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AMBIENTLY UNIVERSAL SETS IN E^n

BY

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ABSTRACT. For each closed set X in E^n of dimension at most $n - 3$, we show that X fails to be ambiently universal with respect to Cantor sets in E^n; i.e., we find a Cantor set Y in E^n so that for any self-homeomorphism h of E^n, $h(Y)$ is not contained in X. This result answers a question posed by H. G. Bothe and completes the understanding of ambiently universal sets in E^n.

1. Introduction. Let M and N be subsets of E^n. We say that M is *ambiently embeddable* in N if there is a homeomorphism h of E^n onto itself so that $h(M)$ is a subset of N. Let F be a family of sets in some fixed E^n and U some fixed subset of E^n. We call U an *ambiently universal set for the family* F if each set in F is ambiently embeddable in U. For $0 \leq m \leq n$, a compact m-dimensional subset X in E^n is called a *compact ambiently universal m-dimensional* set if every compactum of dimension $\leq m$ in E^n is ambiently embeddable in X.

H. G. Bothe has made an extensive study of the existence of compact ambiently universal sets in E^n [**Bo$_1$**, **Bo$_2$**, **Bo$_3$**]. It seems to be well known that Bothe constructed a one-dimensional continuum in E^3 [**Bo$_4$**] similar to the McMillan-Row continuum [**M-R**]. However, it does not seem to be well known that his purpose was to exhibit a one-dimensional compactum in E^3 that is not ambiently embeddable in the Menger universal curve. We give a summary of Bothe's results.

In each E^n, Bothe constructed compact m-dimensional sets M_n^m, $0 \leq m \leq n$. For $n = 2m + 1$, M_n^m is the Menger universal set [**H-W**, p. 64]. He then showed that M_n^m is a compact ambiently universal m-dimensional set in E^n ($n \neq 3$) if and only if $m > n - 3$. For E^3, M_3^m is a compact ambiently universal m-dimensional set if and only if $m = 2$ or 3. Bothe defined a condition on a set X in E^n which is now known as the dimension of embedding of X and sometimes written dem X [**Ed**]. He then showed that M_n^m is an ambiently universal set for all compact subsets X of E^n for which dim $X =$ dem X and dim $X \leq m$. Bothe showed that there does not exist a compact ambiently universal 0-dimensional set nor a compact ambiently universal 1-dimensional set in E^3. His proof that there is no compact ambiently universal 0-dimensional set in E^3 answered a question posed by R.H. Bing [**Bi$_1$**]. Interestingly, this problem has received recent attention and a new proof by M. Starbird and his students [**S**].

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The question of the existence of compact ambiently universal m-dimensional sets in E^n for $n > 3$ and $m \leqslant n - 3$ has remained open [$\mathbf{Bo_1}$, p. 204] until now. Our main theorem states that for each closed set X in E^n, dim $X \leqslant n - 3$, there is a Cantor set Y in E^n so that Y is not ambiently embeddable in X. Hence, it is an easy corollary that there do not exist compact ambiently universal m-dimensional sets in E^n, $m \leqslant n - 3$.

I thank John Walsh for patiently listening and for suggestions that are reflected in §7.

2. Definitions and notation. We use S^n, B^n, and E^n to denote the n-sphere, the n-ball, and Euclidean n-space, respectively. We let dim X and diam X denote the dimension of X and the diameter of X, respectively. We will assume all manifolds are PL, piecewise-linear, subsets of E^n whenever possible. We will also assume that PL subsets of E^n are in general position whenever possible. All of the homology will be done with integer coefficients. If M is a manifold we let Bd M and Int M denote the boundary and interior of M, respectively. If M is a compact orientable 2-manifold we also let Bd M denote the boundary of M oriented in a manner consistent with an orientation on M. If J is the union of a finite collection of oriented simple closed curves and $f: J \to E^n - A$ is a map, we say f links A in case f, thought of as a 1-cycle, is not null homologous in $E^n - A$. If $J \subseteq E^n - A$ and the inclusion map from J into $E^n - A$ links A, we simply say J links A.

3. Antoine's necklace. We briefly review a specific construction of Antoine's necklace [A]. A *solid torus* is a topological space homeomorphic to $B^2 \times S^1$. Consider the embedding of four solid tori T_1, T_2, T_3, T_4 in a solid torus T as shown in Figure 1. We call this embedding an Antoine embedding.

