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BOUNDS FOR INTEGRAL SOLUTIONS OF DIAGONAL CUBIC EQUATIONS

BY

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ABSTRACT. It was proved by Davenport [3] that for the nonzero integral λ_i the equation $\lambda_1 x_1^3 + \cdots + \lambda_8 x_8^3 = 0$ always has a nontrivial integral solution. In this paper, we investigate the bounds of nontrivial integral solutions in terms of $\lambda_1, \ldots, \lambda_8$.

1. Introduction. Pitman and Ridout [9] proved that for every $\theta > 0$, there exists a constant c_θ with the following property. If $\lambda_1, \ldots, \lambda_9$ are nonzero integers, then the equation

$$\lambda_1 x_1^3 + \cdots + \lambda_9 x_9^3 = 0 \tag{1}$$

has a solution in nonzero integers $x_1, \ldots, x_9$, such that

$$|\lambda_1 x_1^3| + \cdots + |\lambda_9 x_9^3| < c_\theta |\lambda_1 \cdots \lambda_9|^{(3/2)+\theta}.$$

They conjectured [9] that it should be possible to obtain bounds for integral solutions of the equation

$$\lambda_1 x_1^3 + \cdots + \lambda_8 x_8^3 = 0 \tag{2}$$

where $\lambda_1, \ldots, \lambda_8$ are nonzero integers.

However, their method cannot be directly extended. In this paper we shall use I. Danicic's [7] idea and improve Davenport and Roth's [6] results to overcome the difficulties and prove the following

THEOREM 1. *For every $\theta > 0$, there exists a constant c_θ, depending on θ only, with the following property. If $\lambda_1, \ldots, \lambda_8$ are nonzero integers, and not all of the same sign, then* (2) *has a solution in positive integers $x_1, \ldots, x_8$ such that*

$$|\lambda_1 x_1^3| + \cdots + |\lambda_8 x_8^3| < c_\theta |\lambda_1 \cdots \lambda_8|^{(35/8)+\theta}.$$

2. Notation and general lemmas. Let $\lambda_i (i = 1, \ldots, 8)$ be given nonzero integers such that $P \geqslant |\lambda_i|$ for all i. We write

$$\Pi = \prod_{i=1}^{8} |\lambda_i|$$

and define

$$S_i(\alpha) = \sum_{x_i} e(\lambda_i x_i^3 \alpha), \qquad i = 1, \ldots, 8,$$

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where $e(y) = \exp(2\pi yi)$ and x_i runs through all integral values in the range

$$P \leqslant |\lambda_i|^{1/3}x_i \leqslant 2P, \qquad i = 1,\dots,4, \tag{3}$$
$$P^{4/5} \leqslant |\lambda_j|^{1/3}x_j \leqslant 2P^{4/5}, \qquad j = 5,\dots,8.$$

Throughout the paper, the letters a, q, a_i, q_i always denote integers. In the following lemmas, δ denotes a fixed small positive number and ε denotes an arbitrary small positive number, not the same throughout. The constants implied by the notation 0, $\ll$, $\gg$ are always independent of the λ_i and of P, and without loss of generality, we assume the constants are greater than or equal to 1. In this section they depend only on δ and ε; in later sections they will depend only on θ, δ and ε and so ultimately on θ, since δ and ε will be determined by θ.

LEMMA 2.1. (i) *If p is prime and $(a, p) = 1$, then*

$$S(a, p) \leqslant 2P^{1/2}.$$

(ii) *If $(a, q) = 1$, then $S(a, q) \ll q^{2/3}$ where $S(a, k) = \sum_{x=1}^{k} e(ax^3/k)$.*

PROOF. Both (i) and (ii) are particular cases of Lemmas 12 and 15, respectively, of [4].

LEMMA 2.2. *If $|\beta| \leqslant \frac{1}{2}$, then*

$$I(\beta) = \sum_{m=P^3}^{8P^3} \frac{1}{3} m^{-2/3} e(\beta m) \ll P\min\left(1, P^{-3}|\beta|^{-1}\right)$$

and

$$I'(\beta) = \sum_{m=P^{12/5}}^{8P^{12/5}} \frac{1}{3} m^{-2/3} e(\beta m) \ll P^{4/5}\min\left(1, P^{-12/5}|\beta|^{-1}\right).$$

PROOF. See [1, Lemma 3].

Now for all $\alpha \in [0, 1] = I$, $\lambda_i\alpha$ can be represented in the form

$$\tag{4} \begin{cases} \lambda_i\alpha = (a_i/q_i) + \beta_i \quad \text{where } (a_i, q_i) = 1, \\ 0 < q_i \leqslant \left(|\lambda_i|^{-1/3}P\right)^{2+\delta}, \quad |\beta_i| \leqslant q_i^{-1}\left(|\lambda_i|^{-1/3}P\right)^{-2-\delta}, \qquad i = 1,\dots,4, \\ 0 < q_i \leqslant \left(|\lambda_i|^{-1/3}P^{4/5}\right)^{2+\delta}, \quad |\beta_i| \leqslant q_i^{-1}\left(|\lambda_i|^{-1/3}P^{4/5}\right)^{-2-\delta}, \qquad i = 5,\dots,8. \end{cases}$$

