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# SPECTRAL DECOMPOSITION WITH MONOTONIC SPECTRAL RESOLVENTS

BY

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ABSTRACT. The spectral decomposition problem of a Banach space over the complex field entails two kinds of constructive elements: (1) the open sets of the field and (2) the invariant subspaces (under a given linear operator) of the Banach space. The correlation between these two structures, in the framework of a spectral decomposition, is the spectral resolvent concept. Special properties of the spectral resolvent determine special types of spectral decompositions. In this paper, we obtain conditions for a spectral resolvent to have various monotonic properties.

**1. Introduction.** A spectral decomposition of a Banach space $X$, by a bounded linear operator $T\colon X \to X$,

(a) expresses $X$ as a finite linear sum of $T$-invariant subspaces $X_i$;

(b) represents $T$ as the sum of its restrictions $T_i = T \mid X_i$;

(c) localizes the spectrum $\sigma(T_i)$ of each $T_i$ in the closure of a given open set $G_i$, which intersects the spectrum $\sigma(T)$ of $T$.

The relationship between the invariant subspaces $X_i$ and the open sets $G_i$, formalized under the name of spectral resolvent, has been the study of some recent works [**1, 2, 8**]. In this paper, we investigate conditions under which the spectral resolvent possesses certain specific monotonic properties. Such conditions and subsequence properties infer the corresponding spectral decompositions.

For a bounded linear operator $T$, which maps an abstract Banach space $X$ over the complex field $\mathbf{C}$ into itself, we use the following notation: spectrum $\sigma(T)$, point spectrum $\sigma_p(T)$, resolvent set $\rho(T)$, the unbounded component of the resolvent set $\rho_\infty(T)$, and the resolvent operator $R(\cdot\,; T)$. If $T$ has the single valued extension property then, for $x \in X$, $\sigma_T(x)$ denotes the local spectrum, $\rho_T(x)$ the local resolvent set and $x(\cdot)$ the local resolvent function.

For a subspace (closed linear manifold) $Y$ of $X$, $T \mid Y$ is the restriction of $T$ to $Y$ and $T/Y$ is the coinduced operator on the quotient space $X/Y$. $\operatorname{Inv} T$ denotes the lattice of the invariant subspaces of $X$ under $T$. $T^*$ is the conjugate of $T$. If $A$ is a subset of $X$ then $A^\perp$ denotes the annihilator of $A$ in the dual space $X^*$. Given a set $S$, we write $\bar{S}$ for the closure, $S^c$ for the complement, $d(\lambda, S)$ for the distance from a point $\lambda$ to $S$, and express by $\operatorname{cov} S$, the collection of all finite open covers of $S$. $\mathfrak{G}$ stands for the family of all open subsets of $\mathbf{C}$. An open set $\Delta$ is called a Cauchy

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domain if it has a finite number of components and the boundary $\Gamma = \partial\Delta$ is a positively oriented finite system of closed, nonintersecting, rectifiable Jordan curves.

Throughout this paper $T$ is a bounded linear operator mapping the underlying Banach space $X$ into itself.

1.1. Definition. A spectral decomposition of $X$ by $T$ is a finite system $\{(G_i, X_i)\} \subset \mathfrak{G} \times \operatorname{Inv} T$, satisfying the following conditions:

(i) $\{G_i\} \in \operatorname{cov} \sigma(T)$;

(ii) $X = \sum_i X_i$;

(iii) $\sigma(T \mid X_i) \subset \overline{G}_i$, for all $i$.

1.2. Definition [1]. A map $E\colon \mathfrak{G} \to \operatorname{Inv} T$ is called a spectral resolvent of $T$ if it satisfies the following conditions:

(I) $E(\varnothing) = \{0\}$;

(II) for any $\{G_i\} \in \operatorname{cov} \sigma(T)$, $\{(G_i, E(G_i))\}$ is a spectral decomposition of $X$ by $T$.

Although the spectral resolvent fails to be unique, the properties they have in common characterize specific types of spectral decompositions. In this vein, we mention that an operator $T$ having a spectral resolvent possesses the single valued extension property [1] and, moreover, it is decomposable [8] in the sense of Foiaş [4].

The following types of invariant subspaces will be involved in our study.