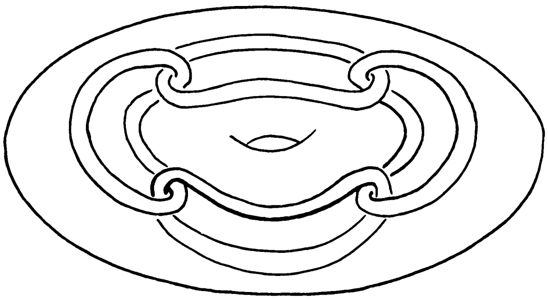

FIGURE 1

We construct Antoine's necklace, a Cantor set in E^3, $A = \bigcap M_i$, where for each nonnegative integer i, M_i is a collection of 4^i disjoint solid tori. We let M_0 be an unknotted solid torus in E^3. The collection M_{i+1} is obtained by taking an Antoine embedding of solid tori in each component of M_i. By exercising care so that the diameters of the components of M_i approach zero as i gets large, the set $A = \bigcap M_i$ will be a Cantor set.

4. I-inessential disks with holes and ramification techniques. Let H be a disk with holes and $f\colon H \to M$ a map into a manifold M so that $f(\mathrm{Bd}\, H) \subseteq \mathrm{Bd}\, M$. Following Daverman [**D$_2$**] we call the map f *I-inessential* (interior inessential) if there is a map $\tilde{f}$ from H into Bd M with $f|\,\mathrm{Bd}\, H = \tilde{f}|\,\mathrm{Bd}\, H$. We now state without proof a relationship between I-inessential maps and Antoine Cantor sets (see [**D$_2$**] for a more detailed discussion).

LEMMA 4.1. *Let H be a disk with holes and $f\colon H \to \dot{M}$, $f(\mathrm{Bd}\, H) \subseteq \mathrm{Bd}\, M$, be a map where M is a component in some stage of the construction of Antoine's necklace in E^3. If $f(H)$ misses the Cantor set, then the map f is I-inessential.*

Let M be a closed manifold. Consider the manifold $B^2 \times M$. For a positive integer m, choose m pairwise disjoint subdisks $D_1,\ldots,D_m$ of Int B^2 and form m "parallel" copies of $B^2 \times M$ by taking $D_1 \times M,\ldots,D_m \times M$. We call the set $\bigcup D_i \times M$ an m-fold ramification of $B^2 \times M$ [**D$_1$, Ea**].

Let $\alpha = a_0, a_1, a_2,\ldots$ be a sequence of positive integers. We construct a ramified Antoine's necklace with respect to α as the intersection of nested manifolds M_0, M_1, $M_2,\ldots$. The set M_0 is a single unknotted solid torus in E^3. Let i be a nonnegative integer. The manifold M_{2i+1} is obtained by taking an a_i-fold ramification of each component of M_{2i}. The manifold M_{2i+2} is obtained by taking an Antoine embedding of solid tori in each component of M_{2i+1}. Exercising due care to insure that the diameters of the components get small as i gets large yields the desired ramified Antoine's necklace as the intersection of the M_i.

We call the M_i a *special defining sequence* for the ramified Antoine's necklace. Notice that in M_{2i} we can find 4^i components which we designate by $\tilde{M}_{2i}$ so that $\tilde{M}_{2i}$ is embedded in M_0 in the same manner as the ith stage of the Antoine necklace construction is embedded in the first stage.

We now give the obvious generalization of Lemma 4.1 to a ramified Antoine's necklace.

LEMMA 4.2. *Let H be a disk with holes and $f\colon H \to M$, $f(\mathrm{Bd}\, H) \subset \mathrm{Bd}\, M$, be a map where M is a component in some stage of the construction of a ramified Antoine's necklace in E^3. If $f(H)$ misses the Cantor set, then the map f is I-inessential.*

5. There is no ambiently universal Cantor set in E^3. In this section we prove our main result in E^3. Bothe [**Bo$_2$**] and more recently Starbird and his students [**S**] have proved this theorem. Our proof uses techniques found in both of the previous proofs. This section will serve as a warm-up for the proof in higher dimensions since the strategy is similar. However, some of the techniques will be discarded when we approach the proof in higher dimensions.

THEOREM 5.1. *For each closed* 0*-dimensional set X in E^3 there is a Cantor set Y so that Y is not ambiently embeddable in X.*

PROOF. Let X be a fixed 0-dimensional set. Let T_i, $i = 0, 1, 2,\ldots,$ be a sequence of all unknotted PL solid tori in E^3 all of whose vertices have rational coordinates. In T_i choose $4^i + 1$ disjoint meridional disks. For each disk it is possible to find a compact 3-manifold that contains the disk in its interior, misses all other disks, and the boundary of the 3-manifold misses X. Let a_i be a positive integer so that for any disk D of the $4^i + 1$ meridional disks, it is possible to find a compact 3-manifolds N so that $D \subset \text{Int } N$, N misses the other 4^i meridional disks, $\text{Bd } N \cap X = \varnothing$, and the number of handles in the 2-manifold $\text{Bd } N$ is less than a_i.

