BOUNDS FOR INTEGRAL SOLUTIONS
OF DIAGONAL CUBIC EQUATIONS

BY

KA-HIN LEUNG[1]

ABSTRACT. It was proved by Davenport [3] that for the nonzero integral λ_i the equation $\lambda_1 x_1^3 + \cdots + \lambda_8 x_8^3 = 0$ always has a nontrivial integral solution. In this paper, we investigate the bounds of nontrivial integral solutions in terms of $\lambda_1, \ldots, \lambda_8$.

1. Introduction. Pitman and Ridout [9] proved that for every $\theta > 0$, there exists a constant c_θ with the following property. If $\lambda_1, \ldots, \lambda_9$ are nonzero integers, then the equation

$$(1) \qquad \lambda_1 x_1^3 + \cdots + \lambda_9 x_9^3 = 0$$

has a solution in nonzero integers $x_1, \ldots, x_9$, such that

$$|\lambda_1 x_1^3| + \cdots + |\lambda_9 x_9^3| < c_\theta |\lambda_1 \cdots \lambda_9|^{(3/2)+\theta}.$$

They conjectured [9] that it should be possible to obtain bounds for integral solutions of the equation

$$(2) \qquad \lambda_1 x_1^3 + \cdots + \lambda_8 x_8^3 = 0$$

where $\lambda_1, \ldots, \lambda_8$ are nonzero integers.

However, their method cannot be directly extended. In this paper we shall use I. Danicic's [7] idea and improve Davenport and Roth's [6] results to overcome the difficulties and prove the following

THEOREM 1. *For every $\theta > 0$, there exists a constant c_θ, depending on θ only, with the following property. If $\lambda_1, \ldots, \lambda_8$ are nonzero integers, and not all of the same sign, then (2) has a solution in positive integers $x_1, \ldots, x_8$ such that*

$$|\lambda_1 x_1^3| + \cdots + |\lambda_8 x_8^3| < c_\theta |\lambda_1 \cdots \lambda_8|^{(35/8)+\theta}.$$

2. Notation and general lemmas. Let $\lambda_i (i = 1, \ldots, 8)$ be given nonzero integers such that $P \geqslant |\lambda_i|$ for all i. We write

$$\Pi = \prod_{i=1}^{8} |\lambda_i|$$

and define

$$S_i(\alpha) = \sum_{x_i} e(\lambda_i x_i^3 \alpha), \qquad i = 1, \ldots, 8,$$

where $e(y) = \exp(2\pi yi)$ and x_i runs through all integral values in the range

$$(3) \qquad P \leqslant |\lambda_i|^{1/3} x_i \leqslant 2P, \qquad i = 1,\ldots,4,$$

$$P^{4/5} \leqslant |\lambda_j|^{1/3} x_j \leqslant 2P^{4/5}, \qquad j = 5,\ldots,8.$$

Throughout the paper, the letters a, q, a_i, q_i always denote integers. In the following lemmas, δ denotes a fixed small positive number and ε denotes an arbitrary small positive number, not the same throughout. The constants implied by the notation 0, $\ll$, $\gg$ are always independent of the λ_i and of P, and without loss of generality, we assume the constants are greater than or equal to 1. In this section they depend only on δ and ε; in later sections they will depend only on θ, δ and ε and so ultimately on θ, since δ and ε will be determined by θ.

LEMMA 2.1. (i) *If p is prime and $(a, p) = 1$, then*

$$S(a, p) \leqslant 2P^{1/2}.$$

(ii) *If $(a, q) = 1$, then $S(a, q) \ll q^{2/3}$ where $S(a, k) = \sum_{x=1}^{k} e(ax^3/k)$.*

PROOF. Both (i) and (ii) are particular cases of Lemmas 12 and 15, respectively, of [4].

LEMMA 2.2. *If $|\beta| \leqslant \frac{1}{2}$, then*

$$I(\beta) = \sum_{m=P^3}^{8P^3} \frac{1}{3} m^{-2/3} e(\beta m) \ll P \min\left(1, P^{-3} |\beta|^{-1}\right)$$

and

$$I'(\beta) = \sum_{m=P^{12/5}}^{8P^{12/5}} \frac{1}{3} m^{-2/3} e(\beta m) \ll P^{4/5} \min\left(1, P^{-12/5} |\beta|^{-1}\right).$$

PROOF. See [1, Lemma 3].

Now for all $\alpha \in [0, 1] = I$, $\lambda_i \alpha$ can be represented in the form

$$(4)$$
$$\begin{cases} \lambda_i \alpha = (a_i/q_i) + \beta_i \quad \text{where } (a_i, q_i) = 1, \\ 0 < q_i \leqslant \left(|\lambda_i|^{-1/3} P\right)^{2+\delta}, \quad |\beta_i| \leqslant q_i^{-1}\left(|\lambda_i|^{-1/3} P\right)^{-2-\delta}, \quad i = 1,\ldots,4, \\ 0 < q_i \leqslant \left(|\lambda_i|^{-1/3} P^{4/5}\right)^{2+\delta}, \quad |\beta_i| \leqslant q_i^{-1}\left(|\lambda_i|^{-1/3} P^{4/5}\right)^{-2-\delta}, \quad i = 5,\ldots,8. \end{cases}$$

LEMMA 2.3. *Suppose that $\lambda_i \alpha$ is in the form of (4). Then:*
(i) *If $1 \leqslant i \leqslant 4$,*

$$S_i(\alpha) = |\lambda_i|^{-1/3} q_i^{-1} S(a_i, q_i) I(\pm \beta_i/\lambda_i) + O\left(q_i^{2/3+\varepsilon}\right)$$

$$\ll |\lambda_i|^{-1/3} q_i^{-1/3} P \min\left(1, P^{-3} |\lambda_i/\beta_i|\right).$$

(ii) *If $5 \leqslant i \leqslant 8$,*

$$S_i(\alpha) = |\lambda_i|^{-1/3} q_i^{-1} S(a_i, q_i) I'(\pm \beta_i/\lambda_i) + O\left(q_i^{2/3+\varepsilon}\right)$$

$$\ll |\lambda_i|^{-1/3} q_i^{-1/3} P^{4/5} \min\left(1, P^{-12/5} |\lambda_i/\beta_i|\right)$$

where $\pm$ is the sign of λ_i.