1.3. Definition [5]. A subspace $Y$ of $X$ is said to be analytically invariant under $T$ if, for every function $f\colon D \to X$ analytic on some open $D \subset \mathbf{C}$, the condition

$$(\lambda - T)f(\lambda) \in Y \quad \text{on } D$$

implies that $f(\lambda) \in Y$ on $D$.

An analytically invariant subspace is also invariant under $T$ [6].

1.4. Definition [4]. $Y \in \operatorname{Inv} T$ is said to be a spectral maximal space of $T$ if, for any $Z \in \operatorname{Inv} T$, the inclusion $\sigma(T \mid Z) \subset \sigma(T \mid Y)$ implies that $Z \subset Y$.

If $T$ has the single valued extension property then, for any set $S \subset \mathbf{C}$,

$$X_T(S) = \{x \in X\colon \sigma_T(x) \subset S\}$$

is a linear manifold in $X$. If $T$ is a decomposable operator then, for any $G \in \mathfrak{G}$, $\overline{X_T(G)}$ is an analytically invariant subspace under $T$ [5] and, for any closed $F \subset \mathbf{C}$, $X_T(F)$, in particular $X_T(\overline{G})$, is a spectral maximal space of $T$ [4]. Moreover, for a decomposable $T$, we have

$$\overline{G \cap \sigma(T)} \subset \sigma\left[T \mid X_T(\overline{G})\right] \subset \overline{G} \cap \sigma(T). \tag{1.1}$$

1.5. Definition [9]. $Y \in \operatorname{Inv} T$ is said to be a $T$-absorbent space if, for every $y \in Y$ and all $\lambda \in \sigma(T \mid Y)$, the equation $(\lambda - T)x = y$ has all solutions $x$, if any, contained in $Y$.

If $T$ has the single valued extension property, then every $T$-absorbent space is analytically invariant under $T$.

1.6. Proposition [2]. *Let $\{(G_i, X_i)\}_{i=1,2}$ be a spectral decomposition of $X$ by $T$ in terms of $T$-absorbent spaces $X_1$ and $X_2$. Then*

$$\sigma(T \mid X_1 \cap X_2) \subset \sigma(T \mid X_1) \cap \sigma(T \mid X_2).$$

1.7. PROPOSITION. *If, for* $X_1$, $X_2 \in \operatorname{Inv} T$, $X = X_1 + X_2$ *then*

$$\sigma(T) \subset \sigma(T \mid X_1) \cup \sigma(T \mid X_2) \cup \sigma_p(T). \tag{1.2}$$

*In particular, if T has the single valued extension property, then*

$$\sigma(T) \subset \sigma(T \mid X_1) \cup \sigma(T \mid X_2).$$

PROOF. Let $\lambda \in \rho(T \mid X_1) \cap \rho(T \mid X_2) - \sigma_p(T)$ and $x \in X$. There is a representation for $x$, $x = x_1 + x_2$ with $x_i \in X_i$, $i = 1, 2$. For $y_i = R(\lambda; T \mid X_i)x_i$, $i = 1, 2$, and $y = y_1 + y_2$ we have

$$(\lambda - T)y = (\lambda - T)y_1 + (\lambda - T)y_2 = x_1 + x_2 = x$$

and hence $\lambda - T$ is surjective. Furthermore, since $\lambda \notin \sigma_p(T)$, we have $\lambda \in \rho(T)$. The last statement of the proposition follows from [**3**, Theorem 2]. □

Property (1.1) of $X_T(\cdot)$ has an interesting variant in terms of a spectral resolvent $E$, expressed by [**8**, Proposition 16]. For completeness, we recall that property and provide it with a shorter proof.