LEMMA 2.3. *Suppose that $\lambda_i\alpha$ is in the form of (4). Then:*

(i) *If $1 \leqslant i \leqslant 4$,*

$$\begin{aligned} S_i(\alpha) &= |\lambda_i|^{-1/3} q_i^{-1} S(a_i, q_i) I(\pm\beta_i/\lambda_i) + O\left(q_i^{2/3+\varepsilon}\right) \\ &\ll |\lambda_i|^{-1/3} q_i^{-1/3} P \min\left(1, P^{-3}|\lambda_i/\beta_i|\right). \end{aligned}$$

(ii) *If $5 \leqslant i \leqslant 8$,*

$$\begin{aligned} S_i(\alpha) &= |\lambda_i|^{-1/3} q_i^{-1} S(a_i, q_i) I'(\pm\beta_i/\lambda_i) + O\left(q_i^{2/3+\varepsilon}\right) \\ &\ll |\lambda_i|^{-1/3} q_i^{-1/3} P^{4/5} \min\left(1, P^{-12/5}|\lambda_i/\beta_i|\right) \end{aligned}$$

where $\pm$ is the sign of λ_i.

PROOF. This is essentially the same as [**1**, Lemmas 7–10].

LEMMA 2.4. *Suppose that* $\lambda_i\alpha$ *satisfies* (4) *and, in particular,*

$$\left(|\lambda_i|^{-1/3}P\right)^{-1-\delta} < q_i \leqslant \left(|\lambda_i|^{-1/3}P\right)^{2+\delta} \quad \text{if } 1 \leqslant i \leqslant 4,$$

$$\left(|\lambda_i|^{-1/3}P^{4/5}\right)^{1-\delta} < q_i \leqslant \left(|\lambda_i|^{-1/3}P^{4/5}\right)^{2+\delta} \quad \text{if } 5 \leqslant i \leqslant 8.$$

Then

$$S_i(\alpha) \ll \left(|\lambda_i|^{-1/3}P\right)^{3/4+\delta} \quad \text{if } 1 \leqslant i \leqslant 4, \tag{5}$$

$$S_i(\alpha) \ll \left(|\lambda_i|^{-1/3}P^{4/5}\right)^{3/4+\delta} \quad \text{if } 5 \leqslant i \leqslant 8.$$

PROOF. See [**1**, Lemma 13].

Before we proceed, we rearrange λ_i so that

$$|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_j|, \qquad j = 3,\ldots,8, \tag{6}$$

and if $\lambda_1\lambda_2 > 0$, choose λ_3 s.t. $\lambda_1\lambda_3 < 0$ and define, for any $g \geqslant 0$, $h \geqslant 0$,

$$K_{ij}(g,h) = \int_I |S_i(\alpha)|^g |S_j(\alpha)|^h \, d\alpha \tag{7}$$

for $i = 1,\ldots,4$, $j = 5,\ldots,8$. In proving the following lemmas, (6) is assumed.

LEMMA 2.5.

(i) $$K_{ij}(0,4) \ll \left(|\lambda_j|^{-1/3}P^{4/5}\right)^{2+\varepsilon}.$$

(ii) $$K_{ij}(0,8) \ll \left(|\lambda_j|^{-1/3}P^{4/5}\right)^{5+\varepsilon}.$$

PROOF. See [**9**, Lemma 5].

LEMMA 2.6. *Let* $N'(m)$ *be the number of solutions of* $x^3 + y^3 - x'^3 - y'^3 \equiv 0$ mod m, *where* $1 \leqslant x, y, x', y' \leqslant m$. *Then* $N'(m) \ll m^{7/2+\varepsilon}$.

PROOF.

$$\begin{aligned} N'(m) &= \int_I \left| \sum_{x=1}^{m} e(\alpha x^3) \right|^4 \sum_{x=-2m^2}^{2m^2} e(mx\alpha)\, d\alpha \\ &\ll \left(\int_I \left| \sum_{x=1}^{m} e(\alpha x^3) \right|^8 d\alpha \int_I \left| \sum_{x=-2m^2}^{2m^2} e(mx\alpha) \right|^2 d\alpha \right)^{1/2} \\ &\ll m^{7/2+\varepsilon} \quad \text{by Hua's inequality.} \end{aligned}$$

LEMMA 2.7.

(i) $$K_{ij}(2,4) \ll |\lambda_i|^{-1/3} |\lambda_j|^{-2/3} P^{13/5+\varepsilon}.$$

(ii) $$K_{ij}(2,6) \ll |\lambda_i|^{-1/3} |\lambda_j|^{-7/6} P^{19/5+\varepsilon}.$$

PROOF. It is essentially the same as [**6**, Lemmas 5 and 6] for $|\lambda_i| \leqslant |\lambda_j|$. Since we may have $|\lambda_i| \geqslant |\lambda_j|$, we need Lemma 2.6 to improve the result. We omit (ii) and prove (i) only. The integral is equal to the number of solutions

$$\lambda_i(x^3 - x'^3) + \lambda_j(y^3 + z^3 - y'^3 - z'^3) = 0 \tag{8}$$

in integers satisfying

$$P \leqslant |\lambda_i|^{1/3}x, \quad |\lambda_i|^{1/3}x' \leqslant 2P, \quad P^{4/5} \leqslant |\lambda_j|^{1/3}(y, z, y', z') \leqslant 2P^{4/5}.$$

