow{\sim} q_1 \perp Tq_2 \perp H(Q)$ over $R[T, T^{-1}]$. Otherwise, $\varphi\psi^{-1} \in O_{S_\lambda[T,T^{-1}]}(q_1 \perp Tq_2 \perp H(Q))$. Since $\lambda \in \operatorname{rad} S$, S_λ is a field and by [**10**, Lemma 1.3],

$$\varphi\psi^{-1} = \tau\eta, \quad \tau \in O_{S_\lambda[T,T^{-1}]}(H(Q)), \eta \in EO_{S_\lambda[T,T^{-1}]}(q_1 \perp Tq_2, H(Q)).$$

By Lemma 3.3, there exist

$$\eta_1 \in O_{S[T,T^{-1}]}(q_1 \perp Tq_2 \perp H(Q)), \quad \eta_2 \in O_{R_\lambda[T,T^{-1}]}(q_1 \perp Tq_2 \perp H(Q))$$

such that $\tau\eta = \eta_1\tau\eta_2$. Modifying φ and ψ by η_1 and η_2 respectively, we may assume that $\varphi\psi^{-1} = \tau$. Then identifying quadratic spaces over R with triples in $\mathfrak{N}$,

$$q \simeq (q_1 \perp Tq_2 \perp H(Q), \mathrm{Id} \perp \tau, q_1 \perp Tq_2 \perp H(Q)) \simeq q_1 \perp Tq_2 \perp q_3$$

where $q_3 \simeq (H(Q), \tau, H(Q))$. Since discriminant of q_3 is locally -1, disc $q_3 = -1$ and in view of [**1**, Proposition 5.1], $q_3 \simeq H(Q')$, $Q' \in \operatorname{Pic} R[T, T^{-1}] = \operatorname{Pic} R$. Thus $q \simeq (q_1 \perp H(Q')) \perp Tq_2$ and this completes the proof of the theorem.

REMARK 3.6. Theorem 3.5 is false for anisotropic quadratic spaces over Laurent extensions of Dedekind domains. Let q be the rank 4 quadratic space over $\mathbf{R}[X, Y]$ given by the reduced norm on the nonfree projective ideal P of $\mathbf{H}[X, Y]$ defined in [7] as the kernel of the surjective homomorphism

$$\begin{array}{ccc} \mathbf{H}[X,Y]^2 & \to & \mathbf{H}[X,Y], \\ (1,0) & \mapsto & X+i, \\ (0,1) & \mapsto & Y+j. \end{array}$$

In fact q is stably isometric to $\langle 1,1,1,1\rangle$, but not extended from $\mathbf{R}[X]$ and in fact over $\mathbf{R}[X]_{(1+x^2)}[Y]$, q remains nonextended from $\mathbf{R}[X]_{(1+x^2)}$ [**8**, Theorem 2.1]. Let $R = \mathbf{R}[X]_{(1+x^2)}$ and let $\tilde{q} = q \otimes_{\mathbf{R}[X,Y]} R[Y, Y^{-1}]$. Suppose $\tilde{q} \xrightarrow{\sim} q_1 \perp Tq_2$, q_1, q_2 quadratic spaces over R. Let K be the quotient field of R. Then, $\tilde{q}$ is stably isometric to $\langle 1,1,1,1\rangle$ and hence over $K[Y, Y^{-1}]$, $q_1 \perp Tq_2 \simeq \langle 1,1,1,1\rangle$. Since these forms are anisotropic, it follows that $q_2 = 0$. Thus, $\tilde{q}$ is stably extended from R. In this case, it should be extended from $\tilde{q}(1) \xrightarrow{\sim} \langle 1,1,1,1\rangle$. Suppose φ: $\tilde{q} \xrightarrow{\sim} \langle 1,1,1,1\rangle$ is an isometry over $R[Y, Y^{-1}]$. Since modulo Y, q and $\langle 1,1,1,1\rangle$ are anisotropic, in view of [**5**, Proposition 1.1], φ is defined over $R[Y]$ contradicting that over $R[Y]$, $q \not\simeq \langle 1,1,1,1\rangle$.

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