1.8. PROPOSITION. *If T has a spectral resolvent E then, for any* $G \in \mathfrak{G}$,

$$\overline{G \cap \sigma(T)} \subset \sigma[T \mid E(G)]. \tag{1.3}$$

PROOF. Let $\lambda \in \overline{G \cap \sigma(T)}$ be given and let $H \in \mathfrak{G}$ be such that $\{G, H\} \in \operatorname{cov} \sigma(T)$ with $\lambda \notin \overline{H}$. Then $X = E(G) + E(H)$ and Proposition 1.7 implies

$$\sigma(T) \subset \sigma[T \mid E(G)] \cup \sigma[T \mid E(H)]. \tag{1.4}$$

Since $\lambda \in [G \cap \sigma(T)] - \overline{H}$, it follows from (1.4) that $\lambda \in \sigma[T \mid E(G)]$ and hence inclusion (1.3) holds. □

If $T$ has a spectral resolvent $E$, then $T$ has a maximal spectral resolvent $E_m$ in the sense that, for every $G \in \mathfrak{G}$ and all spectral resolvents $E$ of $T$,

$$E(G) \subset E_m(G) = X_T(\overline{G}).$$

Since, clearly $\overline{X_T(G)} \subset X_T(\overline{G})$, where the inclusion may be proper, some spectral resolvents $E$ may be such that

$$\overline{X_T(G)} \subset E(G) \subset X_T(\overline{G}) \quad \text{for all } G \in \mathfrak{G}. \tag{1.5}$$

Condition (1.5) endows $E$ with some remarkable properties, which will be the topic of the following sections.

## 2. Monotonic spectral resolvents.

2.1. DEFINITION. A spectral resolvent $E$ is said to be monotonic if $G_1, G_2 \in \mathfrak{G}$ and $\overline{G}_1 \subset G_2$ imply that $E(G_1) \subset E(G_2)$.

Note that (1.5) is a sufficient condition for a spectral resolvent $E$ of $T$ to be monotonic. In fact, if the open sets $G_1$, $G_2$ are such that $\overline{G}_1 \subset G_2$, then (1.5) implies the inclusions

$$E(G_1) \subset X_T(\overline{G}_1) \subset \overline{X_T(G_2)} \subset E(G_2).$$

2.2. THEOREM. *Let $T$ have a spectral resolvent $E$. If for any pair $G_1, G_2 \in \mathfrak{G}$, $E$ satisfies condition*

$$\sigma[T \mid E(G_1) \cap E(G_2)] \subset \bar{G}_1 \cap \bar{G}_2 \tag{2.1}$$

*then property* (1.5) *holds and $E$ is monotonic.*

PROOF. Given $G_1 \in \mathfrak{G}$, let $x \in X_T(G_1)$. Choose $G_2 \in \mathfrak{G}$ such that $\{G_1, G_2\} \in \operatorname{cov} \sigma(T)$ and $\sigma_T(x) \cap \bar{G}_2 = \varnothing$ (this is possible because $\sigma_T(x)$ is closed and is contained in $G_1$). To avoid repetitions, we divide the remainder of the proof in two parts.

*Part* A. There is a representation of $x$,

$$x = x_1 + x_2 \quad \text{with } x_i \in E(G_i),\ i = 1, 2.$$

In view of some elementary properties, the local spectra of $x_1$ and $x_2$ are contained in some pertinent sets

$$\sigma_T(x_1) \subset \sigma_T(x) \cup (\bar{G}_1 \cap \bar{G}_2), \qquad \sigma_T(x_2) \subset \bar{G}_1 \cap \bar{G}_2. \tag{2.2}$$

For $\lambda \in \rho_T(x) \cap (\bar{G}_1 \cap \bar{G}_2)^c = H$, we have $x(\lambda) = x_1(\lambda) + x_2(\lambda)$. Let $\Delta$ be a Cauchy domain with boundary $\Gamma$ such that $\sigma_T(x) \subset \Delta$ and $\bar{\Delta} \subset (\bar{G}_1 \cap \bar{G}_2)^c$. The functional calculus gives

$$x = \frac{1}{2\pi i} \int_\Gamma x(\lambda)\, d\lambda = \frac{1}{2\pi i} \int_\Gamma x_1(\lambda)\, d\lambda. \tag{2.3}$$

For every $\lambda_0 \in \Gamma$, there is a neighborhood $V \subset H$ of $\lambda_0$ and there are functions $f_i$: $V \to E(G_i)$ $(i = 1, 2)$ analytic on $V$ such that

$$x_1(\lambda) = f_1(\lambda) + f_2(\lambda) \quad \text{on } V. \tag{2.4}$$

It follows from

$$(\lambda - T)x_1(\lambda) = x_1 \quad \text{on } \rho_T(x_1),$$

that the function $g$: $V \to E(G_1) \cap E(G_2)$ defined by

$$g(\lambda) = x_1 - (\lambda - T)f_1(\lambda) = (\lambda - T)f_2(\lambda)$$

is analytic on $V$.