For these solutions with $x = x'$, the number of pairs of y, z, y', z' satisfying (8) $\ll (|\lambda_j|^{-1/3}P^{4/5})^{2+\varepsilon}$. The number of this kind of solution is therefore

$$\ll |\lambda_i|^{-1/3}|\lambda_j|^{-2/3}P^{13/5+\varepsilon}.$$

Now we estimate the number N of solutions with $x' > x$. Let $x' = x + t$. (8) becomes

$$\lambda_i(3x^2t + 3xt^2 + t^3) - \lambda_j(y^3 + z^3 - y'^3 - z'^3) = 0, \tag{9}$$

and we observe that $0 < t \leqslant |\lambda_i|^{-1/3}P^{2/5}$. Let $N(t, y', z')$ denote the number of solutions for prescribed values of t, y', z'. Then

$$N = \sum_{t, y', z'} N(t, y', z') \leqslant \Big(\sum_{t, y', z'} 1\Big)^{1/2}\Big(\sum_{t, y', z'} N^2(t, y', z')\Big)^{1/2} = (N_1)^{1/2}(N_2)^{1/2},$$

say. It is obvious that $N_1 \ll |\lambda_i|^{-1/3}|\lambda_j|^{-2/3}P^2$. Also N_2 represents the number of solutions of

$$\lambda_i(3x_1^2t + 3x_1t^2 + t^3) - \lambda_j(y_1^3 + z_1^3 - y'^3 - z'^3) = 0,$$
$$\lambda_i(3x_2^2t + 3x_2t^2 + t^3) - \lambda_j(y_2^3 + z_2^3 - y'^3 - z'^3) = 0.$$

The number of solutions of these simultaneous equations with $x_1 = x_2$ is $\ll P^\varepsilon N$. With regard to the solution for $x_1 \neq x_2$, that would imply

$$3\lambda_i t(x_1 - x_2)(x_1 + x_2 + t) = \lambda_j(y_1^3 + z_1^3 - y_2^3 - z_2^3)$$

for values of y_1, z_1, y_2, z_2 such that the right-hand side is a multiple of λ_i. We can determine the values of x_1, x_2, t with p^ε possibilities, and the values of y', z' are then determined by either of the two equations with p^ε possibilities. Thus the number of solutions with $x_1 \neq x_2$ is $\ll (|\lambda_j|^{-1/3}P^{4/5})^4 m^{-4}N'(m)P^\varepsilon$, where $m = |\lambda_i/(\lambda_i, \lambda_j)|$ and $N'(m)$ is defined as in Lemma 2.6. Hence

$$\ll |\lambda_j|^{-4/3}P^{16/5}(|\lambda_i|/|\lambda_j|)^{-1/3+\varepsilon} = |\lambda_i|^{-1/3}|\lambda_j|^{-1}P^{16/5+\varepsilon}.$$

By a similar argument applied to (9),

$$N \ll |\lambda_i|^{-1/3}|\lambda_j|^{-1}P^{16/5+\varepsilon}.$$

It follows that $N_2 \ll |\lambda_i|^{-1/3}|\lambda_j|^{-1}P^{16/5+\varepsilon}$, giving the desired result

$$N \ll |\lambda_i|^{-1/3}|\lambda_j|^{-2/3}P^{13/5+\varepsilon}.$$

$x' < x$ is similar.

LEMMA 2.8. *Let* $A_i = \{\alpha \in I\colon S_i(\alpha) \gg |\lambda_i|^{-1/4}P^{3/4+2\delta}\}$, $i = 1, \ldots, 4$. *Then*:

(i) $$\int_{A_i} |S_i(\alpha)|^6\, d\alpha \ll |\lambda_i|^{-1}P^{3+\varepsilon}.$$

(ii) $$\int_{A_i} |S_i(\alpha)|^5\,d\alpha \ll |\lambda_i|^{-3/4} P^{9/4-2\delta+\varepsilon}.$$

PROOF. (i) For every α in A_i, we determine a_i, q_i such that (4) is satisfied, so by Lemma 2.4, $0 < q_i \leqslant (|\lambda_i|^{-1/3}P)^{1-\delta}$ and, by Lemma 2.3,

$$|\lambda_i|^{-1/4}P^{3/4+2\delta} \ll |S_i(\alpha)| \ll q_i^{-1/3}|\lambda_i|^{-1/3}P\min\left(1, P^{-3}|\lambda_i/\beta_i|\right)$$

which imply

(10) $$0 < q_i \ll |\lambda_i|^{-1/4}P^{3/4-6\delta}, \quad |\beta_i/\lambda_i| \ll q_i^{-1/3}|\lambda_i|^{-1/12}P^{-11/4-2\delta}.$$

Let F_{a_i,q_i} denote the interval of values of α satisfying (4) and (10). Since $A_i \subset \bigcup F_{a_i,q_i}$, where q_i satisfies (10) and $|a_i| \leqslant |\lambda_i q_i|$,

$$\int_{A_i} |S_i(\alpha)|^6\,d\alpha \ll \sum_{q_i}\sum_{a_i}\int_{F_{a_i,q_i}} |S_i(\alpha)|^6\,d\alpha,$$

where the summations a_i, q_i are as before.

$$\int_{F_{a_i,q_i}} |S_i(\alpha)|^6\,d\alpha \ll \int_0^{P^{-3}|\lambda_i|} |\lambda_i|^{-3}q_i^{-2}P^6\,d\beta_i + \int_{P^{-3}|\lambda_i|}^{\infty} P^{-12}|\lambda_i|^3 q_i^{-2}\beta_i^{-6}\,d\beta_i$$
$$\ll |\lambda_i|^{-2}q_i^{-2}P^3.$$

So

$$\int_{A_i} |S_i(\alpha)|^6\,d\alpha \ll \sum_{q_i}\sum_{a_i} |\lambda_i|^{-2}q_i^{-2}P^3 \ll |\lambda_i|^{-1}P^{3+\varepsilon}.$$

(ii) follows easily from (i).