Let Y be a ramified Antoine's necklace in E^3 with respect to the sequence a_i, and let M_i be a special defining sequence for Y. We now show $h(Y) \not\subset X$ for any homeomorphism h.

Suppose $h(Y) \subset X$ for some self-homeomorphism h of E^3. Without loss of generality $h(M_0)$ is a PL solid torus, and we suppose $h(M_0) = T_i$. To further simplify the proof we also suppose that h is the identity homeomorphism.

We choose the union of 4^i components of M_{2i}, denoted $\tilde{M}_{2i}$, so that $\tilde{M}_{2i}$ is embedded in M_0 in the same manner as the ith stage of the Antoine necklace construction is embedded in the 0th stage. Now each meridional disk in $M_0 = T_i$ must contain a meridional simple closed curve of some component of $\tilde{M}_{2i}$. Hence of the prechosen $4^i + 1$ meridional disks of T_i, we can find two disks D_1 and D_2, and a component W of $\tilde{M}_{2i}$ so that $W \cap D_1$ and $W \cap D_2$ each contain a meridional simple closed curve of $\text{Bd } W$.

Let N be a compact PL 3-manifold so that $D_1 \subset \text{Int } N$, $N \cap D_2 = \varnothing$, $\text{Bd } N \cap X = \varnothing$, and the number of handles of $\text{Bd } N$ is less than a_i. In $\text{Bd } N$ we find a 2-manifold M such that $\text{Bd } M$ links W. The proof of this fact is somewhat technical and we defer the proof to the end of this section.

Let $W(1), W(2), \ldots, W(a_i)$ be the components of M_{2i+1} that lie inside W. We assume that $W(j)$ is in general position with respect to the 2-manifold M. Since the number of handles of M is less than the number of handles of $\text{Bd } N$, some one of the $W(j)$, which we designate W', must have the property that $W' \cap M$ is a 2-manifold with no handles; i.e., each component of $W' \cap M$ is a disk with holes. Furthermore, since $Y \cap M = \varnothing$, the inclusion of each component of $W' \cap M$ into W' is 1-inessential. This implies that $\text{Bd } M$ does not link W'. Since W' is "parallel" to W, $\text{Bd } M$ must link both W and W' or neither W nor W'. We are led to a contradiction from our supposition that $h(Y) \subset X$. We are therefore forced to conclude that $h(Y) \not\subset X$. Our proof is now complete with the exception of the following lemma.

Lemma 5.2. *Let D_1 and D_2 be disks in E^3, N a compact 3-manifold $D_1 \subset \text{Int } N$, $N \cap D_2 = \varnothing$, and W a solid torus so that $W \cap D_1$ and $W \cap D_2$ each contain a meridional simple closed curve of* $\text{Bd } W$. *Then there is a 2-manifold M contained in* $\text{Bd } N$ *so that* $\text{Bd } M$ *links W.*

Proof. Let $\tilde{W}$ be a small regular neighborhood of W so that $\text{Bd } \tilde{W}$ is in general position with $\text{Bd } N$. Let J_1 and J_2 be meridional simple closed curves of $\tilde{W}$ so that $J_1 \subset \text{Int } N$, $J_2 \cap N = \varnothing$, and J_1, J_2 each bound homologically in the complement of $\text{Bd } N$. Choose an annulus A in $\text{Bd } \tilde{W}$ with boundary components J_1 and J_2. Let J be the collection of all simple closed curves, of $A \cap \text{Bd } N$. Orient the simple closed curves of J consistent with some orientation on $A \cap N$. Hence, as a 1-cycle, J is homologous to J_1 in $\text{Bd } \tilde{W}$, and J links W in E^3. Using $A \cap N$ it is easy to see that J

bounds a 2-cycle in N. Similarly, J bounds a 2-cycle in closure $(E^3 - N)$. Therefore, a Mayer-Vietoris argument shows that J also bounds homologically in Bd N.

