

## BOUNDS FOR INTEGRAL SOLUTIONS OF DIAGONAL CUBIC EQUATIONS

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ABSTRACT. It was proved by Davenport [3] that for the nonzero integral  $\lambda_i$ , the equation  $\lambda_1 x_1^3 + \cdots + \lambda_8 x_8^3 = 0$  always has a nontrivial integral solution. In this paper, we investigate the bounds of nontrivial integral solutions in terms of  $\lambda_1, \dots, \lambda_8$ .

**1. Introduction.** Pitman and Ridout [9] proved that for every  $\theta > 0$ , there exists a constant  $c_\theta$  with the following property. If  $\lambda_1, \dots, \lambda_9$  are nonzero integers, then the equation

$$(1) \quad \lambda_1 x_1^3 + \cdots + \lambda_9 x_9^3 = 0$$

has a solution in nonzero integers  $x_1, \dots, x_9$ , such that

$$|\lambda_1 x_1^3| + \cdots + |\lambda_9 x_9^3| < c_\theta |\lambda_1 \cdots \lambda_9|^{(3/2)+\theta}.$$

They conjectured [9] that it should be possible to obtain bounds for integral solutions of the equation

$$(2) \quad \lambda_1 x_1^3 + \cdots + \lambda_8 x_8^3 = 0$$

where  $\lambda_1, \dots, \lambda_8$  are nonzero integers.

However, their method cannot be directly extended. In this paper we shall use I. Danicic's [7] idea and improve Davenport and Roth's [6] results to overcome the difficulties and prove the following

**THEOREM 1.** *For every  $\theta > 0$ , there exists a constant  $c_\theta$ , depending on  $\theta$  only, with the following property. If  $\lambda_1, \dots, \lambda_8$  are nonzero integers, and not all of the same sign, then (2) has a solution in positive integers  $x_1, \dots, x_8$  such that*

$$|\lambda_1 x_1^3| + \cdots + |\lambda_8 x_8^3| < c_\theta |\lambda_1 \cdots \lambda_8|^{(35/8)+\theta}.$$

**2. Notation and general lemmas.** Let  $\lambda_i (i = 1, \dots, 8)$  be given nonzero integers such that  $P \geq |\lambda_i|$  for all  $i$ . We write

$$\Pi = \prod_{i=1}^8 |\lambda_i|$$

and define

$$S_i(\alpha) = \sum_{x_i} e(\lambda_i x_i^3 \alpha), \quad i = 1, \dots, 8,$$

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where  $e(y) = \exp(2\pi yi)$  and  $x_i$  runs through all integral values in the range

$$(3) \quad \begin{aligned} P \leq |\lambda_i|^{1/3} x_i \leq 2P, \quad & i = 1, \dots, 4, \\ P^{4/5} \leq |\lambda_j|^{1/3} x_j \leq 2P^{4/5}, \quad & j = 5, \dots, 8. \end{aligned}$$

Throughout the paper, the letters  $a, q, a_i, q_i$  always denote integers. In the following lemmas,  $\delta$  denotes a fixed small positive number and  $\epsilon$  denotes an arbitrary small positive number, not the same throughout. The constants implied by the notation  $0, \ll, \gg$  are always independent of the  $\lambda_i$  and of  $P$ , and without loss of generality, we assume the constants are greater than or equal to 1. In this section they depend only on  $\delta$  and  $\epsilon$ ; in later sections they will depend only on  $\theta, \delta$  and  $\epsilon$  and so ultimately on  $\theta$ , since  $\delta$  and  $\epsilon$  will be determined by  $\theta$ .

LEMMA 2.1. (i) If  $p$  is prime and  $(a, p) = 1$ , then

$$S(a, p) \leq 2P^{1/2}.$$

(ii) If  $(a, q) = 1$ , then  $S(a, q) \ll q^{2/3}$  where  $S(a, k) = \sum_{x=1}^k e(ax^3/k)$ .

PROOF. Both (i) and (ii) are particular cases of Lemmas 12 and 15, respectively, of [4].

LEMMA 2.2. If  $|\beta| \leq \frac{1}{2}$ , then

$$I(\beta) = \sum_{m=P^3}^{8P^3} \frac{1}{3} m^{-2/3} e(\beta m) \ll P \min(1, P^{-3} |\beta|^{-1})$$

and

$$I'(\beta) = \sum_{m=P^{12/5}}^{8P^{12/5}} \frac{1}{3} m^{-2/3} e(\beta m) \ll P^{4/5} \min(1, P^{-12/5} |\beta|^{-1}).$$

PROOF. See [1, Lemma 3].

Now for all  $\alpha \in [0, 1] = I$ ,  $\lambda_i \alpha$  can be represented in the form

$$(4) \quad \begin{cases} \lambda_i \alpha = (a_i/q_i) + \beta_i \quad \text{where } (a_i, q_i) = 1, \\ 0 < q_i \leq (|\lambda_i|^{-1/3} P)^{2+\delta}, \quad |\beta_i| \leq q_i^{-1} (|\lambda_i|^{-1/3} P)^{-2-\delta}, \quad i = 1, \dots, 4, \\ 0 < q_i \leq (|\lambda_i|^{-1/3} P^{4/5})^{2+\delta}, \quad |\beta_i| \leq q_i^{-1} (|\lambda_i|^{-1/3} P^{4/5})^{-2-\delta}, \quad i = 5, \dots, 8. \end{cases}$$

LEMMA 2.3. Suppose that  $\lambda_i \alpha$  is in the form of (4). Then:

(i) If  $1 \leq i \leq 4$ ,

$$\begin{aligned} S_i(\alpha) &= |\lambda_i|^{-1/3} q_i^{-1} S(a_i, q_i) I(\pm \beta_i/\lambda_i) + O(q_i^{2/3+\epsilon}) \\ &\ll |\lambda_i|^{-1/3} q_i^{-1/3} P \min(1, P^{-3} |\lambda_i/\beta_i|). \end{aligned}$$

(ii) If  $5 \leq i \leq 8$ ,

$$\begin{aligned} S_i(\alpha) &= |\lambda_i|^{-1/3} q_i^{-1} S(a_i, q_i) I'(\pm \beta_i/\lambda_i) + O(q_i^{2/3+\epsilon}) \\ &\ll |\lambda_i|^{-1/3} q_i^{-1/3} P^{4/5} \min(1, P^{-12/5} |\lambda_i/\beta_i|) \end{aligned}$$

where  $\pm$  is the sign of  $\lambda_i$ .

PROOF. This is essentially the same as [1, Lemmas 7–10].

LEMMA 2.4. Suppose that  $\lambda_i \alpha$  satisfies (4) and, in particular,

$$\begin{aligned} (|\lambda_i|^{-1/3}P)^{-1-\delta} < q_i \leq (|\lambda_i|^{-1/3}P)^{2+\delta} \quad \text{if } 1 \leq i \leq 4, \\ (|\lambda_i|^{-1/3}P^{4/5})^{1-\delta} < q_i \leq (|\lambda_i|^{-1/3}P^{4/5})^{2+\delta} \quad \text{if } 5 \leq i \leq 8. \end{aligned}$$

Then

$$\begin{aligned} (5) \quad S_i(\alpha) &\ll (|\lambda_i|^{-1/3}P)^{3/4+\delta} \quad \text{if } 1 \leq i \leq 4, \\ S_i(\alpha) &\ll (|\lambda_i|^{-1/3}P^{4/5})^{3/4+\delta} \quad \text{if } 5 \leq i \leq 8. \end{aligned}$$

PROOF. See [1, Lemma 13].

Before we proceed, we rearrange  $\lambda_i$  so that

$$(6) \quad |\lambda_1| \geq |\lambda_2| \geq |\lambda_j|, \quad j = 3, \dots, 8,$$

and if  $\lambda_1 \lambda_2 > 0$ , choose  $\lambda_3$  s.t.  $\lambda_1 \lambda_3 < 0$  and define, for any  $g \geq 0, h \geq 0$ ,

$$(7) \quad K_{ij}(g, h) = \int_I |S_i(\alpha)|^g |S_j(\alpha)|^h d\alpha$$

for  $i = 1, \dots, 4, j = 5, \dots, 8$ . In proving the following lemmas, (6) is assumed.

LEMMA 2.5.

$$\begin{aligned} (i) \quad K_{ij}(0, 4) &\ll (|\lambda_j|^{-1/3}P^{4/5})^{2+\epsilon}. \\ (ii) \quad K_{ij}(0, 8) &\ll (|\lambda_j|^{-1/3}P^{4/5})^{5+\epsilon}. \end{aligned}$$

PROOF. See [9, Lemma 5].

LEMMA 2.6. Let  $N'(m)$  be the number of solutions of  $x^3 + y^3 - x'^3 - y'^3 \equiv 0 \pmod m$ , where  $1 \leq x, y, x', y' \leq m$ . Then  $N'(m) \ll m^{7/2+\epsilon}$ .

PROOF.

$$\begin{aligned} N'(m) &= \int_I \left| \sum_{x=1}^m e(\alpha x^3) \right|^4 \left| \sum_{x=-2m^2}^{2m^2} e(m\alpha x) \right|^2 d\alpha \\ &\ll \left( \int_I \left| \sum_{x=1}^m e(\alpha x^3) \right|^8 d\alpha \int_I \left| \sum_{x=-2m^2}^{2m^2} e(m\alpha x) \right|^2 d\alpha \right)^{1/2} \\ &\ll m^{7/2+\epsilon} \quad \text{by Hua's inequality.} \end{aligned}$$

LEMMA 2.7.

$$\begin{aligned} (i) \quad K_{ij}(2, 4) &\ll |\lambda_i|^{-1/3} |\lambda_j|^{-2/3} P^{13/5+\epsilon}. \\ (ii) \quad K_{ij}(2, 6) &\ll |\lambda_i|^{-1/3} |\lambda_j|^{-7/6} P^{19/5+\epsilon}. \end{aligned}$$

PROOF. It is essentially the same as [6, Lemmas 5 and 6] for  $|\lambda_i| \leq |\lambda_j|$ . Since we may have  $|\lambda_i| \geq |\lambda_j|$ , we need Lemma 2.6 to improve the result. We omit (ii) and

prove (i) only. The integral is equal to the number of solutions

$$(8) \quad \lambda_i(x^3 - x'^3) + \lambda_j(y^3 + z^3 - y'^3 - z'^3) = 0$$

in integers satisfying

$$P \leq |\lambda_i|^{1/3}x, \quad |\lambda_i|^{1/3}x' \leq 2P, \quad P^{4/5} \leq |\lambda_j|^{1/3}(y, z, y', z') \leq 2P^{4/5}.$$