LEMMA 2.9. *Assume that $P > \Pi^{35/24}$. Then:*
(i) *For $i = 1,\dots,4$, $j = 5,\dots,8$,*

$$\int_{A_i} |S_i(\alpha)|^3|S_j(\alpha)|^4\,d\alpha \ll |\lambda_i|^{-9/16}|\lambda_j|^{-7/9}P^{67/20+2\delta+\varepsilon} \quad \textit{if } i = 1,$$
$$\ll |\lambda_i|^{-2/3}|\lambda_j|^{-7/9}P^{67/20+2\delta+\varepsilon} \quad \textit{if } i \neq 1.$$

(ii)
$$K_{ij}(3,4) \ll |\lambda_i|^{-7/12}|\lambda_j|^{-2/3}P^{67/20+2\delta+\varepsilon}, \qquad i = 2,3,4, j = 5,\dots,8.$$

PROOF. (i)

$$\int_{A_i} |S_i(\alpha)|^3|S_j(\alpha)|^4\,d\alpha \ll K_{ij}(2,6)^{2/3}\left(\int_{A_i} |S_i(\alpha)|^5\,d\alpha\right)^{1/3}$$
$$\ll |\lambda_i|^{-17/36}|\lambda_j|^{-7/9}P^{197/60+\varepsilon}.$$

Using $P > \Pi^{35/24}$, which implies $P > |\lambda_1|^{35/24}$ and $P > |\lambda_i|^{35/12}$ if $i \neq 1$, (i) follows.
(ii)

$$K_{ij}(3,4) \ll \int_{A_i} |S_i(\alpha)|^3|S_j(\alpha)|^4\,d\alpha + \int_{I\setminus A_i} |S_i(\alpha)|^3|S_j(\alpha)|^4\,d\alpha$$
$$\ll |\lambda_i|^{-7/12}|\lambda_j|^{-2/3}P^{67/20+2\delta+\varepsilon} + |\lambda_i|^{-1/4}P^{3/4+2\delta}K_{ij}(2,4).$$

By (i) and Lemma 2.7(i), (ii) follows.

3. Minor arcs for the proof of Theorem 1. We let $\mathcal{N}(P)$ be the number of integral solutions of (2) which lie in the 'box' determined by (3). We shall show that for a given positive number θ, there exists a positive constant D_θ, independent of λ_i, such that if $P^{1-\theta} > D_\theta \Pi^{35/24}$, then $\mathcal{N}(P) > 0$. So we may take P such that $D_\theta \Pi^{35/24} < P^{1-\theta} < 2D_\theta \Pi^{35/24}$, which will imply there exists a solution of (2) in nonzero integers with

$$\sum_i |\lambda_i x_i^3| < 8\left(2D_\theta \Pi^{35/24}\right)^{3/(1-\theta)} < K_\theta \Pi^{35/8(1-\theta)},$$

where K_θ depends on θ only, so for every $\phi > 0$, we can find $0 < \theta < 1$ such that $35/8(1-\theta) < 35/8 + \phi$, and hence Theorem 1 follows.

We now use

$$\mathcal{N}(P) = \int_I V(\alpha)\, d\alpha, \quad \text{where } V(\alpha) = \prod_{i=1}^{8} S_i(\alpha)$$

and estimate $\mathcal{N}(P)$ will be of the form $c\Pi^{-1/3-\varepsilon}P^{21/5}$ where $c > 0$, and the error terms are substantially smaller than $\Pi^{-1/3}P^{21/5}$ provided P is substantially larger than $\Pi^{35/24}$.

By Dirichlet's theorem on diophantine approximation, for each α in I and i, there exists a rational approximation a_i/q_i to $\lambda_i\alpha$ such that (4) holds. Let $S = \bigcup_{i=1}^{8} B_i$ where

$$B_i = \left\{\alpha \in I\colon S_i(\alpha) \ll |\lambda_i|^{-1/4}P^{3/4+2\delta}\right\}, \qquad 1 \leqslant i \leqslant 4,$$
$$B_j = \left\{\alpha \in I\colon S_j(\alpha) \ll |\lambda_j|^{-1/4}P^{3/5+2\delta}\right\}, \qquad 5 \leqslant j \leqslant 8.$$

LEMMA 3.1. $\int_S |V(\alpha)|\, d\alpha \ll \Pi^{-3/16}P^{41/10+4\delta+\varepsilon}$.

PROOF. We may assume that the constants implied in defining A_i and B_i are the same for $i, \ldots, 4$. Then $A_i = I \setminus B_i$, $i = 1, \ldots, 4$.

Let $A = A_1 \cap A_2 \cap A_3 \cap A_4$ and

$$S = (B_1 \cap B_2) \cup (B_1 \cap A_2) \cup (B_2 \cap A_1) \cup (B_3 \cap A_1 \cap A_2)$$
$$\cup (B_4 \cap A_1 \cap A_2) \cup \bigcup_{j=5}^{8} (B_j \cap A).$$

(i)

$$\int_{B_1 \cap B_2} |V(\alpha)|\, d\alpha \ll |\lambda_1|^{-1/4} |\lambda_2|^{-1/4} P^{3/2+4\delta} \left(\prod_{i=3}^{4} \prod_{j=5}^{8} K_{ij}(2,4)\right)^{1/8}$$
$$\ll \Pi^{-3/16}P^{41/10+4\delta+\varepsilon}.$$

By Lemma 2.7 and notice that $|\lambda_1|, |\lambda_2| \geqslant |\lambda_j|$ for $j = 3, 4, \ldots, 8$.