We now use geometric interpretations of homology theory to show the existence of the desired 2-manifold M with Bd $M \subset J$. We assume the Bd N has an oriented triangulation so that J is contained in the 1-skeleton. By simplicial homology theory $J = \partial \Sigma \alpha_i \sigma_i$ where α_i are integers and the σ_i are all of the finitely many oriented 2-simplexes of the triangulation of Bd N. Let $\alpha = \max\{\alpha_i\}$ and set $M' = \bigcup \{\sigma_i \mid \alpha_i = \alpha\}$. One readily checks that M' is a 2-manifold, Bd $M' \subset J$, and Bd M' is oriented in the same manner as J. If Bd M' links W, let $M = M'$; otherwise, consider $J' = J -$ Bd M'. Then J' must link W and be null homologous in Bd N. Since J' has fewer components than J, an inductive argument on the number of components of J yields the desired manifold M.

6. Bing Cantor sets. R. H. Bing's proof that the sewing of two Alexander horned spheres yields S^3 [**Bi$_2$**] consisted in showing that a Cantor set could be described in E^3 as the intersection of manifolds M_i, $i = 0, 1, 2, \ldots$. The manifold M_0 is an unknotted solid torus. Each component of M_i is a solid torus and contains two components of M_{i+1} which are embedded as shown in Figure 2. Bing's clever proof showed that the M_i could be chosen in such a manner that the diameters of the components of M_i tend to zero as i gets large. The intersection of the M_i yields a Cantor set in E^3 which we call a *Bing Cantor set*.

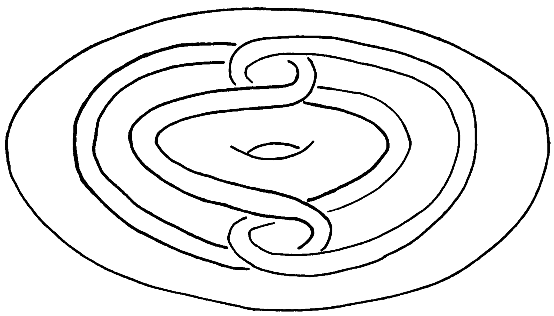

FIGURE 2

Consider the 3-cell A in $E^3_+ = \{(x, y, z) \in E^3 \mid z \geqslant 0\}$ that contains two properly embedded arcs A_1 and A_2 as shown in Figure 3. Notice that Figure 2 can be obtained from Figure 3 by reflecting through the x-y plane and thickening the resulting simple closed curves. By spinning E^3_+ into E^n ($n \geqslant 3$) [**C-D**] we then obtain as the spin of A a manifold T homeomorphic to $B^2 \times S^{n-2}$ that contains the spin of A_1 and the spin of A_2, two geometrically linked $(n-2)$-spheres, denoted by S_1 and S_2, in the interior of T. Notice that there are obvious $(n-1)$-cells D_1, D_2 in Int T so that $S_i = \text{Bd}\, D_i$ $(i = 1, 2)$ and $D_i \cap S_j$ $[(i, j) = (1, 2), (2, 1)]$ is homeomorphic to S^{n-3}. Observe that $D_1 \cup S_2$ contains a core of T so that any map $f\colon X \to E^n - (D_1 \cup S_2)$ is homotopic to a map to $E^n - \text{Int}\, T$, the homotopy fixing points in $f^{-1}(E^n - \text{Int}\, T)$. For $n > 3$ it is also true that a map $g\colon S^1 \to D_i - S_j$ $[(i, j) = (1, 2), (2, 1)]$ is null homotopic in $D_i - S_j$ if and only if g does not link S_j in E^n. This last fact is not true for $n = 3$.

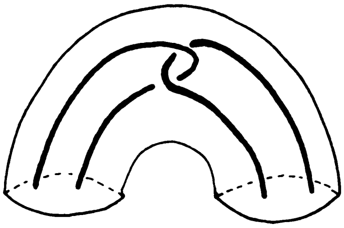

FIGURE 3

Let T_1 and T_2 be disjoint regular neighborhoods of S_1 and S_2 in Int T. We construct generalizations of the Bing Cantor set in E^n as the intersection of nested manifolds W_i, $i = 0, 1, 2, \ldots$. The manifold W_0 is an unknotted $B^2 \times S^{n-2}$ in E^n, and each component of W_i contains two components of W_{i+1} which are embedded in W_i in the same manner as $T_1 \cup T_2$ is embedded in T. It is a consequence of **[C-D**, §8] that the W_i may be chosen so that the diameters of the components tend to zero as i gets large. We also call the intersection of the W_i a Bing Cantor set.