*Part* B. Since $V \subset (\bar{G}_1 \cap \bar{G}_2)^c \subset \rho[T \mid E(G_1) \cap E(G_2)]$, the function $h$: $V \to E(G_1) \cap E(G_2)$ defined by

$$h(\lambda) = R[\lambda; T \mid E(G_1) \cap E(G_2)]\, g(\lambda)$$

is analytic on $V$. We have

$$(\lambda - T)h(\lambda) = g(\lambda) = (\lambda - T)f_2(\lambda) \quad \text{on } V$$

and hence the single valued extension property of $T$ implies that

$$f_2(\lambda) = h(\lambda) \in E(G_1) \cap E(G_2) \subset E(G_1) \quad \text{on } V.$$

Thus, by (2.4) $x_1(\lambda) \in E(G_1)$ on $V$ and, in particular, $x_1(\lambda_0) \in E(G_1)$. Since $\lambda_0$ is arbitrary on $\Gamma$, it follows from (2.3) that $x \in E(G_1)$. Thus, $X_T(G_1) \subset E(G_1)$ and this establishes (1.5). Consequently, $E$ is a monotonic spectral resolvent. □

2.3. COROLLARY. *Let $E$ be a spectral resolvent of $T$. If for each $G \in \mathfrak{G}$, any one of the following conditions holds, then $E$ is monotonic.*

(1) $\sigma[T^* \mid E(G)^{\perp}] \subset G^c$;

(2) $\sigma[T/E(G)] \subset G^c$;

(3) *$E(G)$ is analytically invariant*;

(4) *$E(G)$ is $T$-absorbent.*

PROOF. Conditions (1)–(3) are equivalent **[1]**. Moreover, since $T$ has the single valued extension property, every $T$-absorbent space is analytically invariant under $T$. Thus, it suffices to prove the statement of the corollary under hypothesis (4). Given $G_1, G_2 \in \mathfrak{G}$, Proposition 1.6 implies

$$\sigma[T \mid E(G_1) \cap E(G_2)] \subset \sigma[T \mid E(G_1)] \cap \sigma[T \mid E(G_2)] \subset \overline{G}_1 \cap \overline{G}_2.$$

Now, Theorem 2.2 concludes the proof. □

2.4. COROLLARY. *Let $T$ have a spectral resolvent $E$. If $\sigma(T)$ has empty interior and $\rho_\infty(T) = \rho(T)$ (in particular, if $\sigma(T)$ is contained on an open Jordan curve), then $E$ is monotonic.*

PROOF. It suffices to show that for every $G \in \mathfrak{G}$, $E(G)$ is analytically invariant under $T$. Let $f: D \to X$ be analytic on an open $D \subset \mathbf{C}$ such that for every $G \in \mathfrak{G}$,

$$(\lambda - T)f(\lambda) \in E(G) \quad \text{on } D.$$

Since $\sigma(T)$ has empty interior, $D - \sigma(T)$ is a nonempty open set. Then, since $\rho_\infty(T) = \rho(T)$, we have

$$f(\lambda) = R(\lambda; T)(\lambda - T)f(\lambda) \in E(G) \quad \text{for all } \lambda \in D - \sigma(T)$$

and $f(\lambda) \in E(G)$ on $D$, by analytic continuation. □

As a summary of this section, the "spectral inclusion property" (1.5) and the "spectral invariance property" (2.1) proved to be sufficient conditions for a spectral resolvent $E$ to be monotonic. By strengthening the monotonic spectral resolvent concept, (1.5) is heightened to a necessary and sufficient condition for the validity of the new monotonic attribute of a spectral resolvent.

**3. Strongly monotonic spectral resolvents.**

3.1. DEFINITION. A spectral resolvent $E$ is said to be strongly monotonic if $G, G_1, G_2 \in \mathfrak{G}$ and $\overline{G}_1 \cap \overline{G}_2 \subset G$ imply $E(G_1) \cap E(G_2) \subset E(G)$.