(ii)

$$\int_{B_1 \cap A_2} |V(\alpha)|\, d\alpha$$
$$\ll |\lambda_1|^{-1/4}P^{3/4+2\delta} \left(\prod_{j=5}^{8} \int_{A_2} |S_2(\alpha)|^3 |S_j(\alpha)|^4\, d\alpha\right)^{1/12} \left(\prod_{i=3}^{4} \prod_{j=5}^{8} K_{ij}(3,4)\right)^{1/12}.$$

By Lemma 2.9,

$$\ll |\lambda_1|^{-1/4}|\lambda_2|^{-2/9}|\lambda_3|^{-7/36}|\lambda_4|^{-7/36}|\lambda_5\cdots\lambda_8|^{-19/108}P^{41/10+4\delta+\varepsilon}$$
$$\ll \Pi^{-3/16}P^{41/10+4\delta+\varepsilon}.$$

Similarly for $B_2 \cap A_1$.

(iii)

$$\int_{B_3\cap(A_1\cap A_2)} |V(\alpha)|\,d\alpha$$
$$\ll |\lambda_3|^{-1/4}P^{3/4+2\delta}\left(\prod_{\substack{k=1\\k\neq 3}}^{4}\prod_{j=5}^{8}\int_{A_1\cap A_2}|S_k(\alpha)|^3|S_j(\alpha)|^4\,d\alpha\right)^{1/12}.$$

By Lemma 2.9,

$$\ll |\lambda_1|^{-3/16}|\lambda_2|^{-2/9}|\lambda_3|^{-1/4}|\lambda_4|^{-7/36}|\lambda_5\cdots\lambda_8|^{-5/27}P^{41/10+4\delta+\varepsilon}$$
$$\ll \Pi^{-3/16}P^{41/10+4\delta+\varepsilon}.$$

Similarly for $B_4 \cap A_1 \cap A_2$.

(iv) For $j = 5,\ldots,8$,

$$\int_{B_j\cap A}|V(\alpha)|\,d\alpha \ll |\lambda_j|^{-1/4}P^{3/5+2\delta}\left(\prod_{k=1}^{4}\prod_{\substack{m=5\\m\neq j}}^{8}\int_{A_k}|S_k(\alpha)|^4|S_m(\alpha)|^3\,d\alpha\right)^{1/12}$$
$$\ll |\lambda_j|^{-1/4}P^{3/5+2\delta}\left(\prod_{k=1}^{4}\prod_{\substack{m=5\\m\neq j}}^{8}K_{k,m}(2,6)\int_{A_k}|S_k(\alpha)|^6\,d\alpha\right)^{1/24}.$$

By Lemmas 2.7, 2.8 and using the fact that $P > \Pi$,

$$\ll \Pi^{-1/6}P^{4+2\delta} \ll \Pi^{-3/16}P^{41/10+4\delta+\varepsilon}.$$

Summing up (i) to (iv) the desired result follows.

4. Major arcs for the proof of Theorem 1. We define

$$M = \{\alpha \in S\colon S_i(\alpha) \gg |\lambda_i|^{-1/4}P^{3/4+2\delta},\ i = 1,\ldots,4,$$
$$\text{and } S_j(\alpha) \gg |\lambda_j|^{-1/4}P^{3/5+2\delta}, j = 5,\ldots,8\}.$$

Since we have assumed the constants implied are greater than or equal to 1, by Lemma 2.4, the rational approximation a_i/q_i which satisfies (4) must satisfy

$$q_i < \left(|\lambda_i|^{-1/3}P\right)^{1-\delta},\quad i = 1,\ldots,4;\qquad q_j < \left(|\lambda_j|^{-1/3}P^{4/5}\right)^{1-\delta},\quad j = 5,\ldots,8.$$

By Lemma 2.3, this will imply that

(11)

$$0 < q_i < \left(|\lambda_i|^{-1/3}P\right)^{3/4},\qquad |\beta_i/\lambda_i| < |\lambda_i|^{-1/12}P^{-11/4-2\delta},\quad i = 1,\ldots,4,$$
$$0 < q_j < \left(|\lambda_j|^{-1/3}P^{4/5}\right)^{3/4},\qquad |\beta_j/\lambda_j| < |\lambda_j|^{-1/12}P^{-11/5-2\delta},\quad j = 5,\ldots,8.$$

We consider several lemmas before we proceed.