Of course the ramification process of §4 can be applied to Bing Cantor sets to obtain *ramified Bing Cantor sets.*

7. The I-inessential property revisited. Let H be a disk with holes and $f\colon H \to M$, $f(\mathrm{Bd}\, H) \subset \mathrm{Bd}\, M$, a map where M is a component in some stage of the construction of a Bing Cantor set in E^n ($n > 3$). The arguments of §4 can be generalized to show that if $f(H)$ misses the Bing Cantor set, then the map f is I-inessential. However, we will have need of the above fact when H is a compact 2-dimensional polyhedron that behaves like a disk with holes. By a 2-dimensional polyhedron we will always mean a polyhedron that is strictly 2-dimensional, i.e., each open subset is 2-dimensional.

DEFINITION 7.1. Let P be a compact 2-dimensional polyhedron. For some fixed triangulation T of P, let *boundary* of P be the union of all 1-simplexes of T that are the face of exactly one 2-simplex of T. Clearly this is independent of the choice of the triangulation since we could also define the boundary to be the closure of the set $\{x \in P \mid H_2(P, P - x) = 0\}$ as is done in defining the boundary of a homology manifold **[Sp**, p. 277]. We let Bd P denote the boundary of P, and we define the *interior* of P, denoted Int P, to be $P - \mathrm{Bd}\, P$.

DEFINITION 7.2. Let Q be a compact 2-dimensional polyhedron. We call Q a *pseudo disk with holes* if for each $\alpha \in H_1(Q)$ there is a nonzero integer m so that $m\alpha$ is in the image of $H_1(\mathrm{Bd}\, Q)$ under the inclusion induced homomorphism.

DEFINITION 7.3. Let f be a map of a pseudo disk with holes Q into a manifold M so that $f(\mathrm{Bd}\, Q) \subseteq \mathrm{Bd}\, M$. We call the map f *I-inessential* (interior inessential) if there is a map $\tilde{f}$ from Q into Bd M with $\tilde{f} \mid \mathrm{Bd}\, Q = f \mid \mathrm{Bd}\, Q$. Otherwise the map f is I-essential.

THEOREM 7.4. *Let Q be a pseudo disk with holes. If P is a compact 2-dimensional subpolyhedron of Q so that* Int P *is an open subset of Q, then P is a pseudo disk with holes.*

PROOF. Let γ be a 1-cycle in P. By hypothesis there is a nonzero integer m so that $m\gamma$ is homologous in Q to a 1-cycle β of Bd Q. Notice that $\mathrm{Bd}\, Q \subset Q - \mathrm{Int}\, P$.

Consider the following Mayer-Vietoris sequence:

$$\to H_1(\mathrm{Bd}\, P) \to H_1(P) \oplus H_1(Q - \mathrm{Int}\, P) \to H_1(Q) \to .$$

Since the element represented by $m\gamma \oplus (-\beta)$ is sent to zero, we can find a 1-cycle δ in Bd P that is homologous to $m\gamma$ in P.

THEOREM 7.5. *Let P be a compact 2-dimensional polyhedron. Suppose $Q_1, Q_2, \ldots, Q_n$ are disjoint 2-dimensional subpolyhedra with* Int Q_i *an open subset of P for each i. If $n >$* rank $H_1(P)$, *then some Q_i is a pseudo disk with holes.*

PROOF. Suppose each Q_i is not a pseudo disk with holes. Choose γ_i a 1-cycle in Q_i so that no nonzero multiple of γ_i represents an element of $H_1(Q_i)$ that is the image of $H_1(\mathrm{Bd}\, Q_i)$ under the inclusion induced homomorphism. Since $n >$ rank $H_1(P)$, there exist integers $m_1, m_2, \ldots, m_n$, not all zero, so that $m_1\gamma_1 + m_2\gamma_2 + \cdots + m_n\gamma_n$ is null homologous in P. Without loss of generality, assume $m_1 \neq 0$. A Mayer-Vietoris argument applied to Q and $P - \mathrm{Int}\, Q_1$ shows there is a 1-cycle β in $Q_1 \cap (P - \mathrm{Int}\, Q_1) = \mathrm{Bd}\, Q_1$ that is homologous to $m_1\gamma_1$ in Q_1. This contradicts the choice of γ_1 and the theorem is proved.

Let $T, T_1, T_2, S_1, S_2, D_1, D_2$ be defined as in §6.