PROOF. This is essentially the same as [**1**, Lemmas 7–10].

LEMMA 2.4. *Suppose that* $\lambda_i \alpha$ *satisfies* (4) *and, in particular,*

$$\left(|\lambda_i|^{-1/3}P\right)^{-1-\delta} < q_i \leq \left(|\lambda_i|^{-1/3}P\right)^{2+\delta} \quad \text{if } 1 \leq i \leq 4,$$

$$\left(|\lambda_i|^{-1/3}P^{4/5}\right)^{1-\delta} < q_i \leq \left(|\lambda_i|^{-1/3}P^{4/5}\right)^{2+\delta} \quad \text{if } 5 \leq i \leq 8.$$

Then

(5)
$$S_i(\alpha) \ll \left(|\lambda_i|^{-1/3}P\right)^{3/4+\delta} \quad \text{if } 1 \leq i \leq 4,$$

$$S_i(\alpha) \ll \left(|\lambda_i|^{-1/3}P^{4/5}\right)^{3/4+\delta} \quad \text{if } 5 \leq i \leq 8.$$

PROOF. See [**1**, Lemma 13].

Before we proceed, we rearrange λ_i so that

(6)
$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_j|, \qquad j = 3,\dots,8,$$

and if $\lambda_1 \lambda_2 > 0$, choose λ_3 s.t. $\lambda_1 \lambda_3 < 0$ and define, for any $g \geq 0$, $h \geq 0$,

(7)
$$K_{ij}(g, h) = \int_I |S_i(\alpha)|^g |S_j(\alpha)|^h \, d\alpha$$

for $i = 1,\dots,4$, $j = 5,\dots,8$. In proving the following lemmas, (6) is assumed.

LEMMA 2.5.

(i)
$$K_{ij}(0,4) \ll \left(|\lambda_j|^{-1/3}P^{4/5}\right)^{2+\varepsilon}.$$

(ii)
$$K_{ij}(0,8) \ll \left(|\lambda_j|^{-1/3}P^{4/5}\right)^{5+\varepsilon}.$$

PROOF. See [**9**, Lemma 5].

LEMMA 2.6. *Let* $N'(m)$ *be the number of solutions of* $x^3 + y^3 - x'^3 - y'^3 \equiv 0$ *mod* m, *where* $1 \leq x, y, x', y' \leq m$. *Then* $N'(m) \ll m^{7/2+\varepsilon}$.

PROOF.

$$N'(m) = \int_I \left| \sum_{x=1}^{m} e(\alpha x^3) \right|^4 \sum_{x=-2m^2}^{2m^2} e(mx\alpha) \, d\alpha$$

$$\ll \left(\int_I \left| \sum_{x=1}^{m} e(\alpha x^3) \right|^8 d\alpha \int_I \left| \sum_{x=-2m^2}^{2m^2} e(mx\alpha) \right|^2 d\alpha \right)^{1/2}$$

$$\ll m^{7/2+\varepsilon} \quad \text{by Hua's inequality.}$$

LEMMA 2.7.

(i)
$$K_{ij}(2,4) \ll |\lambda_i|^{-1/3} |\lambda_j|^{-2/3} P^{13/5+\varepsilon}.$$

(ii)
$$K_{ij}(2,6) \ll |\lambda_i|^{-1/3} |\lambda_j|^{-7/6} P^{19/5+\varepsilon}.$$

PROOF. It is essentially the same as [**6**, Lemmas 5 and 6] for $|\lambda_i| \leq |\lambda_j|$. Since we may have $|\lambda_i| \geq |\lambda_j|$, we need Lemma 2.6 to improve the result. We omit (ii) and

prove (i) only. The integral is equal to the number of solutions

$$(8) \qquad \lambda_i(x^3 - x'^3) + \lambda_j(y^3 + z^3 - y'^3 - z'^3) = 0$$

in integers satisfying

$$P \leqslant |\lambda_i|^{1/3}x, \quad |\lambda_i|^{1/3}x' \leqslant 2P, \quad P^{4/5} \leqslant |\lambda_j|^{1/3}(y, z, y', z') \leqslant 2P^{4/5}.$$