LEMMA 4.1. (i) *If* $P > 4|\lambda_i|^{1/2}$, *then there is at most one approximation* a_i/q_i *to* $\lambda_i\alpha$ *such that* (4) *and* (11) *hold.*

(ii) *If* $P > 4|\lambda_j\lambda_k|$ *and* (4) *and* (11) *hold for* $i = j, k$ $(j \neq k)$, *then*

$$a_j/\lambda_j q_j = a_k/\lambda_k q_k.$$

(iii) *If* (4) *and* (11) *hold, then*

$$|\beta_i| < q_i^{-1}\left(|\lambda_i|^{-1/3}P\right)^{-2-\delta} \qquad i = 1,\dots,4,$$
$$|\beta_j| < q_j^{-1}\left(|\lambda_i|^{-1/3}P^{4/5}\right)^{-2-\delta}, \qquad j = 5,\dots,8.$$

PROOF. The proofs are essentially the same as [9, Lemma 8]. We prove (iii) only.
(iii)

$$\begin{aligned}|\beta_i|q_i &\leqslant |\lambda_i||\lambda_i|^{-1/12}P^{-11/4-2\delta}\left(|\lambda_i|^{-1/4}P^{3/4}\right) \qquad (i = 1,\dots,4)\\ &< \left(|\lambda_i|^{-1/3}P\right)^{-2-\delta}.\end{aligned}$$

Similarly for $j = 5,\dots,8$.

Suppose that $\alpha \in M$ and for each i let a_i/q_i be an approximation which satisfies (4) and (11). Since we have assumed $P > \Pi^{35/24}$, by Lemma 4.1(i), a_i/q_i are unique for $\lambda_i\alpha$ and, by (ii), $a_j/\lambda_j q_j = a_k/\lambda_k q_k$, for all j, k. Hence there exist unique integers a, q such that $(a, q) = 1$, $q > 0$ and

$$a_i/q_i = \lambda_i a/q \tag{12}$$

for all i. Let

$$\delta_i = (\lambda_i, q), \qquad i = 1,\dots,8. \tag{13}$$

Then

$$a_i = \lambda_i a/\delta_i, \qquad q_i = q/\delta_i. \tag{14}$$

By (11), we have

$$0 < q \leqslant \min_{\substack{1\leqslant i\leqslant 4\\ 5\leqslant j\leqslant 8}} \left\{\delta_i|\lambda_i|^{-1/4}P^{3/4}, \delta_j|\lambda_j|^{-1/4}P^{3/5}\right\} \tag{15}$$

and

$$|\alpha - a/q| \leqslant |\lambda_1|^{-1/12}P^{-11/4-2\delta}. \tag{16}$$

Since $P > \Pi^{35/24}$ and (6) is satisfied, this implies $P > |\lambda_1|^{35/24}$ and $P > |\lambda_j|^{35/8}$ for $j = 3,\dots,8$.

$$\begin{aligned}P^{19/35} &\leqslant \min\left\{|P^{24/35}|^{-1/4}P^{3/4}, (P^{8/35})^{-1/4}P^{3/5}\right\}\\ &\leqslant \min_{\substack{1\leqslant i\leqslant 4\\ 5\leqslant j\leqslant 8}} \left\{\delta_i|\lambda_i|^{-1/4}P^{3/4}, \delta_j|\lambda_j|^{-1/4}P^{3/5}\right\}\\ &\leqslant (P^{8/35})^{3/4}P^{3/5} \leqslant P^{4/5}.\end{aligned} \tag{17}$$

We define $I_{a,q}$ to be the interval determined by (16). By previous argument, if $\alpha \in M$, then every rational approximation a_i/q_i satisfying (4) must satisfy (11), which implies $a \in I_{a,q}$ for some a, q such that $0 \leqslant a \leqslant q - 1$ and q satisfies (15).

Conversely, suppose that a, q are integers such that $(a, q) = 1$, $q > 0$, $0 \leqslant a \leqslant q - 1$ and (15) holds, where the δ_i are defined by (13). If α belongs to the interval $I_{a,q}$ defined as above, then $\alpha \in I$ and the approximation a_i/q_i, such that $(a_i, q_i) = 1$ and $q_i > 1$, defined by (12), satisfy (11) and (14), and these intervals do not overlap because of the uniqueness of the a_i/q_i.

By Lemma 4.1(iii), we can always use Lemma 2.3 to estimate $S_i(\alpha)$ for α in M. We can now estimate the contribution from M to $\mathfrak{N}(P)$.

LEMMA 4.2. *If* $\delta \leqslant 1/30$,

$$\int_M V(\alpha)\, d\alpha = \Pi^{-1/3}\mathfrak{S}R(P) + O(P^{39/10}) \tag{18}$$

$$where\ \mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{a=0\\(a,q)=1}}^{q-1} (q_1 \cdots q_8)^{-1} S(a_1, q_1), \ldots, S(a_8, q_8),$$

the a_i, q_i *are defined by* (13) *and* (14), *and* $R(P) \gg P^{21/5}$.

PROOF. Suppose $\alpha \in M$ and define a_i/q_i by (13) and (14) and write

$$\beta = \alpha - a/q = \beta_i/\lambda_i, \qquad i = 1, \ldots, 8.$$

Then by Lemma 2.3,

$$V(\alpha) = \Pi^{-1/3}\left(\prod_{i=1}^{4} q_i^{-1}S(a_i, q_i)I(\pm\beta)\right)\left(\prod_{j=5}^{8} q_j^{-1}S(a_j, q_j)I'(\pm\beta)\right) + E,$$

where $\pm$ is the sign of λ_i and the error E satisfies

$$E \ll \sum_{i=1}^{8} q_i^{2/3+\varepsilon}\left(\prod_{\substack{j=1\\j\neq i}}^{8} |\lambda_j|^{-1/3} q_j^{-1/3}\right) P^{32/5}\left(\min\left(1, P^{-3}|\beta|^{-1}\right)\right)^2$$

$$\ll q^{-5/3+\varepsilon}\left(\min\left(1, P^{-3}|\beta|^{-1}\right)\right)^2.$$