Evidently, every strongly monotonic spectral resolvent is monotonic. As an example, if $T$ has a spectral resolvent $E$ then its maximal spectral resolvent $E_m$ is strongly monotonic. Indeed, $G, G_1, G_2 \in \mathfrak{G}$ and $\overline{G}_1 \cap \overline{G}_2 \subset G$ imply

$$E_m(G_1) \cap E_m(G_2) = X_T(\overline{G}_1) \cap X_T(\overline{G}_2) = X_T(\overline{G}_1 \cap \overline{G}_2) \subset X_T(\overline{G}) = E_m(G).$$

3.2. THEOREM. *Let $E$ be a spectral resolvent of $T$. $E$ is strongly monotonic if and only if (1.5) holds for every $G \in \mathfrak{G}$.*

PROOF. We only have to prove the "only if" part. Assume that $E$ is strongly monotonic. Given $G \in \mathfrak{G}$, let $x \in X_T(G)$. Let $\{G_1, G_2\} \in \operatorname{cov} \sigma(T)$ be such that

$$\sigma_T(x) \subset G_1 \subset \overline{G} \subset G \quad \text{and} \quad \sigma_T(x) \cap \overline{G}_2 = \varnothing.$$

Follow verbatim Part A of the proof of Theorem 2.2. Let $K \in \mathfrak{G}$ be such that

$$\overline{G}_1 \cap \overline{G}_2 \subset K \subset \overline{K} \subset G, \quad \overline{K} \cap \sigma_T(x) = \varnothing \quad \text{and} \quad V \cap \overline{K} = \varnothing .$$

$E$ being strongly monotonic, we have $g(\lambda) \in E(K)$ on $V$. The function $h\colon V \to E(K)$ defined by $h(\lambda) = R[\lambda; T \mid E(K)]g(\lambda)$ is analytic on $V$ and

$$(\lambda - T)h(\lambda) = (\lambda - T)f_2(\lambda) \quad \text{on } V.$$

By the single valued extension property of $T$,

$$f_2(\lambda) = h(\lambda) \in E(K) \quad \text{on } V.$$

$E$ being monotonic, we have

$$x_1(\lambda) \in E(G_1) + E(K) \subset E(G) \quad \text{on } V$$

and, in particular, $x_1(\lambda_0) \in E(G)$. Since $\lambda_0$ is arbitrary on $\Gamma$, it follows from (2.3) that $x \in E(G)$. Since $x$ is arbitrary in $X_T(G)$, the proof concludes with $\overline{X_T(G)} \subset E(G)$. □

Another characterization of a strongly monotonic spectral resolvent involves the range of the local resolvent function.

3.3. THEOREM. *Let $E$ be a spectral resolvent of $T$. The following assertions are equivalent*:

(i) *$E$ is strongly monotonic*;

(ii) $G_1, G_2 \in \mathfrak{G}$, $\overline{G}_1 \subset G_2$ *and* $x \in E(G_1)$ *imply* $\{x(\lambda)\colon \lambda \in \rho_T(x)\} \subset E(G_2)$.

PROOF. (i) ⇒ (ii): Let $G_1, G_2 \in \mathfrak{G}$ be such that $\overline{G}_1 \subset G_2$. By Theorem 3.2, we have

$$E(G_1) \subset X_T(\overline{G}_1) \subset X_T(G_2) \subset E(G_2). \tag{3.1}$$

Let $x \in E(G_1)$ be given. Then $x \in X_T(\overline{G}_1)$ and since $X_T(\overline{G}_1)$ is a spectral maximal space of $T$, (3.1) implies

$$\{x(\lambda)\colon \lambda \in \rho_T(x)\} \subset X_T(\overline{G}_1) \subset E(G_2).$$

(ii) ⇒ (i): Let $G \subset \mathbf{C}$ be an open set and let $x \in X_T(G)$. Choose $G_1 \in \mathfrak{G}$ such that $\sigma_T(x) \subset G_1 \subset \overline{G}_1 \subset G$. Let $G_2 \in \mathfrak{G}$ satisfy conditions

$$\sigma(T) \subset G_1 \cup G_2, \quad \sigma_T(x) \cap \overline{G}_2 = \varnothing .$$