For these solutions with $x = x'$, the number of pairs of y, z, y', z' satisfying (8) $\ll (|\lambda_j|^{-1/3}P^{4/5})^{2+\varepsilon}$. The number of this kind of solution is therefore

$$\ll |\lambda_i|^{-1/3}|\lambda_j|^{-2/3}P^{13/5+\varepsilon}.$$

Now we estimate the number N of solutions with $x' > x$. Let $x' = x + t$. (8) becomes

$$(9) \qquad \lambda_i(3x^2t + 3xt^2 + t^3) - \lambda_j(y^3 + z^3 - y'^3 - z'^3) = 0,$$

and we observe that $0 < t \leqslant |\lambda_i|^{-1/3}P^{2/5}$. Let $N(t, y', z')$ denote the number of solutions for prescribed values of t, y', z'. Then

$$N = \sum_{t, y', z'} N(t, y', z') \leqslant \left(\sum_{t, y', z'} 1 \right)^{1/2} \left(\sum_{t, y', z'} N^2(t, y', z') \right)^{1/2} = (N_1)^{1/2}(N_2)^{1/2},$$

say. It is obvious that $N_1 \ll |\lambda_i|^{-1/3}|\lambda_j|^{-2/3}P^2$. Also N_2 represents the number of solutions of

$$\lambda_i(3x_1^2t + 3x_1t^2 + t^3) - \lambda_j(y_1^3 + z_1^3 - y'^3 - z'^3) = 0,$$
$$\lambda_i(3x_2^2t + 3x_2t^2 + t^3) - \lambda_j(y_2^3 + z_2^3 - y'^3 - z'^3) = 0.$$

The number of solutions of these simultaneous equations with $x_1 = x_2$ is $\ll P^\varepsilon N$. With regard to the solution for $x_1 \neq x_2$, that would imply

$$3\lambda_i t(x_1 - x_2)(x_1 + x_2 + t) = \lambda_j(y_1^3 + z_1^3 - y_2^3 - z_2^3)$$

for values of y_1, z_1, y_2, z_2 such that the right-hand side is a multiple of λ_i. We can determine the values of x_1, x_2, t with p^ε possibilities, and the values of y', z' are then determined by either of the two equations with p^ε possibilities. Thus the number of solutions with $x_1 \neq x_2$ is $\ll (|\lambda_j|^{-1/3}P^{4/5})^4 m^{-4}N'(m)P^\varepsilon$, where $m = |\lambda_i/(\lambda_i, \lambda_j)|$ and $N'(m)$ is defined as in Lemma 2.6. Hence

$$\ll |\lambda_j|^{-4/3}P^{16/5}(|\lambda_i|/|\lambda_j|)^{-1/3+\varepsilon} = |\lambda_i|^{-1/3}|\lambda_j|^{-1}P^{16/5+\varepsilon}.$$

By a similar argument applied to (9),

$$N \ll |\lambda_i|^{-1/3}|\lambda_j|^{-1}P^{16/5+\varepsilon}.$$

It follows that $N_2 \ll |\lambda_i|^{-1/3}|\lambda_j|^{-1}P^{16/5+\varepsilon}$, giving the desired result

$$N \ll |\lambda_i|^{-1/3}|\lambda_j|^{-2/3}P^{13/5+\varepsilon}.$$

$x' < x$ is similar.

LEMMA 2.8. *Let* $A_i = \{\alpha \in I: S_i(\alpha) \gg |\lambda_i|^{-1/4}P^{3/4+2\delta}\}$, $i = 1, \ldots, 4$. *Then*:

$$(i) \qquad \int_{A_i} |S_i(\alpha)|^6 \, d\alpha \ll |\lambda_i|^{-1}P^{3+\varepsilon}.$$

(ii)
$$\int_{A_i} |S_i(\alpha)|^5 \, d\alpha \ll |\lambda_i|^{-3/4} P^{9/4 - 2\delta + \varepsilon}.$$

PROOF. (i) For every α in A_i, we determine a_i, q_i such that (4) is satisfied, so by Lemma 2.4, $0 < q_i \le (|\lambda_i|^{-1/3} P)^{1-\delta}$ and, by Lemma 2.3,

$$|\lambda_i|^{-1/4} P^{3/4 + 2\delta} \ll |S_i(\alpha)| \ll q_i^{-1/3} |\lambda_i|^{-1/3} P \min(1, P^{-3}|\lambda_i/\beta_i|)$$

which imply

(10)
$$0 < q_i \ll |\lambda_i|^{-1/4} P^{3/4 - 6\delta}, \quad |\beta_i/\lambda_i| \ll q_i^{-1/3}|\lambda_i|^{-1/12} P^{-11/4 - 2\delta}.$$

Let F_{a_i, q_i} denote the interval of values of α satisfying (4) and (10). Since $A_i \subset \bigcup F_{a_i, q_i}$, where q_i satisfies (10) and $|a_i| \le |\lambda_i q_i|$,

$$\int_{A_i} |S_i(\alpha)|^6 \, d\alpha \ll \sum_{q_i} \sum_{a_i} \int_{F_{a_i,q_i}} |S_i(\alpha)|^6 \, d\alpha,$$

where the summations a_i, q_i are as before.

$$\int_{F_{a_i,q_i}} |S_i(\alpha)|^6 \, d\alpha \ll \int_0^{P^{-3}|\lambda_i|} |\lambda_i|^{-3} q_i^{-2} P^6 \, d\beta_i + \int_{P^{-3}|\lambda_i|}^{\infty} P^{-12} |\lambda_i|^3 q_i^{-2} \beta_i^{-6} \, d\beta_i$$

$$\ll |\lambda_i|^{-2} q_i^{-2} P^3.$$

So

$$\int_{A_i} |S_i(\alpha)|^6 \, d\alpha \ll \sum_{q_i} \sum_{a_i} |\lambda_i|^{-2} q_i^{-2} P^3 \ll |\lambda_i|^{-1} P^{3 + \varepsilon}.$$

(ii) follows easily from (i).