THEOREM 7.6. *Let $f\colon Q \to T \subset E^n$ $(n > 3)$ be an I-essential map from a pseudo disk with holes so that f is in general position with respect to $T_1 \cup T_2$. Let $Q_i = f^{-1}(T_i)$, $i = 1, 2$. Then $f \mid Q_i\colon Q_i \to T_i$ is an I-essential map from a pseudo disk with holes for $i = 1$ or $i = 2$.*

PROOF. By Theorem 7.4 one easily checks that Q_i is either empty or a pseudo disk with holes for each i. If $f \mid Q_i$ fails to be I-essential for each i, then we may find a new map $f_1\colon Q \to T - (S_1 \cup S_2)$ that agrees with f on Bd Q and is in general position with respect to the $(n-1)$-cell D_1.

Let $\Gamma = f_1^{-1}(D_1)$. We now show $f_1 \mid \Gamma\colon \Gamma \to \mathrm{Int}\, D_1 - S_2$ induces the trivial homomorphism between the first homology groups. Suppose not. Then there is a 1-cycle γ in Γ so that $f_1(\gamma)$ is a 1-cycle in Int D_1 that links $D_1 \cap S_2$. Hence $f_1(\gamma)$ links S_2 in E^n. Since Q is a pseudo disk with holes, there is a 1-cycle β in Bd Q so that β is homologous to $m\gamma$ in Q for some nonzero integer m. Since $m \neq 0$, $mf_1(\gamma)$ also links S_2. Because the support of $f_1(\beta)$ is in Bd M, $f_1(\beta)$ does not link S_2. But this is impossible since $f_1(\beta)$ is homologous to $mf(\gamma)$ in $E^n - S_2$. Hence, f_1 induces the trivial homomorphism on first homology.

Since the fundamental group of Int $D_1 - S_2$ is the same as the first homology group, we find that $f_1 \mid \Gamma\colon \Gamma \to \mathrm{Int}\, D_1 - S_2$ is homotopic to a constant map. Using this fact and the collar structure on D_1, we can find a new map f_2 of Q into T that agrees with f_1 on Bd Q and misses $D_1 \cup S_2$. Recall that $D_1 \cup S_2$ contains a core of T, and we can modify f_2 to get a map $f_3\colon Q \to \mathrm{Bd}\, T$ agreeing with f_2 on Bd Q. This contradicts the fact that f is I-essential, and our theorem is proved.

Theorem 7.6 and standard techniques now give our required generalization of the first paragraph of this section which we now state.

THEOREM 7.7. *Let Q be a pseudo disk with holes and $f\colon Q \to M$, $f(\mathrm{Bd}\, Q) \subset \mathrm{Bd}\, M$, a map where M is a component in some stage of the construction of a (ramified) Bing Cantor set in E^n $(n > 3)$. If $f(Q)$ misses the Cantor set, then the map f is I-inessential.*

THEOREM 7.8. *Let M be a PL n-manifold in E^n and $f\colon P \to E^n$ a map from a polyhedron. Suppose $Q = f^{-1}(M)$ is a subpolyhedron of P that is a pseudo disk with holes whose interior is open in P and $f | Q\colon Q \to M$ is I-inessential. If γ is a 1-cycle in $P - Q$ that bounds homologically in P, then $f(\gamma)$ bounds homologically in $E^n - M$.*

PROOF. Since $f | Q$ is I-inessential and $\mathrm{Int}\, Q$ is open in P, we can find a new map $f'\colon P \to E^n - \mathrm{Int}\, T$, agreeing with f on $P - \mathrm{Int}\, Q$. Since γ bounds in P, $f'(\gamma)$ bounds in $f'(P)$. Hence $f'(\gamma)$ bounds in $E^n - \mathrm{Int}\, T$ and, therefore, in $E^n - T$. But $f(\gamma) = f'(\gamma)$ and our proof is complete.

8. Linking Bing Cantor sets. Let T, T_1, T_2, S_1, S_2, D_1, D_2 be as defined in §6.

THEOREM 8.1. *Let $f\colon M \to E^n$ $(n > 3)$ be a PL map in general position from a compact orientable 2-manifold M so that $f | \mathrm{Bd}\, M$ links $\mathrm{Int}\, T$. Then there is a simple closed curve J in M so that $f | J$ links either T_1 or T_2.*

PROOF. We first consider the case where $f(M)$ misses $T_1 \cup T_2$. We assume that f is in general position with respect to D_1 so that $f^{-1}(D_1)$ is a finite collection of disjoint simple closed curves in M. If f restricted to any of the simple closed curves links T_2 we are done. Otherwise, f restricted to each simple closed curve is null homotopic in $D_1 - S_2$, and the manifold M can be surgered to obtain a new manifold M' and a map $f'\colon M' \to E^n$ so that $\mathrm{Bd}\, M = \mathrm{Bd}\, M'$, $f | \mathrm{Bd}\, M = f' | \mathrm{Bd}\, M'$, and $f'(M')$ misses $D_1 \cup S_2$. But this implies that $f | \mathrm{Bd}\, M = f | \mathrm{Bd}\, M'$ does not link $\mathrm{Int}\, T$ which is a contradiction.