Then $x$ has a representation $x = x_1 + x_2$ with $x_i \in E(G_i)$, $i = 1, 2$. As obtained in an earlier proof, we have (2.2)

$$\sigma_T(x_1) \subset \sigma_T(x) \cup (\overline{G}_1 \cap \overline{G}_2), \quad \sigma_T(x_2) \subset \overline{G}_1 \cap \overline{G}_2 .$$

Let $\Delta$ be a Cauchy domain with boundary $\Gamma \subset \rho_T(x) \cap (\overline{G}_1 \cap \overline{G}_2)^c$, such that $\sigma_T(x) \subset \Delta$ and $\overline{\Delta} \cap (\overline{G}_1 \cap \overline{G}_2) = \varnothing$. Then

$$x = \frac{1}{2\pi i}\int_\Gamma x(\lambda)\, d\lambda = \frac{1}{2\pi i}\int_\Gamma x_1(\lambda)\, d\lambda. \tag{3.2}$$

Since $x_1 \in E(G_1)$ and $\overline{G}_1 \subset G$, hypothesis (ii) implies

$$\{x_1(\lambda)\colon \lambda \in \rho_T(x)\} \subset E(G).$$

Then, by (3.2), $x \in E(G)$ and hence $X_T(G) \subset E(G)$. Now, Theorem 3.2 concludes the proof. □

A further characterization of a strongly monotonic spectral resolvent can be obtained in terms of a localization property of the spectral resolvent. The following definition generalizes the concept of "almost localized spectrum" [**10**].

3.4. DEFINITION. A spectral resolvent $E$ is said to be almost localized if $G, G_1, G_2 \in \mathfrak{G}$ and $\overline{G} \subset G_1 \cup G_2$ imply $E(G) \subset E(G_1) + E(G_2)$.

The following result is due to Radjabalipour [**7**].

3.5. PROPOSITION. *If $T$ is decomposable then, for every closed set $F$ and $\{H_1, H_2\} \in$ $\operatorname{cov} F$, the following inclusion holds*:

$$X_T(F) \subset X_T(\overline{H}_1) + X_T(\overline{H}_2). \tag{3.3}$$

Since, for every open cover $\{H_1, H_2\}$ of $F$, there is $\{G_1, G_2\} \in \operatorname{cov} F$ with $\overline{H}_1 \subset G_1$ and $\overline{H}_2 \subset G_2$, property (3.3) can be expressed as

$$X_T(F) \subset \overline{X_T(G_1)} + \overline{X_T(G_2)}. \tag{3.4}$$

3.6. THEOREM. *Let $T$ have a spectral resolvent $E$. Then $E$ is strongly monotonic if and only if $E$ is almost localized.*

PROOF. In view of Theorem 3.2, we have to show that the following conditions are equivalent:

(i) $\overline{X_T(G)} \subset E(G)$ for all $G \in \mathfrak{G}$;

(ii) $G, G_1, G_2 \in \mathfrak{G}$ and $\overline{G} \subset G_1 \cup G_2$ imply $E(G) \subset E(G_1) + E(G_2)$.

(i) $\Rightarrow$ (ii): Let $G, G_1, G_2 \in \mathfrak{G}$ be such that $\overline{G} \subset G_1 \cup G_2$. Since $T$ is decomposable, (3.4) implies

$$E(G) \subset X_T(\overline{G}) \subset \overline{X_T(G_1)} + \overline{X_T(G_2)} \subset E(G_1) + E(G_2).$$

(ii) $\Rightarrow$ (i): Given $G \in \mathfrak{G}$, let $x \in X_T(G)$. Further, let $H_0$ be a relatively compact, open neighborhood of $\sigma(T)$. Then

$$x \in X = E(H_0) \quad \text{and} \quad \sigma_T(x) \subset \sigma(T) \subset H_0.$$

Let $\varepsilon$ be arbitrary , with $0 < \varepsilon < \sup_{\lambda \in \partial H_0} d[\lambda, \sigma_T(x)]$. Define the open sets

$$H = \{\lambda \in \mathbf{C}: d[\lambda, \sigma_T(x)] < \varepsilon\}, \quad H' = \left\{\lambda \in \mathbf{C}: d(\lambda, H_0) < \frac{\varepsilon}{6}\right\}.$$