Since $|\lambda_i|^{-1}q \leqslant q_i \leqslant q$ for all i by (13) and (14),

$$\int_M E\, d\alpha \ll \sum_q{}' \sum_a{}' \int_{I_{a,q}} q^{-5/3+\varepsilon} P^{32/5} \min\left(1, P^{-6}|\beta|^{-2}\right) d\alpha$$

$$\ll \sum_q{}' \sum_a{}' q^{-5/3+\varepsilon} P^{17/5} \ll P^{19/5} \quad \text{by (17)},$$

where $\Sigma'_q \Sigma'_a$ denotes the range of q defined in (15), $(a, q) = 1$ and $0 \leqslant a < q$. Since the $I_{a,q}$ do not overlap, it follows that

$$\int_M V(\alpha)\, d\alpha = \Pi^{-1/3} \sum_q{}' \sum_a{}' (q_1 \cdots q_8)^{-1} S(a_1, q_8), \ldots, S(a_8, q_8) \tag{19}$$

$$\times \int_{I_{a,q}} \prod_{i=1}^{4} I(\pm\beta) \prod_{j=5}^{8} I'(\beta)\, d\alpha + O(P^{19/5}).$$

The error caused by replacing $I_{a,q}$ in (19) by the interval $|\alpha - a/q| \leqslant \frac{1}{2}$ is

$$\ll \Pi^{-1/3} \sum_q{}' \sum_a{}' (q_1 \cdots q_8)^{-1/3} \int_{I^*_{a,q}} P^{36/5} \min\left(1, P^{-3}|\beta|^{-1}\right)^4 d\alpha,$$

where $I^*_{a,q}$ is the set $\alpha = a/q + \beta$ such that

$$\tfrac{1}{2} \geqslant |\beta| > |\lambda_1|^{-1/12} P^{-11/4-2\delta} \geqslant P^{-17/6-2\delta} \geqslant P^{-3},$$

since $P > |\lambda_1|$ and $2\delta \leqslant \frac{1}{6}$.

Now, for any pair a, q,

$$\Pi^{-1/3}(q_1 \cdots q_8)^{-1/3} P^{-24/5} \int_{I^*_{a,q}} \beta^{-4}\, d\alpha \ll q^{-8/3} P^{37/10+6\delta}.$$

Hence the total error is certainly

$$\ll P^{37/10+6\delta} \sum_{q=1}^{\infty} \sum_{a=0}^{q-1} q^{-8/3} \ll P^{39/10}$$

provided that $6\delta \leqslant \frac{1}{5}$. Thus we have

$$\int_M V(\alpha)\, d\alpha = \Pi^{-1/3} \sum_q{}' \sum_a{}' (q_1 \cdots q_8)^{-1} S(a_1, q_1) \cdots S(a_8, q_8) R(P) + O(P^{19/5}), \tag{20}$$

where

$$R(P) = \int_{-1/2}^{1/2} \prod_{i=1}^{4} I(\beta) \prod_{j=5}^{8} I'(\beta)\, d\beta = 3^{-8} \sum_{m_1, \ldots, m_8} (m_1 \cdots m_8)^{-2/3} \tag{21}$$

by definition of $I(\beta)$ and $I'(\beta)$, where

$$\begin{aligned} P^3 &\leqslant m_i \leqslant 8P^3, & i &= 1, \ldots, 4, \\ P^{12/5} &\leqslant m_j \leqslant 8P^{12/5}, & j &= 5, \ldots, 8, \end{aligned} \tag{22}$$

and $\pm m_1 \pm m_2 \pm \cdots \pm m_8 = 0$.

Since either λ_2 or λ_3 has sign different then λ_1, without loss of generality, we assume $\lambda_2\lambda_1 < 0$. So now the last condition is

$$m_1 = m_2 \mp m_3 \mp m_4 \mp \cdots \mp m_8 \tag{23}$$

for each $(m_2, \ldots, m_4)$ such that

$$4P^3 \leqslant m_2 \leqslant 5P^3, \quad P^3 \leqslant m_i \leqslant 5P^3/4, \qquad i = 3, 4,$$

and the integer m_1 defined by (23) satisfies $P^3 \leqslant m_1 \leqslant 8P^3$. Thus the number of solutions of (23) such that (22) holds is $\gg P^9 P^{48/5}$. For any such solution we have

$$(m_1 \cdots m_8)^{-2/3} \geqslant 2^{-16} P^{-8} P^{-32/5}.$$

Thus $R(P) \gg P^{21/5}$ as required.

The error caused by extending the range of summation for q to infinity is

$$\ll \Pi^{-1/3}\sum_q \sum_a (q_1 \cdots q_8)^{-1/3} P^{36/5} \int_{-1/2}^{1/2} \min\left(1, P^{-6}|\beta|^{-2}\right) d\beta$$
$$\ll \Pi^{-1/3}\sum_q \sum_a (q_1 \cdots q_8)^{-1/3} P^{21/5},$$

where q is summed over the range $q > P^{19/35}$, and a is as before. We consider a particular set of divisors $\delta_1, \ldots, \delta_8$ of $\lambda_1, \ldots, \lambda_8$, repectively. The contribution to the above from those pairs a, q corresponding to $\delta_1, \ldots, \delta_8$ is

$$\ll \Pi^{-1/3}(\delta_1 \cdots \delta_8)^{1/3} P^{21/5} \sum_{q > P^{19/35}} q^{-5/3}.$$

Thus the contribution from these a, q is

$$\ll \Pi^{-1/3}P^{21/5}(\delta_1 \cdots \delta_8)^{1/3} P^{-38/105} \ll \Pi^{-1/3}P^{21/5}\Pi^{1/3}P^{-38/105} \ll P^{39/10-\varepsilon}.$$

Since $\Pi \ll P$, the number of different possibilities for $\delta_1 \cdots \delta_8$ is $O(P^{\varepsilon})$, and it follows that the total error is $\ll P^{39/10}$. Hence the result followed.