LEMMA 2.9. *Assume that* $P > \Pi^{35/24}$. *Then*:
(i) *For* $i = 1, \ldots, 4, j = 5, \ldots, 8$,

$$\int_{A_i} |S_i(\alpha)|^3 |S_j(\alpha)|^4 \, d\alpha \ll |\lambda_i|^{-9/16} |\lambda_j|^{-7/9} P^{67/20 + 2\delta + \varepsilon} \quad \text{if } i = 1,$$

$$\ll |\lambda_i|^{-2/3} |\lambda_j|^{-7/9} P^{67/20 + 2\delta + \varepsilon} \quad \text{if } i \ne 1.$$

(ii)
$$K_{ij}(3, 4) \ll |\lambda_i|^{-7/12} |\lambda_j|^{-2/3} P^{67/20 + 2\delta + \varepsilon}, \quad i = 2, 3, 4, j = 5, \ldots, 8.$$

PROOF. (i)

$$\int_{A_i} |S_i(\alpha)|^3 |S_j(\alpha)|^4 \, d\alpha \ll K_{ij}(2, 6)^{2/3} \left(\int_{A_i} |S_i(\alpha)|^5 \, d\alpha \right)^{1/3}$$

$$\ll |\lambda_i|^{-17/36} |\lambda_j|^{-7/9} P^{197/60 + \varepsilon}.$$

Using $P > \Pi^{35/24}$, which implies $P > |\lambda_1|^{35/24}$ and $P > |\lambda_i|^{35/12}$ if $i \ne 1$, (i) follows.
(ii)

$$K_{ij}(3, 4) \ll \int_{A_i} |S_i(\alpha)|^3 |S_j(\alpha)|^4 \, d\alpha + \int_{I \setminus A_i} |S_i(\alpha)|^3 |S_j(\alpha)|^4 \, d\alpha$$

$$\ll |\lambda_i|^{-7/12} |\lambda_j|^{-2/3} P^{67/20 + 2\delta + \varepsilon} + |\lambda_i|^{-1/4} P^{3/4 + 2\delta} K_{ij}(2, 4).$$

By (i) and Lemma 2.7(i), (ii) follows.

3. Minor arcs for the proof of Theorem 1. We let $\mathfrak{N}(P)$ be the number of integral solutions of (2) which lie in the 'box' determined by (3). We shall show that for a given positive number θ, there exists a positive constant D_θ, independent of λ_i, such that if $P^{1-\theta} > D_\theta \Pi^{35/24}$, then $\mathfrak{N}(P) > 0$. So we may take P such that $D_\theta \Pi^{35/24} < P^{1-\theta} < 2D_\theta \Pi^{35/24}$, which will imply there exists a solution of (2) in nonzero integers with

$$\sum_i |\lambda_i x_i^3| < 8\left(2D_\theta \Pi^{35/24}\right)^{3/(1-\theta)} < K_\theta \Pi^{35/8(1-\theta)},$$

where K_θ depends on θ only, so for every $\phi > 0$, we can find $0 < \theta < 1$ such that $35/8(1-\theta) < 35/8 + \phi$, and hence Theorem 1 follows.

We now use

$$\mathfrak{N}(P) = \int_I V(\alpha)\, d\alpha, \quad \text{where } V(\alpha) = \prod_{i=1}^{8} S_i(\alpha)$$

and estimate $\mathfrak{N}(P)$ will be of the form $c\Pi^{-1/3-\varepsilon}P^{21/5}$ where $c > 0$, and the error terms are substantially smaller than $\Pi^{-1/3}P^{21/5}$ provided P is substantially larger than $\Pi^{35/24}$.

By Dirichlet's theorem on diophantine approximation, for each α in I and i, there exists a rational approximation a_i/q_i to $\lambda_i \alpha$ such that (4) holds. Let $S = \bigcup_{i=1}^{8} B_i$ where

$$B_i = \left\{\alpha \in I : S_i(\alpha) \ll |\lambda_i|^{-1/4}P^{3/4+2\delta}\right\}, \quad 1 \le i \le 4,$$

$$B_j = \left\{\alpha \in I : S_j(\alpha) \ll |\lambda_j|^{-1/4}P^{3/5+2\delta}\right\}, \quad 5 \le j \le 8.$$

LEMMA 3.1. $\int_S |V(\alpha)|\, d\alpha \ll \Pi^{-3/16}P^{41/10+4\delta+\varepsilon}$.

PROOF. We may assume that the constants implied in defining A_i and B_i are the same for $i, \ldots, 4$. Then $A_i = I \setminus B_i$, $i = 1, \ldots, 4$.

Let $A = A_1 \cap A_2 \cap A_3 \cap A_4$ and

$$S = (B_1 \cap B_2) \cup (B_1 \cap A_2) \cup (B_2 \cap A_1) \cup (B_3 \cap A_1 \cap A_2)$$

$$\cup (B_4 \cap A_1 \cap A_2) \cup \bigcup_{j=5}^{8}(B_j \cap A).$$

(i)

$$\int_{B_1 \cap B_2} |V(\alpha)|\, d\alpha \ll |\lambda_1|^{-1/4}|\lambda_2|^{-1/4}P^{3/2+4\delta}\left(\prod_{i=3}^{4}\prod_{j=5}^{8} K_{ij}(2,4)\right)^{1/8}$$

$$\ll \Pi^{-3/16}P^{41/10+4\delta+\varepsilon}.$$

By Lemma 2.7 and notice that $|\lambda_1|, |\lambda_2| \gg |\lambda_j|$ for $j = 3, 4, \ldots, 8$.