For these solutions with  $x = x'$ , the number of pairs of  $y, z, y', z'$  satisfying (8)  $\ll (|\lambda_j|^{-1/3}P^{4/5})^{2+\epsilon}$ . The number of this kind of solution is therefore

$$\ll |\lambda_i|^{-1/3} |\lambda_j|^{-2/3} P^{13/5+\epsilon}.$$

Now we estimate the number  $N$  of solutions with  $x' > x$ . Let  $x' = x + t$ . (8) becomes

$$(9) \quad \lambda_i(3x^2t + 3xt^2 + t^3) - \lambda_j(y^3 + z^3 - y'^3 - z'^3) = 0,$$

and we observe that  $0 < t \leq |\lambda_i|^{-1/3}P^{2/5}$ . Let  $N(t, y', z')$  denote the number of solutions for prescribed values of  $t, y', z'$ . Then

$$N = \sum_{t, y', z'} N(t, y', z') \leq \left( \sum_{t, y', z'} 1 \right)^{1/2} \left( \sum_{t, y', z'} N^2(t, y', z') \right)^{1/2} = (N_1)^{1/2} (N_2)^{1/2},$$

say. It is obvious that  $N_1 \ll |\lambda_i|^{-1/3} |\lambda_j|^{-2/3} P^2$ . Also  $N_2$  represents the number of solutions of

$$\lambda_i(3x_1^2t + 3x_1t^2 + t^3) - \lambda_j(y_1^3 + z_1^3 - y'^3 - z'^3) = 0,$$

$$\lambda_i(3x_2^2t + 3x_2t^2 + t^3) - \lambda_j(y_2^3 + z_2^3 - y'^3 - z'^3) = 0.$$

The number of solutions of these simultaneous equations with  $x_1 = x_2$  is  $\ll P^\epsilon N$ . With regard to the solution for  $x_1 \neq x_2$ , that would imply

$$3\lambda_i t(x_1 - x_2)(x_1 + x_2 + t) = \lambda_j(y_1^3 + z_1^3 - y_2^3 - z_2^3)$$

for values of  $y_1, z_1, y_2, z_2$  such that the right-hand side is a multiple of  $\lambda_j$ . We can determine the values of  $x_1, x_2, t$  with  $p^\epsilon$  possibilities, and the values of  $y', z'$  are then determined by either of the two equations with  $p^\epsilon$  possibilities. Thus the number of solutions with  $x_1 \neq x_2$  is  $\ll (|\lambda_j|^{-1/3}P^{4/5})^4 m^{-4} N'(m) P^\epsilon$ , where  $m = |\lambda_i/(\lambda_i, \lambda_j)|$  and  $N'(m)$  is defined as in Lemma 2.6. Hence

$$\ll |\lambda_j|^{-4/3} P^{16/5} (|\lambda_i|/|\lambda_j|)^{-1/3+\epsilon} = |\lambda_i|^{-1/3} |\lambda_j|^{-1} P^{16/5+\epsilon}.$$

By a similar argument applied to (9),

$$N \ll |\lambda_i|^{-1/3} |\lambda_j|^{-1} P^{16/5+\epsilon}.$$

It follows that  $N_2 \ll |\lambda_i|^{-1/3} |\lambda_j|^{-1} P^{16/5+\epsilon}$ , giving the desired result

$$N \ll |\lambda_i|^{-1/3} |\lambda_j|^{-2/3} P^{13/5+\epsilon}.$$

$x' < x$  is similar.

**LEMMA 2.8.** Let  $A_i = \{\alpha \in I: S_i(\alpha) \gg |\lambda_i|^{-1/4} P^{3/4+2\delta}\}$ ,  $i = 1, \dots, 4$ . Then:

$$(i) \quad \int_{A_i} |S_i(\alpha)|^6 d\alpha \ll |\lambda_i|^{-1} P^{3+\epsilon}.$$

$$(ii) \quad \int_{A_i} |S_i(\alpha)|^5 d\alpha \ll |\lambda_i|^{-3/4} P^{9/4-2\delta+\epsilon}.$$

PROOF. (i) For every  $\alpha$  in  $A_i$ , we determine  $a_i, q_i$  such that (4) is satisfied, so by Lemma 2.4,  $0 < q_i \leq (|\lambda_i|^{-1/3} P)^{1-\delta}$  and, by Lemma 2.3,

$$|\lambda_i|^{-1/4} P^{3/4+2\delta} \ll |S_i(\alpha)| \ll q_i^{-1/3} |\lambda_i|^{-1/3} P \min(1, P^{-3} |\lambda_i/\beta_i|)$$

which imply

$$(10) \quad 0 < q_i \ll |\lambda_i|^{-1/4} P^{3/4-6\delta}, \quad |\beta_i/\lambda_i| \ll q_i^{-1/3} |\lambda_i|^{-1/12} P^{-11/4-2\delta}.$$

Let  $F_{a_i, q_i}$  denote the interval of values of  $\alpha$  satisfying (4) and (10). Since  $A_i \subset \cup F_{a_i, q_i}$ , where  $q_i$  satisfies (10) and  $|a_i| \leq |\lambda_i q_i|$ ,

$$\int_{A_i} |S_i(\alpha)|^6 d\alpha \ll \sum_{q_i} \sum_{a_i} \int_{F_{a_i, q_i}} |S_i(\alpha)|^6 d\alpha,$$

where the summations  $a_i, q_i$  are as before.

$$\begin{aligned} \int_{F_{a_i, q_i}} |S_i(\alpha)|^6 d\alpha &\ll \int_0^{P^{-3}|\lambda_i|} |\lambda_i|^{-3} q_i^{-2} P^6 d\beta_i + \int_{P^{-3}|\lambda_i|}^{\infty} P^{-12} |\lambda_i|^3 q_i^{-2} \beta_i^{-6} d\beta_i \\ &\ll |\lambda_i|^{-2} q_i^{-2} P^3. \end{aligned}$$

So

$$\int_{A_i} |S_i(\alpha)|^6 d\alpha \ll \sum_{q_i} \sum_{a_i} |\lambda_i|^{-2} q_i^{-2} P^3 \ll |\lambda_i|^{-1} P^{3+\epsilon}.$$

(ii) follows easily from (i).