For every $\lambda \in \overline{H}' \cap H^c$, let $D_\lambda = \{\mu \in \mathbf{C}: |\mu - \lambda| < \varepsilon/3\}$. Then $\{D_\lambda: \lambda \in \overline{H}' \cap H^c\}$ is an open cover of $\overline{H}' \cap H^c$. Since $\overline{H}' \cap H^c$ is compact, there is a finite collection $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \overline{H}' \cap H^c$ such that

$$\overline{H}' \cap H^c \subset \bigcup_{i=1}^{n} D_i, \quad \text{where } D_i = D_\lambda \text{ for } \lambda = \lambda_i.$$

For $1 \leqslant i \leqslant n$, define

$$K_i = \left\{\mu \in \mathbf{C}: |\mu - \lambda_i| < \frac{2}{3}\varepsilon\right\}, \quad \Delta_i = \left\{\mu \in C: |\mu - \lambda_i| < \frac{\varepsilon}{2}\right\}.$$

Clearly, $\overline{K}_i \cap \sigma_T(x) = \varnothing$, $1 \leqslant i \leqslant n$. Put

$$H_1 = \left\{\lambda \in \mathbf{C}: d(\lambda, H_0) < \frac{\varepsilon}{9n}\right\} - \overline{\Delta}_1.$$

It is easy to see that $\overline{H}_1 \cap \overline{D}_1 = \emptyset$. Since

$$\overline{H}_0 \subset H_1 \cup \overline{\Delta}_1 \subset H_1 \cup K_1,$$

we have

$$x \in E(H_0) \subset E(H_1) + E(K_1).$$

For $G_1 = H_1$, $G_2 = K_1$, follow Part A of the proof of Theorem 2.2. Note that the boundary $\Gamma$ of the Cauchy domain $\Delta$ in Part A, verifies inclusions

$$\Gamma \subset \rho_\infty[T | E(K_1)] \subset \rho[T | E(H_1) \cap E(K_1)].$$

The function $h: V \to E(H_1) \cap E(K_1)$, defined by

$$h(\lambda) = R[\lambda; T | E(H_1) \cap E(K_1)] g(\lambda)$$

verifies equality

$$(\lambda - T) h(\lambda) = (\lambda - T) f_2(\lambda) \quad \text{on } V,$$

which implies

$$f_2(\lambda) = h(\lambda) \in E(H_1) \cap E(K_1) \quad \text{on } V.$$

Thus, with reference to Part A, (2.4) implies that $x_1(\lambda) \in E(H_1)$ on $V$, and hence $x_1(\lambda_0) \in E(H_1)$. $\lambda_0 \in \Gamma$ being arbitrary, $x \in E(H_1)$ by (2.3).

Inductively, define

$$H_k = \{\lambda \in \mathbf{C}: d(\lambda, H_{k-1}) < \varepsilon/9n\} - \overline{\Delta}_k, \qquad 1 \leqslant k \leqslant n.$$

Then $\{H_k, K_k\}$ covers $\overline{H}_{k-1}$ and $\overline{H}_k \cap \overline{D}_i = \emptyset$, $1 \leqslant i \leqslant k$. In view of hypothesis (ii), $E(H_{k-1}) \subset E(H_k) + E(K_k)$, and the hypothesis $x \in E(H_{k-1})$ of the induction gives $x \in E(H_k) + E(K_k)$. As for $k = 1$, by using Part A of the proof of Theorem 2.2 and a conveniently defined function $h: V \to E(H_k) \cap E(K_k)$, we obtain $x \in E(H_k)$. Thus, by the inductive process, we obtain an open set $H_n$ with the properties

$$x \in E(H_n) \quad \text{and} \quad \overline{H}_n \subset H' - \left( \bigcup_{i=1}^{n} \overline{D}_i \right) \subset H.$$

$E$ being monotonic, $E(H_n) \subset E(H)$ and hence $x \in E(H)$. Since $\varepsilon$ is arbitrarily small, we may choose it such that $\overline{H} \subset G$. Then $E(H) \subset E(G)$ and hence $x \in E(G)$. Since $x \in X_T(G)$ is arbitrary, we obtain $\overline{X_T(G)} \subset E(G)$. □

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