(ii)

$$\int_{B_1 \cap A_2} |V(\alpha)|\, d\alpha$$

$$\ll |\lambda_1|^{-1/4}P^{3/4+2\delta}\left(\prod_{j=5}^{8}\int_{A_2}|S_2(\alpha)|^3|S_j(\alpha)|^4\, d\alpha\right)^{1/12}\left(\prod_{i=3}^{4}\prod_{j=5}^{8} K_{ij}(3,4)\right)^{1/12}.$$

By Lemma 2.9,

$$\ll |\lambda_1|^{-1/4}|\lambda_2|^{-2/9}|\lambda_3|^{-7/36}|\lambda_4|^{-7/36}|\lambda_5 \cdots \lambda_8|^{-19/108}P^{41/10+4\delta+\varepsilon}$$

$$\ll \Pi^{-3/16}P^{41/10+4\delta+\varepsilon}.$$

Similarly for $B_2 \cap A_1$.

(iii)

$$\int_{B_3 \cap (A_1 \cap A_2)} |V(\alpha)|\, d\alpha$$

$$\ll |\lambda_3|^{-1/4}P^{3/4+2\delta}\left(\prod_{\substack{k=1 \\ k\neq 3}}^{4}\prod_{j=5}^{8}\int_{A_1 \cap A_2}|S_k(\alpha)|^3|S_j(\alpha)|^4\,d\alpha\right)^{1/12}.$$

By Lemma 2.9,

$$\ll |\lambda_1|^{-3/16}|\lambda_2|^{-2/9}|\lambda_3|^{-1/4}|\lambda_4|^{-7/36}|\lambda_5 \cdots \lambda_8|^{-5/27}P^{41/10+4\delta+\varepsilon}$$

$$\ll \Pi^{-3/16}P^{41/10+4\delta+\varepsilon}.$$

Similarly for $B_4 \cap A_1 \cap A_2$.

(iv) For $j = 5,\ldots,8$,

$$\int_{B_j \cap A}|V(\alpha)|\,d\alpha \ll |\lambda_j|^{-1/4}P^{3/5+2\delta}\left(\prod_{k=1}^{4}\prod_{\substack{m=5 \\ m\neq j}}^{8}\int_{A_k}|S_k(\alpha)|^4|S_m(\alpha)|^3\,d\alpha\right)^{1/12}$$

$$\ll |\lambda_j|^{-1/4}P^{3/5+2\delta}\left(\prod_{k=1}^{4}\prod_{\substack{m=5 \\ m\neq j}}^{8}K_{k,m}(2,6)\int_{A_k}|S_k(\alpha)|^6\,d\alpha\right)^{1/24}.$$

By Lemmas 2.7, 2.8 and using the fact that $P > \Pi$,

$$\ll \Pi^{-1/6}P^{4+2\delta} \ll \Pi^{-3/16}P^{41/10+4\delta+\varepsilon}.$$

Summing up (i) to (iv) the desired result follows.

4. Major arcs for the proof of Theorem 1. We define

$$M = \big\{\alpha \in S: S_i(\alpha) \gg |\lambda_i|^{-1/4}P^{3/4+2\delta}, i = 1,\ldots,4,$$

$$\text{and } S_j(\alpha) \gg |\lambda_j|^{-1/4}P^{3/5+2\delta}, j = 5,\ldots,8\big\}.$$

Since we have assumed the constants implied are greater than or equal to 1, by Lemma 2.4, the rational approximation a_i/q_i which satisfies (4) must satisfy

$$q_i < \left(|\lambda_i|^{-1/3}P\right)^{1-\delta}, \quad i = 1,\ldots,4; \qquad q_j < \left(|\lambda_j|^{-1/3}P^{4/5}\right)^{1-\delta}, \quad j = 5,\ldots,8.$$

By Lemma 2.3, this will imply that
(11)

$$0 < q_i < \left(|\lambda_i|^{-1/3}P\right)^{3/4}, \qquad |\beta_i/\lambda_i| < |\lambda_i|^{-1/12}P^{-11/4-2\delta}, \quad i = 1,\ldots,4,$$

$$0 < q_j < \left(|\lambda_j|^{-1/3}P^{4/5}\right)^{3/4}, \qquad |\beta_j/\lambda_j| < |\lambda_j|^{-1/12}P^{-11/5-2\delta}, \quad j = 5,\ldots,8.$$

We consider several lemmas before we proceed.

LEMMA 4.1. (i) *If* $P > 4|\lambda_i|^{1/2}$, *then there is at most one approximation* a_i/q_i *to* $\lambda_i\alpha$ *such that* (4) *and* (11) *hold*.

(ii) *If* $P > 4|\lambda_j\lambda_k|$ *and* (4) *and* (11) *hold for* $i = j, k$ ($j \neq k$), *then*

$$a_j/\lambda_j q_j = a_k/\lambda_k q_k.$$

(iii) *If* (4) *and* (11) *hold, then*

$$|\beta_i| < q_i^{-1}\left(|\lambda_i|^{-1/3}P\right)^{-2-\delta} \qquad i = 1,\ldots,4,$$

$$|\beta_j| < q_j^{-1}\left(|\lambda_i|^{-1/3}P^{4/5}\right)^{-2-\delta}, \qquad j = 5,\ldots,8.$$

PROOF. The proofs are essentially the same as [**9**, Lemma 8]. We prove (iii) only.
(iii)

$$|\beta_i|q_i \leqslant |\lambda_i||\lambda_i|^{-1/12}P^{-11/4-2\delta}\left(|\lambda_i|^{-1/4}P^{3/4}\right) \qquad (i = 1,\ldots,4)$$

$$< \left(|\lambda_i|^{-1/3}P\right)^{-2-\delta}.$$

Similarly for $j = 5,\ldots,8$.