LEMMA 2.9. Assume that  $P > \Pi^{35/24}$ . Then:

(i) For  $i = 1, \dots, 4, j = 5, \dots, 8$ ,

$$\begin{aligned} \int_{A_i} |S_i(\alpha)|^3 |S_j(\alpha)|^4 d\alpha &\ll |\lambda_i|^{-9/16} |\lambda_j|^{-7/9} P^{67/20+2\delta+\epsilon} \quad \text{if } i = 1, \\ &\ll |\lambda_i|^{-2/3} |\lambda_j|^{-7/9} P^{67/20+2\delta+\epsilon} \quad \text{if } i \neq 1. \end{aligned}$$

(ii)

$$K_{ij}(3, 4) \ll |\lambda_i|^{-7/12} |\lambda_j|^{-2/3} P^{67/20+2\delta+\epsilon}, \quad i = 2, 3, 4, j = 5, \dots, 8.$$

PROOF. (i)

$$\begin{aligned} \int_{A_i} |S_i(\alpha)|^3 |S_j(\alpha)|^4 d\alpha &\ll K_{ij}(2, 6)^{2/3} \left( \int_{A_i} |S_i(\alpha)|^5 d\alpha \right)^{1/3} \\ &\ll |\lambda_i|^{-17/36} |\lambda_j|^{-7/9} P^{197/60+\epsilon}. \end{aligned}$$

Using  $P > \Pi^{35/24}$ , which implies  $P > |\lambda_1|^{35/24}$  and  $P > |\lambda_i|^{35/12}$  if  $i \neq 1$ , (i) follows.

(ii)

$$\begin{aligned} K_{ij}(3, 4) &\ll \int_{A_i} |S_i(\alpha)|^3 |S_j(\alpha)|^4 d\alpha + \int_{I \setminus A_i} |S_i(\alpha)|^3 |S_j(\alpha)|^4 d\alpha \\ &\ll |\lambda_i|^{-7/12} |\lambda_j|^{-2/3} P^{67/20+2\delta+\epsilon} + |\lambda_i|^{-1/4} P^{3/4+2\delta} K_{ij}(2, 4). \end{aligned}$$

By (i) and Lemma 2.7(i), (ii) follows.

**3. Minor arcs for the proof of Theorem 1.** We let  $\mathcal{N}(P)$  be the number of integral solutions of (2) which lie in the ‘box’ determined by (3). We shall show that for a given positive number  $\theta$ , there exists a positive constant  $D_\theta$ , independent of  $\lambda_i$ , such that if  $P^{1-\theta} > D_\theta \Pi^{35/24}$ , then  $\mathcal{N}(P) > 0$ . So we may take  $P$  such that  $D_\theta \Pi^{35/24} < P^{1-\theta} < 2D_\theta \Pi^{35/24}$ , which will imply there exists a solution of (2) in nonzero integers with

$$\sum_i |\lambda_i x_i^3| < 8(2D_\theta \Pi^{35/24})^{3/(1-\theta)} < K_\theta \Pi^{35/8(1-\theta)},$$

where  $K_\theta$  depends on  $\theta$  only, so for every  $\phi > 0$ , we can find  $0 < \theta < 1$  such that  $35/8(1 - \theta) < 35/8 + \phi$ , and hence Theorem 1 follows.

We now use

$$\mathcal{N}(P) = \int_I V(\alpha) d\alpha, \quad \text{where } V(\alpha) = \prod_{i=1}^8 S_i(\alpha)$$

and estimate  $\mathcal{N}(P)$  will be of the form  $c\Pi^{-1/3-\varepsilon}P^{21/5}$  where  $c > 0$ , and the error terms are substantially smaller than  $\Pi^{-1/3}P^{21/5}$  provided  $P$  is substantially larger than  $\Pi^{35/24}$ .

By Dirichlet’s theorem on diophantine approximation, for each  $\alpha$  in  $I$  and  $i$ , there exists a rational approximation  $a_i/q_i$  to  $\lambda_i\alpha$  such that (4) holds. Let  $S = \cup_{i=1}^8 B_i$  where

$$B_i = \{ \alpha \in I : S_i(\alpha) \ll |\lambda_i|^{-1/4} P^{3/4+2\delta} \}, \quad 1 \leq i \leq 4,$$

$$B_j = \{ \alpha \in I : S_j(\alpha) \ll |\lambda_j|^{-1/4} P^{3/5+2\delta} \}, \quad 5 \leq j \leq 8.$$

LEMMA 3.1.  $\int_S |V(\alpha)| d\alpha \ll \Pi^{-3/16} P^{41/10+4\delta+\varepsilon}$ .

PROOF. We may assume that the constants implied in defining  $A_i$  and  $B_i$  are the same for  $i, \dots, 4$ . Then  $A_i = I \setminus B_i, i = 1, \dots, 4$ .