5. The singular series. Firstly, we assume that the λ_i are cube-free and no prime divides more than five of them. The following lemma shows that the assumption may be made without loss of generality.

LEMMA 5.1. *If Theorem* 1 *holds provided the λ_i are cube-free nonzero integers such that no prime divides more than five of them, then it holds for every set of nonzero integral λ_i.*

PROOF. The proof is essentially the same as [**9**, Lemma 5].

We now investigate the singular series $\mathfrak{S}$ defined by (18). Since $S(\lambda_i a, q) = \delta_i S(a_i, q_i)$ and $S(\lambda_i a, q)$ is periodic of period q, we may rewrite the series as

$$\sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-8} S(\lambda_1 a, q) \cdots S(\lambda_8 a, q).$$

For p prime, we define

$$\chi(p) = 1 + \sum_{\nu=1}^{\infty} \sum_{a=1}^{p^{\nu}} P^{-8\nu} \prod_{i=1}^{8} S(\lambda_i a, P^{\nu}) \tag{24}$$

and $M(P^{\nu})$ to be the number of solutions (mod p^{ν}) of the congruence

$$\sum_{i=1}^{8} \lambda_i x_i^3 \equiv 0 \pmod{P^{\nu}}. \tag{25}$$

Similar to [**9**, §6], we have

$$\chi(p) = \lim_{\nu \to \infty} P^{-7\nu} M(P^{\nu}) \quad \text{and} \quad \mathfrak{S} = \prod_{P} \chi(p), \tag{26}$$

where the product is over all primes p.

LEMMA 5.2. (i) *If $s \geqslant 3$, p is any prime, and $p \nmid |\lambda_1 \cdots \lambda_s|$, then there is a nontrivial solution (i.e. one with some $x_i \not\equiv 0$) of*

$$\sum_{i=1}^{s} \lambda_i x_i^3 \equiv 0 \pmod{p}. \tag{27}$$

(ii) *Suppose that $p \neq 3$ is a prime such that $\lambda_1, \ldots, \lambda_s$ are not divisible by p and $\lambda_{s+1}, \ldots, \lambda_8$ are divisible by p, and let $\mathfrak{N}(p)$ denote the number of nontrivial solutions of* (27). *Then for all $\nu > 0$,*

$$M(p^\nu) \geqslant p^{7(\nu-1)+(8-s)} \mathfrak{N}(p).$$

(iii) *If $\lambda_1, \ldots, \lambda_8$ are cube-free and $\nu \geqslant 6$, then $M(3^\nu) \geqslant 3^{7(\nu-6)}$.*

PROOF. The proof is essentially the same as [9, Lemma 10].

LEMMA 5.3. (i) *There is an absolute constant $C_1 > 0$ such that*

$$\prod_p \chi(p) \geqslant \frac{1}{2} \quad (p \nmid \Pi,\ p > C_1).$$

(ii) *There is a constant $C_2 = C_2(\varepsilon) > 0$ such that*

$$\prod_p \chi(p) \geqslant \Pi^{-\varepsilon} \qquad (p \nmid \Pi,\ p > C_2).$$

(iii) *We have $\mathfrak{S} \gg \Pi^{-\varepsilon}$.*

(*The products in* (i) *and* (ii) *are over all primes p which satisfy the condition in parentheses.*)

PROOF. See [9, Lemma 11].

6. Completion of the proof of Theorem 1. Assuming that $P > \Pi^{35/24}$ and $\delta \leqslant \frac{1}{30}$, by Lemmas 4.2 (with $\varepsilon = \delta$), 5.2 and 3.1 (since we have $I \setminus M \subset S$), we have

$$\mathfrak{N}(P) = C_\varepsilon \Pi^{-(1/3)-\varepsilon} P^{21/5} + E,$$

where $C_\varepsilon > 0$ and

$$E \ll \Pi^{-3/16} P^{41/10+5\delta} + P^{39/10} \ll \Pi^{-3/16} P^{41/10+5\delta}$$
$$\ll \Pi^{-1/3} P^{21/5} (\Pi^{-35/24} P^{1-\theta})^{-1/10} P^{5\delta-\delta/10}.$$

For given $\theta > 0$, we choose $\varepsilon > 0$ and $\delta > 0$ such that $\varepsilon + 5\delta \leqslant \theta/20$, $\delta < 1/30$. Then $P^{5\delta-\theta/10} \leqslant P^{-\varepsilon-\theta/20}$.

Since $P > \Pi^{35/24}$ certainly holds if $P^{1-\theta} > \Pi^{35/24}$, it now follows from the preceding paragraph that there is a constant D_θ such that $\mathfrak{N}(P) > 0$ provided $P^{1-\theta} > D_\theta \Pi^{35/24}$. By the remarks at the beginning of §3 this completes the proof of Theorem 1.

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