Suppose that $\alpha \in M$ and for each i let a_i/q_i be an approximation which satisfies (4) and (11). Since we have assumed $P > \Pi^{35/24}$, by Lemma 4.1(i), a_i/q_i are unique for $\lambda_i\alpha$ and, by (ii), $a_j/\lambda_j q_j = a_k/\lambda_k q_k$, for all j, k. Hence there exist unique integers a, q such that $(a, q) = 1$, $q > 0$ and

(12) $$a_i/q_i = \lambda_i a/q$$

for all i. Let

(13) $$\delta_i = (\lambda_i, q), \qquad i = 1,\ldots,8.$$

Then

(14) $$a_i = \lambda_i a/\delta_i, \qquad q_i = q/\delta_i.$$

By (11), we have

(15) $$0 < q \leqslant \min_{\substack{1 \leqslant i \leqslant 4 \\ 5 \leqslant j \leqslant 8}} \left\{ \delta_i|\lambda_i|^{-1/4}P^{3/4}, \delta_j|\lambda_j|^{-1/4}P^{3/5} \right\}$$

and

(16) $$|\alpha - a/q| \leqslant |\lambda_1|^{-1/12}P^{-11/4-2\delta}.$$

Since $P > \Pi^{35/24}$ and (6) is satisfied, this implies $P > |\lambda_1|^{35/24}$ and $P > |\lambda_j|^{35/8}$ for $j = 3,\ldots,8$.

(17) $$P^{19/35} \leqslant \min\left\{ |P^{24/35}|^{-1/4}P^{3/4}, \left(P^{8/35}\right)^{-1/4}P^{3/5} \right\}$$

$$\leqslant \min_{\substack{1 \leqslant i \leqslant 4 \\ 5 \leqslant j \leqslant 8}} \left\{ \delta_i|\lambda_i|^{-1/4}P^{3/4}, \delta_j|\lambda_j|^{-1/4}P^{3/5} \right\}$$

$$\leqslant \left(P^{8/35}\right)^{3/4}P^{3/5} \leqslant P^{4/5}.$$

We define $I_{a,q}$ to be the interval determined by (16). By previous argument, if $\alpha \in M$, then every rational approximation a_i/q_i satisfying (4) must satisfy (11), which implies $a \in I_{a,q}$ for some a, q such that $0 \leqslant a \leqslant q - 1$ and q satisfies (15).

Conversely, suppose that a, q are integers such that $(a, q) = 1$, $q > 0$, $0 \leqslant a \leqslant q - 1$ and (15) holds, where the δ_i are defined by (13). If α belongs to the interval $I_{a,q}$ defined as above, then $\alpha \in I$ and the approximation a_i/q_i, such that $(a_i, q_i) = 1$ and $q_i > 1$, defined by (12), satisfy (11) and (14), and these intervals do not overlap because of the uniqueness of the a_i/q_i.

By Lemma 4.1(iii), we can always use Lemma 2.3 to estimate $S_i(\alpha)$ for α in M. We can now estimate the contribution from M to $\mathfrak{N}(P)$.

LEMMA 4.2. *If $\delta \leqslant 1/30$,*

$$(18) \qquad \int_M V(\alpha)\, d\alpha = \Pi^{-1/3} \mathfrak{S} R(P) + O(P^{39/10})$$

$$where \; \mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} (q_1 \cdots q_8)^{-1} S(a_1, q_1), \ldots, S(a_8, q_8),$$

the a_i, q_i are defined by (13) and (14), and $R(P) \gg P^{21/5}$.

PROOF. Suppose $\alpha \in M$ and define a_i/q_i by (13) and (14) and write

$$\beta = \alpha - a/q = \beta_i/\lambda_i, \qquad i = 1, \ldots, 8.$$

Then by Lemma 2.3,

$$V(\alpha) = \Pi^{-1/3} \left(\prod_{i=1}^{4} q_i^{-1} S(a_i, q_i) I(\pm\beta) \right) \left(\prod_{j=5}^{8} q_j^{-1} S(a_j, q_j) I'(\pm\beta) \right) + E,$$

where $\pm$ is the sign of λ_i and the error E satisfies

$$E \ll \sum_{i=1}^{8} q_i^{2/3+\varepsilon} \left(\prod_{\substack{j=1 \\ j\neq i}}^{8} |\lambda_j|^{-1/3} q_j^{-1/3} \right) P^{32/5} \left(\min\left(1, P^{-3} |\beta|^{-1}\right) \right)^2$$

$$\ll q^{-5/3+\varepsilon} \left(\min\left(1, P^{-3} |\beta|^{-1}\right) \right)^2.$$

Since $|\lambda_i|^{-1} q \leqslant q_i \leqslant q$ for all i by (13) and (14),

$$\int_M E\, d\alpha \ll \sum_q{}' \sum_a{}' \int_{I_{a,q}} q^{-5/3+\varepsilon} P^{32/5} \min\left(1, P^{-6} |\beta|^{-2}\right) d\alpha$$

$$\ll \sum_q{}' \sum_a{}' q^{-5/3+\varepsilon} P^{17/5} \ll P^{19/5} \quad \text{by (17),}$$

where $\sum_q' \sum_a'$ denotes the range of q defined in (15), $(a, q) = 1$ and $0 \leqslant a < q$. Since the $I_{a,q}$ do not overlap, it follows that

$$(19) \qquad \int_M V(\alpha)\, d\alpha = \Pi^{-1/3} \sum_q{}' \sum_a{}' (q_1 \cdots q_8)^{-1} S(a_1, q_8), \ldots, S(a_8, q_8)$$

$$\times \int_{I_{a,q}} \prod_{i=1}^{4} I(\pm\beta) \prod_{j=5}^{8} I'(\beta)\, d\alpha + O(P^{19/5}).$$