Let  $A = A_1 \cap A_2 \cap A_3 \cap A_4$  and

$$S = (B_1 \cap B_2) \cup (B_1 \cap A_2) \cup (B_2 \cap A_1) \cup (B_3 \cap A_1 \cap A_2) \\ \cup (B_4 \cap A_1 \cap A_2) \cup \bigcup_{j=5}^8 (B_j \cap A).$$

(i)

$$\int_{B_1 \cap B_2} |V(\alpha)| d\alpha \ll |\lambda_1|^{-1/4} |\lambda_2|^{-1/4} P^{3/2+4\delta} \left( \prod_{i=3}^4 \prod_{j=5}^8 K_{ij}(2, 4) \right)^{1/8} \\ \ll \Pi^{-3/16} P^{41/10+4\delta+\varepsilon}.$$

By Lemma 2.7 and notice that  $|\lambda_1|, |\lambda_2| \geq |\lambda_j|$  for  $j = 3, 4, \dots, 8$ .

(ii)

$$\int_{B_1 \cap A_2} |V(\alpha)| d\alpha \\ \ll |\lambda_1|^{-1/4} P^{3/4+2\delta} \left( \prod_{j=5}^8 \int_{A_2} |S_2(\alpha)|^3 |S_j(\alpha)|^4 d\alpha \right)^{1/12} \left( \prod_{i=3}^4 \prod_{j=5}^8 K_{ij}(3, 4) \right)^{1/12}.$$

By Lemma 2.9,

$$\begin{aligned} &\ll |\lambda_1|^{-1/4} |\lambda_2|^{-2/9} |\lambda_3|^{-7/36} |\lambda_4|^{-7/36} |\lambda_5 \dots \lambda_8|^{-19/108} P^{41/10+4\delta+\epsilon} \\ &\ll \Pi^{-3/16} P^{41/10+4\delta+\epsilon}. \end{aligned}$$

Similarly for  $B_2 \cap A_1$ .

(iii)

$$\begin{aligned} &\int_{B_3 \cap (A_1 \cap A_2)} |V(\alpha)| d\alpha \\ &\ll |\lambda_3|^{-1/4} P^{3/4+2\delta} \left( \prod_{\substack{k=1 \\ k \neq 3}}^4 \prod_{\substack{j=5 \\ j \neq 3}}^8 \int_{A_1 \cap A_2} |S_k(\alpha)|^3 |S_j(\alpha)|^4 d\alpha \right)^{1/12}. \end{aligned}$$

By Lemma 2.9,

$$\begin{aligned} &\ll |\lambda_1|^{-3/16} |\lambda_2|^{-2/9} |\lambda_3|^{-1/4} |\lambda_4|^{-7/36} |\lambda_5 \dots \lambda_8|^{-5/27} P^{41/10+4\delta+\epsilon} \\ &\ll \Pi^{-3/16} P^{41/10+4\delta+\epsilon}. \end{aligned}$$

Similarly for  $B_4 \cap A_1 \cap A_2$ .

(iv) For  $j = 5, \dots, 8$ ,

$$\begin{aligned} &\int_{B_j \cap A} |V(\alpha)| d\alpha \ll |\lambda_j|^{-1/4} P^{3/5+2\delta} \left( \prod_{k=1}^4 \prod_{\substack{m=5 \\ m \neq j}}^8 \int_{A_k} |S_k(\alpha)|^4 |S_m(\alpha)|^3 d\alpha \right)^{1/12} \\ &\ll |\lambda_j|^{-1/4} P^{3/5+2\delta} \left( \prod_{k=1}^4 \prod_{\substack{m=5 \\ m \neq j}}^8 K_{k,m}(2, 6) \int_{A_k} |S_k(\alpha)|^6 d\alpha \right)^{1/24}. \end{aligned}$$

By Lemmas 2.7, 2.8 and using the fact that  $P > \Pi$ ,

$$\ll \Pi^{-1/6} P^{4+2\delta} \ll \Pi^{-3/16} P^{41/10+4\delta+\epsilon}.$$

Summing up (i) to (iv) the desired result follows.

**4. Major arcs for the proof of Theorem 1.** We define

$$\begin{aligned} M = \{ \alpha \in S : S_i(\alpha) \gg |\lambda_i|^{-1/4} P^{3/4+2\delta}, i = 1, \dots, 4, \\ \text{and } S_j(\alpha) \gg |\lambda_j|^{-1/4} P^{3/5+2\delta}, j = 5, \dots, 8 \}. \end{aligned}$$

Since we have assumed the constants implied are greater than or equal to 1, by Lemma 2.4, the rational approximation  $a_i/q_i$  which satisfies (4) must satisfy

$$q_i < (|\lambda_i|^{-1/3} P)^{1-\delta}, \quad i = 1, \dots, 4; \quad q_j < (|\lambda_j|^{-1/3} P^{4/5})^{1-\delta}, \quad j = 5, \dots, 8.$$

By Lemma 2.3, this will imply that

(11)

$$\begin{aligned} 0 < q_i < (|\lambda_i|^{-1/3} P)^{3/4}, \quad |\beta_i/\lambda_i| < |\lambda_i|^{-1/12} P^{-11/4-2\delta}, \quad i = 1, \dots, 4, \\ 0 < q_j < (|\lambda_j|^{-1/3} P^{4/5})^{3/4}, \quad |\beta_j/\lambda_j| < |\lambda_j|^{-1/12} P^{-11/5-2\delta}, \quad j = 5, \dots, 8. \end{aligned}$$

We consider several lemmas before we proceed.

LEMMA 4.1. (i) If  $P > 4|\lambda_i|^{1/2}$ , then there is at most one approximation  $a_i/q_i$  to  $\lambda_i\alpha$  such that (4) and (11) hold.