If $f(M)$ meets $T_1 \cup T_2$, we take small regular neighborhoods T_1' and T_2' of T_1 and T_2, respectively, so that f is in general position with respect to T_1' and T_2'. Hence, $f^{-1}(\mathrm{Bd}\, T_1' \cup \mathrm{Bd}\, T_2')$ is a finite collection of disjoint simple closed curves. If f restricted to any of these simple closed curves is not null homotopic in $\mathrm{Bd}\, T_1' \cup \mathrm{Bd}\, T_2'$, we are done (requires $n > 3$). If f restricted to each simple closed curve is null homotopic in $\mathrm{Bd}\, T_1' \cup \mathrm{Bd}\, T_2'$, then M can be surgered on its interior to obtain a 2-manifold M' and a map $f'\colon M' \to E^n$ so that $\mathrm{Bd}\, M = \mathrm{Bd}\, M'$, $f | M \cap M' = f' | M \cap M'$, $M' - M$ is a collection of disks in the interior of M', and $f'(M')$ misses $\mathrm{Bd}\, T_1' \cup \mathrm{Bd}\, T_2'$. By ignoring any components of M' that are sent by f to $\mathrm{Int}(T_1' \cup T_2')$, we may assume that $f'(M')$ misses $T_1 \cup T_2$. By the previous case there is a simple closed curve J in M' so that $f' | J$ links T_1 or T_2. The simple closed curve can be pushed off the disks of $M' - M$ so that $f' | J = f | J$, and the proof is complete.

Let $W_0, W_1, \ldots, W_m$ be nested manifolds in E^n as given in the construction of the Bing Cantor set in §6.

THEOREM 8.2. *Let $A_0 \subset A_1 \subset \cdots \subset A_m$ be absolute neighborhood retracts in E^n $(n > 3)$ so that the inclusion $A_{i-1} \subset A_i$ induces the trivial map on first homology, $1 \leqslant i \leqslant m$. If $f_0\colon S^1 \to A_0$ is a map that links W_0 in E^n, then there is a map $f_i\colon S^1 \to A_i$ that links some component of W_i.*

PROOF. The proof is by induction. Suppose f_i: $S^1 \to A_i$ is given so that f_i links some component T of W_i. Since the inclusion from A_i into A_{i+1} is trivial on first homology, we find an orientable 2-manifold M and a map f: $M \to A_{i+1}$ so that $f |$ Bd M links T. Let T_1', T_2' be small regular neighborhoods of the components T_1, T_2 of W_{i+1} in T, respectively. Let $\tilde{f}$ be a close approximation to f that is in general position, the closeness will be stipulated later. By Theorem 8.2 there is a simple closed curve J in M so that $\tilde{f}|J$ links T_1' or T_2'. Since A_{i+1} is an absolute neighborhood retract, then $\tilde{f}|J$ is homotopic to a map g: $J \to A_{i+1}$. We now assume f is close enough to f so that the image of the homotopy misses $T_1 \cup T_2$. Let h be a homeomorphism from S^1 to J. Then $f_{i+1} = g \circ h$ is the desired map.

9. The n-dimensional theorem ($n > 3$).

THEOREM 9.1. *Each closed set X in E^n ($n > 3$) of dimension at most ($n - 3$) fails to be ambiently universal with respect to the family of Cantor sets in E^n.*

PROOF. Let $J_0, J_1, J_2, \ldots$ be the collection of all polygonal simple closed curves in $E^n - X$ all of whose vertices have rational coordinates. For each i set $A_0^i = J_i$ and find compact 2-dimensional polyhedra $A_1^i \subset A_2^i \subset \cdots \subset A_i^i \subset P_i$ in $E^n - X$ so that each has boundary J_i and the inclusion from A_{j-1}^i into A_j^i and from A_i^i into P_i is trivial on first homology for all j, $1 \leq j \leq i$. It may be helpful to think of P_i as a "grope" as described by J. W. Cannon [**C**]. The construction of P_i depends on the fact that a 1-cycle in $E^n - X$ bounds homologically in $E^n - X$ [**H-W**].