The error caused by replacing $I_{a,q}$ in (19) by the interval $|\alpha - a/q| \le \frac{1}{2}$ is

$$\ll \Pi^{-1/3} \sum_q' \sum_a' (q_1 \cdots q_8)^{-1/3} \int_{I_{a,q}^*} P^{36/5} \min\left(1, P^{-3}|\beta|^{-1}\right)^4 d\alpha,$$

where $I_{a,q}^*$ is the set $\alpha = a/q + \beta$ such that

$$\frac{1}{2} \ge |\beta| > |\lambda_1|^{-1/12} P^{-11/4 - 2\delta} \ge P^{-17/6 - 2\delta} \ge P^{-3},$$

since $P > |\lambda_1|$ and $2\delta \le \frac{1}{6}$.

Now, for any pair a, q,

$$\Pi^{-1/3}(q_1 \cdots q_8)^{-1/3} P^{-24/5} \int_{I_{a,q}^*} \beta^{-4} d\alpha \ll q^{-8/3} P^{37/10 + 6\delta}.$$

Hence the total error is certainly

$$\ll P^{37/10 + 6\delta} \sum_{q=1}^{\infty} \sum_{a=0}^{q-1} q^{-8/3} \ll P^{39/10}$$

provided that $6\delta \le \frac{1}{5}$. Thus we have

$$(20) \quad \int_M V(\alpha)\, d\alpha = \Pi^{-1/3} \sum_q' \sum_a' (q_1 \cdots q_8)^{-1} S(a_1, q_1) \cdots S(a_8, q_8) R(P)$$
$$+ O(P^{19/5}),$$

where

$$(21) \quad R(P) = \int_{-1/2}^{1/2} \prod_{i=1}^{4} I(\beta) \prod_{j=5}^{8} I'(\beta)\, d\beta = 3^{-8} \sum_{m_1, \ldots, m_8} (m_1 \cdots m_8)^{-2/3}$$

by definition of $I(\beta)$ and $I'(\beta)$, where

$$(22) \qquad\qquad P^3 \le m_i \le 8P^3, \qquad i = 1, \ldots, 4,$$
$$P^{12/5} \le m_j \le 8P^{12/5}, \qquad j = 5, \ldots, 8,$$

and $\pm m_1 \pm m_2 \pm \cdots \pm m_8 = 0$.

Since either λ_2 or λ_3 has sign different then λ_1, without loss of generality, we assume $\lambda_2 \lambda_1 < 0$. So now the last condition is

$$(23) \qquad\qquad m_1 = m_2 \mp m_3 \mp m_4 \mp \cdots \mp m_8$$

for each $(m_2, \ldots, m_4)$ such that

$$4P^3 \le m_2 \le 5P^3, \quad P^3 \le m_i \le 5P^3/4, \qquad i = 3, 4,$$

and the integer m_1 defined by (23) satisfies $P^3 \le m_1 \le 8P^3$. Thus the number of solutions of (23) such that (22) holds is $\gg P^9 P^{48/5}$. For any such solution we have

$$(m_1 \cdots m_8)^{-2/3} \ge 2^{-16} P^{-8} P^{-32/5}.$$

Thus $R(P) \gg P^{21/5}$ as required.

The error caused by extending the range of summation for q to infinity is

$$\ll \Pi^{-1/3} \sum_q \sum_a (q_1 \cdots q_8)^{-1/3} P^{36/5} \int_{-1/2}^{1/2} \min(1, P^{-6}|\beta|^{-2}) \, d\beta$$

$$\ll \Pi^{-1/3} \sum_q \sum_a (q_1 \cdots q_8)^{-1/3} P^{21/5},$$

where q is summed over the range $q > P^{19/35}$, and a is as before. We consider a particular set of divisors $\delta_1, \ldots, \delta_8$ of $\lambda_1, \ldots, \lambda_8$, repectively. The contribution to the above from those pairs a, q corresponding to $\delta_1, \ldots, \delta_8$ is

$$\ll \Pi^{-1/3} (\delta_1 \cdots \delta_8)^{1/3} P^{21/5} \sum_{q > P^{19/35}} q^{-5/3}.$$

Thus the contribution from these a, q is

$$\ll \Pi^{-1/3} P^{21/5} (\delta_1 \cdots \delta_8)^{1/3} P^{-38/105} \ll \Pi^{-1/3} P^{21/5} \Pi^{1/3} P^{-38/105} \ll P^{39/10-\varepsilon}.$$

Since $\Pi \ll P$, the number of different possibilities for $\delta_1 \cdots \delta_8$ is $O(P^\varepsilon)$, and it follows that the total error is $\ll P^{39/10}$. Hence the result followed.

5. The singular series. Firstly, we assume that the λ_i are cube-free and no prime divides more than five of them. The following lemma shows that the assumption may be made without loss of generality.

LEMMA 5.1. *If Theorem 1 holds provided the λ_i are cube-free nonzero integers such that no prime divides more than five of them, then it holds for every set of nonzero integral λ_i.*

PROOF. The proof is essentially the same as [9, Lemma 5].