(ii) If  $P > 4|\lambda_j\lambda_k|$  and (4) and (11) hold for  $i = j, k$  ( $j \neq k$ ), then

$$a_j/\lambda_jq_j = a_k/\lambda_kq_k.$$

(iii) If (4) and (11) hold, then

$$|\beta_i| < q_i^{-1}(|\lambda_i|^{-1/3}P)^{-2-\delta} \quad i = 1, \dots, 4,$$

$$|\beta_j| < q_j^{-1}(|\lambda_i|^{-1/3}P^{4/5})^{-2-\delta}, \quad j = 5, \dots, 8.$$

PROOF. The proofs are essentially the same as [9, Lemma 8]. We prove (iii) only.

(iii)

$$\begin{aligned} |\beta_i|q_i &\leq |\lambda_i||\lambda_i|^{-1/12}P^{-11/4-2\delta}(|\lambda_i|^{-1/4}P^{3/4}) \quad (i = 1, \dots, 4) \\ &< (|\lambda_i|^{-1/3}P)^{-2-\delta}. \end{aligned}$$

Similarly for  $j = 5, \dots, 8$ .

Suppose that  $\alpha \in M$  and for each  $i$  let  $a_i/q_i$  be an approximation which satisfies (4) and (11). Since we have assumed  $P > \Pi^{35/24}$ , by Lemma 4.1(i),  $a_i/q_i$  are unique for  $\lambda_i\alpha$  and, by (ii),  $a_j/\lambda_jq_j = a_k/\lambda_kq_k$ , for all  $j, k$ . Hence there exist unique integers  $a, q$  such that  $(a, q) = 1, q > 0$  and

$$(12) \quad a_i/q_i = \lambda_i a/q$$

for all  $i$ . Let

$$(13) \quad \delta_i = (\lambda_i, q), \quad i = 1, \dots, 8.$$

Then

$$(14) \quad a_i = \lambda_i a/\delta_i, \quad q_i = q/\delta_i.$$

By (11), we have

$$(15) \quad 0 < q \leq \min_{\substack{1 \leq i \leq 4 \\ 5 \leq j \leq 8}} \{ \delta_i |\lambda_i|^{-1/4}P^{3/4}, \delta_j |\lambda_j|^{-1/4}P^{3/5} \}$$

and

$$(16) \quad |\alpha - a/q| \leq |\lambda_1|^{-1/12}P^{-11/4-2\delta}.$$

Since  $P > \Pi^{35/24}$  and (6) is satisfied, this implies  $P > |\lambda_1|^{35/24}$  and  $P > |\lambda_j|^{35/8}$  for  $j = 3, \dots, 8$ .

$$\begin{aligned} (17) \quad P^{19/35} &\leq \min \{ |P^{24/35}|^{-1/4}P^{3/4}, (P^{8/35})^{-1/4}P^{3/5} \} \\ &\leq \min_{\substack{1 \leq i \leq 4 \\ 5 \leq j \leq 8}} \{ \delta_i |\lambda_i|^{-1/4}P^{3/4}, \delta_j |\lambda_j|^{-1/4}P^{3/5} \} \\ &\leq (P^{8/35})^{3/4}P^{3/5} \leq P^{4/5}. \end{aligned}$$

We define  $I_{a,q}$  to be the interval determined by (16). By previous argument, if  $\alpha \in M$ , then every rational approximation  $a_i/q_i$  satisfying (4) must satisfy (11), which implies  $a \in I_{a,q}$  for some  $a, q$  such that  $0 \leq a \leq q - 1$  and  $q$  satisfies (15).

Conversely, suppose that  $a, q$  are integers such that  $(a, q) = 1, q > 0, 0 \leq a \leq q - 1$  and (15) holds, where the  $\delta_i$  are defined by (13). If  $\alpha$  belongs to the interval  $I_{a,q}$  defined as above, then  $\alpha \in I$  and the approximation  $a_i/q_i$ , such that  $(a_i, q_i) = 1$  and  $q_i > 1$ , defined by (12), satisfy (11) and (14), and these intervals do not overlap because of the uniqueness of the  $a_i/q_i$ .

By Lemma 4.1(iii), we can always use Lemma 2.3 to estimate  $S_i(\alpha)$  for  $\alpha$  in  $M$ . We can now estimate the contribution from  $M$  to  $\mathfrak{U}(P)$ .

LEMMA 4.2. *If  $\delta \leq 1/30$ ,*

$$(18) \quad \int_M V(\alpha) d\alpha = \Pi^{-1/3} \mathfrak{S} R(P) + O(P^{39/10})$$

$$\text{where } \mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} (q_1 \cdots q_8)^{-1} S(a_1, q_1), \dots, S(a_8, q_8),$$

the  $a_i, q_i$  are defined by (13) and (14), and  $R(P) \gg P^{21/5}$ .

PROOF. Suppose  $\alpha \in M$  and define  $a_i/q_i$  by (13) and (14) and write

$$\beta = \alpha - a/q = \beta_i/\lambda_i, \quad i = 1, \dots, 8.$$

Then by Lemma 2.3,

$$V(\alpha) = \Pi^{-1/3} \left( \prod_{i=1}^4 q_i^{-1} S(a_i, q_i) I(\pm\beta) \right) \left( \prod_{j=5}^8 q_j^{-1} S(a_j, q_j) I'(\pm\beta) \right) + E,$$

where  $\pm$  is the sign of  $\lambda_i$  and the error  $E$  satisfies

$$E \ll \sum_{i=1}^8 q_i^{2/3+\epsilon} \left( \prod_{\substack{j=1 \\ j \neq i}}^8 |\lambda_j|^{-1/3} q_j^{-1/3} \right) P^{32/5} (\min(1, P^{-3} |\beta|^{-1}))^2$$