Let $a_i = \operatorname{rank} H_1(P_i) + 1$. Let Y be a ramified Bing Cantor set with respect to the sequence $a_0, a_1, a_2, \ldots$. Let $M_0, M_1, \ldots$ be a special defining sequence for Y. Recall that M_{2i+1} is obtained by taking an a_i-fold ramification of each component of M_{2i}. The manifold M_{2i+2} has two components in each component of M_{2i+1} that are embedded in the way that $T_1 \cup T_2$ is embedded in T in §6.

We now suppose that h: $E^n \to E^n$ is a homeomorphism so that $h(Y) \subset X$. We assume, for simplicity, in what follows that h is the identity homeomorphism. Some polygonal simple closed curve J_i must link M_0. In M_{2i} we find 2^i components, designated $\tilde{M}_{2i}$, that lie in M_0 in the same manner as W_i is embedded in W_0 in the Bing Cantor set construction. By Theorem 8.2 there is a map f: $S^1 \to A_i^i$ that links some component V of $\tilde{M}_{2i}$. Let $V(1), V(2), \ldots, V(a_i)$ be the components of M_{2i+1} in V. We assume P_i is in general position with each $V(j)$. For each j, $Q_j = V(j) \cap P_i$ is a two complex, Bd $Q_j \subset$ Bd V_j, and Int Q_j is open in P_i. Since $a_i > \operatorname{rank} H_1(P_i)$, Theorem 7.5 shows that Q_j is a pseudo disk with holes for some fixed j. Since Q_j misses X (and therefore Y), the inclusion of Q_j into $V(j)$ is I-inessential. Recall that the inclusion of A_i^i into P_i is trivial on first homology. Therefore, we can invoke Theorem 7.8 to see that the map f does not link $V(j)$. However, this implies that f does not link V, a contradiction.

REFERENCES

[**A**] L. Antoine, *Sur l'homeomorphie de deux figures et de leur voisinages*, J. Math. Pures Appl. **86** (1921), 221–325.

[**Bi$_1$**] R. H. Bing, *Tame Cantor sets in E^3*, Pacific J. Math. **11** (1961), 435–446.

[**Bi$_2$**] ______, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. (2) **56** (1952), 354–362.

[**Bo$_1$**] H. G. Bothe, *Universalmengen bezuglich der Lage im E^n*, Fund. Math. **56** (1964), 203–212.

[**Bo$_2$**] ______, *Zur Lage null-und eindimensionaler Punktmengen in E^3*, Fund. Math. **58** (1966), 1–30.

[**Bo$_3$**] ______, *Zwei Universalmengen im euklidischen Raum*, Math. Nachr. **39** (1969), 117–133.

[**Bo$_4$**] ______, *Ein eindimensionales Kompaktum in E^3, das sich nicht lagetreu in die Mengersche Universalkurve einbetten laesst*, Fund. Math. **54** (1964), 251–258.

[**C**] J. W. Cannon, *The recognition problem: What is a topological manifold?*, Bull. Amer. Math. Soc. **84** (1978), 832–866.

[**C-D**] J. W. Cannon and R. J. Daverman, *Cell-like decompositions arising from mismatched sewings: applications to 4-manifolds*, Fund. Math. **111** (1981), 211–233.

[**D$_1$**] R. J. Daverman, *On the scarcity of tame disks in certain wild cells*, Fund. Math. **79** (1973), 63–77.

[**D$_2$**] ______, *On the absence of tame disks in certain wild cells*, Geometric Topology (Proc. Geometric Topology Conf., Park City, Utah, 1974) (L. C. Glaser and T. B. Rushing, editors), Springer-Verlag, Berlin and New York, 1975, pp. 142–155.

[**Ea**] W. T. Eaton, *A generalization of the dog bone space to E^n*, Proc. Amer. Math. Soc. **39** (1973), 379–387.

[**Ed**] R. D. Edwards, *Demension theory*. I, Geometric Topology (Proc. Geometric Topology Conf., Park City, Utah, 1974) (L. C. Glaser and T. B. Rushing, editors), Springer-Verlag, Berlin and New York, 1975, pp. 195–211.

[**H-W**] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Univ. Press, Princeton, N. J., 1941.

[**M-R**] D. R. McMillan and H. Row, *Tangled embeddings of one-dimensional continua*, Proc. Amer. Math. Soc. **22** (1969), 378–385.

[**Sp**] E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

[**S**] M. Starbird, *There is no universal Cantor set in E^3*, preprint.

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