We now investigate the singular series $\mathfrak{S}$ defined by (18). Since $S(\lambda_i a, q) = \delta_i S(a, q_i)$ and $S(\lambda_i a, q)$ is periodic of period q, we may rewrite the series as

$$\sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-8} S(\lambda_1 a, q) \cdots S(\lambda_8 a, q).$$

For p prime, we define

$$(24) \qquad \chi(p) = 1 + \sum_{\nu=1}^{\infty} \sum_{a=1}^{p^\nu} P^{-8\nu} \prod_{i=1}^{8} S(\lambda_i a, P^\nu)$$

and $M(P^\nu)$ to be the number of solutions (mod p^ν) of the congruence

$$(25) \qquad \sum_{i=1}^{8} \lambda_i x_i^3 \equiv 0 \pmod{P^\nu}.$$

Similar to [9, §6], we have

$$(26) \qquad \chi(p) = \lim_{\nu \to \infty} P^{-7\nu} M(P^\nu) \quad \text{and} \quad \mathfrak{S} = \prod_P \chi(p),$$

where the product is over all primes p.

LEMMA 5.2. (i) *If $s \geq 3$, p is any prime, and $p \nmid |\lambda_1 \cdots \lambda_s|$, then there is a nontrivial solution* (i.e. *one with some $x_i \not\equiv 0$*) *of*

$$(27) \qquad \sum_{i=1}^{s} \lambda_i x_i^3 \equiv 0 \quad (\mathrm{mod}\ p).$$

(ii) *Suppose that $p \neq 3$ is a prime such that $\lambda_1, \ldots, \lambda_s$ are not divisible by p and $\lambda_{s+1}, \ldots, \lambda_8$ are divisible by p, and let $\mathfrak{N}(p)$ denote the number of nontrivial solutions of* (27). *Then for all $\nu > 0$,*

$$M(p^{\nu}) \geq p^{7(\nu-1)+(8-s)} \mathfrak{N}(p).$$

(iii) *If $\lambda_1, \ldots, \lambda_8$ are cube-free and $\nu \geq 6$, then $M(3^{\nu}) \geq 3^{7(\nu-6)}$.*

PROOF. The proof is essentially the same as [**9**, Lemma 10].

LEMMA 5.3. (i) *There is an absolute constant $C_1 > 0$ such that*

$$\prod_p \chi(p) \geq \frac{1}{2} \quad (p \nmid \Pi,\ p > C_1).$$

(ii) *There is a constant $C_2 = C_2(\varepsilon) > 0$ such that*

$$\prod_p \chi(p) \geq \Pi^{-\varepsilon} \quad (p \nmid \Pi,\ p > C_2).$$

(iii) *We have $\mathfrak{S} \gg \Pi^{-\varepsilon}$.*

(*The products in* (i) *and* (ii) *are over all primes p which satisfy the condition in parentheses.*)

PROOF. See [**9**, Lemma 11].

6. Completion of the proof of Theorem 1. Assuming that $P > \Pi^{35/24}$ and $\delta \leq \frac{1}{30}$, by Lemmas 4.2 (with $\varepsilon = \delta$), 5.2 and 3.1 (since we have $I \setminus M \subset S$), we have

$$\mathfrak{N}(P) = C_\varepsilon \Pi^{-(1/3)-\varepsilon} P^{21/5} + E,$$

where $C_\varepsilon > 0$ and

$$E \ll \Pi^{-3/16} P^{41/10+5\delta} + P^{39/10} \ll \Pi^{-3/16} P^{41/10+5\delta}$$

$$\ll \Pi^{-1/3} P^{21/5} (\Pi^{-35/24} P^{1-\theta})^{-1/10} P^{5\delta-\theta/10}.$$

For given $\theta > 0$, we choose $\varepsilon > 0$ and $\delta > 0$ such that $\varepsilon + 5\delta \leq \theta/20$, $\delta < 1/30$. Then $P^{5\delta-\theta/10} \leq P^{-\varepsilon-\theta/20}$.

Since $P > \Pi^{35/24}$ certainly holds if $P^{1-\theta} > \Pi^{35/24}$, it now follows from the preceding paragraph that there is a constant D_θ such that $\mathfrak{N}(P) > 0$ provided $P^{1-\theta} > D_\theta \Pi^{35/24}$. By the remarks at the beginning of §3 this completes the proof of Theorem 1.

REFERENCES

1. H. Davenport, *On Waring's problem for cubes*, Acta Math. **71** (1939), 123–143.
2. _____, *Indefinite quadratic forms in many variables*, Mathematika **3** (1956), 81–101.
3. _____, *Cubic forms in 32 variables*, Philos. Trans. Roy. Soc. London Ser. A **251** (1959), 193–232.
4. _____, *Analytic methods for Diophantine equations and Diophantine inequalities*, Ann Arbor Publ., Ann Arbor, Mich., 1963.

5. H. Davenport and H. Heilbronn, *On indefinite quadratic forms in five variables*, J. London Math. Soc. **21** (1946), 185–193.

6. H. Davenport and K. F. Roth, *The solubility of certain Diophantine inequalities*, Mathematika **2** (1955), 81–96.

7. I. Danicic, *The solubility of certain Diophantine inequalities*, Proc. London Math. Soc. (3) **8** (1958), 161–176.

8. D. J. Lewis, *Cubic congruences*, Michigan Math. J. **4** (1957), 85–95.

9. J. Pitman and D. Ridout, *Diagonal cubic equations and inequalities*, Proc. Roy. Soc. Ser. A **297** (1967), 476–502.

10. T. Nagell, *Introduction to number theory*, Wiley, New York, 1951.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HONG KONG, HONG KONG

Current address: Department of Mathematics, Stanford University, Stanford, California 94305