$$\ll q^{-5/3+\epsilon} (\min(1, P^{-3} |\beta|^{-1}))^2.$$

Since  $|\lambda_i|^{-1} q \leq q_i \leq q$  for all  $i$  by (13) and (14),

$$\int_M E d\alpha \ll \sum'_q \sum'_a \int_{I_{a,q}} q^{-5/3+\epsilon} P^{32/5} \min(1, P^{-6} |\beta|^{-2}) d\alpha$$

$$\ll \sum'_q \sum'_a q^{-5/3+\epsilon} P^{17/5} \ll P^{19/5} \quad \text{by (17),}$$

where  $\sum'_q \sum'_a$  denotes the range of  $q$  defined in (15),  $(a, q) = 1$  and  $0 \leq a < q$ . Since the  $I_{a,q}$  do not overlap, it follows that

$$(19) \quad \int_M V(\alpha) d\alpha = \Pi^{-1/3} \sum'_q \sum'_a (q_1 \cdots q_8)^{-1} S(a_1, q_8), \dots, S(a_8, q_8)$$

$$\times \int_{I_{a,q}} \prod_{i=1}^4 I(\pm\beta) \prod_{j=5}^8 I'(\beta) d\alpha + O(P^{19/5}).$$

The error caused by replacing  $I_{a,q}$  in (19) by the interval  $|\alpha - a/q| \leq \frac{1}{2}$  is

$$\ll \Pi^{-1/3} \sum_q' \sum_a' (q_1 \cdots q_8)^{-1/3} \int_{I_{a,q}^*} P^{36/5} \min(1, P^{-3} |\beta|^{-1})^4 d\alpha,$$

where  $I_{a,q}^*$  is the set  $\alpha = a/q + \beta$  such that

$$\frac{1}{2} \geq |\beta| > |\lambda_1|^{-1/12} P^{-11/4-2\delta} \geq P^{-17/6-2\delta} \geq P^{-3},$$

since  $P > |\lambda_1|$  and  $2\delta \leq \frac{1}{6}$ .

Now, for any pair  $a, q$ ,

$$\Pi^{-1/3} (q_1 \cdots q_8)^{-1/3} P^{-24/5} \int_{I_{a,q}^*} \beta^{-4} d\alpha \ll q^{-8/3} P^{37/10+6\delta}.$$

Hence the total error is certainly

$$\ll P^{37/10+6\delta} \sum_{q=1}^{\infty} \sum_{a=0}^{q-1} q^{-8/3} \ll P^{39/10}$$

provided that  $6\delta \leq \frac{1}{5}$ . Thus we have

$$(20) \quad \int_M V(\alpha) d\alpha = \Pi^{-1/3} \sum_q' \sum_a' (q_1 \cdots q_8)^{-1} S(a_1, q_1) \cdots S(a_8, q_8) R(P) + O(P^{19/5}),$$

where

$$(21) \quad R(P) = \int_{-1/2}^{1/2} \prod_{i=1}^4 I(\beta) \prod_{j=5}^8 I'(\beta) d\beta = 3^{-8} \sum_{m_1, \dots, m_8} (m_1 \cdots m_8)^{-2/3}$$

by definition of  $I(\beta)$  and  $I'(\beta)$ , where

$$(22) \quad \begin{aligned} P^3 \leq m_i \leq 8P^3, & \quad i = 1, \dots, 4, \\ P^{12/5} \leq m_j \leq 8P^{12/5}, & \quad j = 5, \dots, 8, \end{aligned}$$

and  $\pm m_1 \pm m_2 \pm \dots \pm m_8 = 0$ .

Since either  $\lambda_2$  or  $\lambda_3$  has sign different then  $\lambda_1$ , without loss of generality, we assume  $\lambda_2 \lambda_1 < 0$ . So now the last condition is

$$(23) \quad m_1 = m_2 \mp m_3 \mp m_4 \mp \dots \mp m_8$$

for each  $(m_2, \dots, m_8)$  such that

$$4P^3 \leq m_2 \leq 5P^3, \quad P^3 \leq m_i \leq 5P^3/4, \quad i = 3, 4,$$

and the integer  $m_1$  defined by (23) satisfies  $P^3 \leq m_1 \leq 8P^3$ . Thus the number of solutions of (23) such that (22) holds is  $\gg P^9 P^{48/5}$ . For any such solution we have

$$(m_1 \cdots m_8)^{-2/3} \geq 2^{-16} P^{-8} P^{-32/5}.$$

Thus  $R(P) \gg P^{21/5}$  as required.

The error caused by extending the range of summation for  $q$  to infinity is

$$\begin{aligned} &\ll \Pi^{-1/3} \sum_q \sum_a (q_1 \cdots q_8)^{-1/3} P^{36/5} \int_{-1/2}^{1/2} \min(1, P^{-6} |\beta|^{-2}) d\beta \\ &\ll \Pi^{-1/3} \sum_q \sum_a (q_1 \cdots q_8)^{-1/3} P^{21/5}, \end{aligned}$$

where  $q$  is summed over the range  $q > P^{19/35}$ , and  $a$  is as before. We consider a particular set of divisors  $\delta_1, \dots, \delta_8$  of  $\lambda_1, \dots, \lambda_8$ , respectively. The contribution to the above from those pairs  $a, q$  corresponding to  $\delta_1, \dots, \delta_8$  is

$$\ll \Pi^{-1/3} (\delta_1 \cdots \delta_8)^{1/3} P^{21/5} \sum_{q > P^{19/35}} q^{-5/3}.$$

Thus the contribution from these  $a, q$  is

$$\ll \Pi^{-1/3} P^{21/5} (\delta_1 \cdots \delta_8)^{1/3} P^{-38/105} \ll \Pi^{-1/3} P^{21/5} \Pi^{1/3} P^{-38/105} \ll P^{39/10-\epsilon}.$$

Since  $\Pi \ll P$ , the number of different possibilities for  $\delta_1 \cdots \delta_8$  is  $O(P^\epsilon)$ , and it follows that the total error is  $\ll P^{39/10}$ . Hence the result followed.

**5. The singular series.** Firstly, we assume that the  $\lambda_i$  are cube-free and no prime divides more than five of them. The following lemma shows that the assumption may be made without loss of generality.

LEMMA 5.1. *If Theorem 1 holds provided the  $\lambda_i$  are cube-free nonzero integers such that no prime divides more than five of them, then it holds for every set of nonzero integral  $\lambda_i$ .*

PROOF. The proof is essentially the same as [9, Lemma 5].

We now investigate the singular series  $\mathfrak{S}$  defined by (18). Since  $S(\lambda_i a, q) = \delta_i S(a_i, q_i)$  and  $S(\lambda_i a, q)$  is periodic of period  $q$ , we may rewrite the series as

$$\sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a, q)=1}}^q q^{-8} S(\lambda_1 a, q) \cdots S(\lambda_8 a, q).$$

For  $p$  prime, we define

$$(24) \quad \chi(p) = 1 + \sum_{\nu=1}^{\infty} \sum_{a=1}^{p^\nu} P^{-8\nu} \prod_{i=1}^8 S(\lambda_i a, P^\nu)$$

and  $M(P^\nu)$  to be the number of solutions (mod  $p^\nu$ ) of the congruence

$$(25) \quad \sum_{i=1}^8 \lambda_i x_i^3 \equiv 0 \pmod{P^\nu}.$$

Similar to [9, §6], we have

$$(26) \quad \chi(p) = \lim_{\nu \rightarrow \infty} P^{-7\nu} M(P^\nu) \quad \text{and} \quad \mathfrak{S} = \prod_p \chi(p),$$

where the product is over all primes  $p$ .

LEMMA 5.2. (i) *If  $s \geq 3$ ,  $p$  is any prime, and  $p \nmid |\lambda_1 \cdots \lambda_s|$ , then there is a nontrivial solution (i.e. one with some  $x_i \not\equiv 0$ ) of*

$$(27) \quad \sum_{i=1}^s \lambda_i x_i^3 \equiv 0 \pmod{p}.$$

(ii) *Suppose that  $p \neq 3$  is a prime such that  $\lambda_1, \dots, \lambda_s$  are not divisible by  $p$  and  $\lambda_{s+1}, \dots, \lambda_8$  are divisible by  $p$ , and let  $\mathcal{N}(p)$  denote the number of nontrivial solutions of (27). Then for all  $v > 0$ ,*

$$M(p^v) \geq p^{7(v-1)+(8-s)\mathcal{N}(p)}.$$

(iii) *If  $\lambda_1, \dots, \lambda_8$  are cube-free and  $v \geq 6$ , then  $M(3^v) \geq 3^{7(v-6)}$ .*

PROOF. The proof is essentially the same as [9, Lemma 10].

LEMMA 5.3. (i) *There is an absolute constant  $C_1 > 0$  such that*

$$\prod_p \chi(p) \geq \frac{1}{2} \quad (p \nmid \Pi, p > C_1).$$

(ii) *There is a constant  $C_2 = C_2(\epsilon) > 0$  such that*

$$\prod_p \chi(p) \geq \Pi^{-\epsilon} \quad (p \nmid \Pi, p > C_2).$$

(iii) *We have  $\mathfrak{C} \gg \Pi^{-\epsilon}$ .*

(The products in (i) and (ii) are over all primes  $p$  which satisfy the condition in parentheses.)

PROOF. See [9, Lemma 11].

**6. Completion of the proof of Theorem 1.** Assuming that  $P > \Pi^{35/24}$  and  $\delta \leq \frac{1}{30}$ , by Lemmas 4.2 (with  $\epsilon = \delta$ ), 5.2 and 3.1 (since we have  $I \setminus M \subset S$ ), we have

$$\mathcal{N}(P) = C_\epsilon \Pi^{-(1/3)-\epsilon} P^{21/5} + E,$$

where  $C_\epsilon > 0$  and

$$\begin{aligned} E &\ll \Pi^{-3/16} P^{41/10+5\delta} + P^{39/10} \ll \Pi^{-3/16} P^{41/10+5\delta} \\ &\ll \Pi^{-1/3} P^{21/5} (\Pi^{-35/24} P^{1-\theta})^{-1/10} P^{5\delta-\delta/10}. \end{aligned}$$

For given  $\theta > 0$ , we choose  $\epsilon > 0$  and  $\delta > 0$  such that  $\epsilon + 5\delta \leq \theta/20$ ,  $\delta < 1/30$ . Then  $P^{5\delta-\theta/10} \leq P^{-\epsilon-\theta/20}$ .

Since  $P > \Pi^{35/24}$  certainly holds if  $P^{1-\theta} > \Pi^{35/24}$ , it now follows from the preceding paragraph that there is a constant  $D_\theta$  such that  $\mathcal{N}(P) > 0$  provided  $P^{1-\theta} > D_\theta \Pi^{35/24}$ . By the remarks at the beginning of §3 this completes the proof of Theorem 1.

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