TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 278, Number 1, July 1983

THE STRUCTURE OF RINGS WITH FAITHFUL NONSINGULAR MODULES

BY

J. M. ZELMANOWITZ

ABSTRACT. It is shown that the existence of a faithful nonsingular uniform module characterizes rings which have a full linear maximal quotient ring. New information about the structure of these rings is obtained and their maximal quotient rings are constructed in an explicit manner. More generally, rings whose maximal quotient rings are finite direct sums of full linear rings are characterized by the existence of a faithful nonsingular finite dimensional module.

Introduction. The study of prime rings with nonsingular uniform one-sided ideals was initiated in [**5**]. With the treatments in [**8** and **1**], but especially in the latter article, the structure of this class of rings became well understood. In this paper, we will begin by examining rings which possess faithful nonsingular uniform modules but which are not necessarily prime. Our principal discovery is that this is precisely the class of rings whose maximal quotient rings are full linear rings, a class of rings which has been extensively studied from other perspectives (cf. [**6**, **9**]). It is a bit surprising that despite the fact that nonsingular uniform modules play a significant role in module theory (cf. [**3**]), the consequences for the structure of a ring of the existence of a faithful such module appear not to have been suspected.

In [**13**] the structure of the rings with faithful monoform modules is determined. Since a nonsingular uniform module is monoform, the rings described above form a proper subclass of the rings with faithful monoform modules and hence are (in the terminology of [**13**]) "dense" rings of linear transformations. As we will soon see, the structure theory of this subclass is actually much richer.

In the final section of this paper we study rings whose maximal quotient rings are finite direct sums or arbitrary direct products of full linear rings. One sample result (Corollary 5.3) is that a ring has a maximal quotient ring which is a finite direct sum of full linear rings if and only if it possesses a faithful finite dimensional nonsingular module.

An innovation is the introduction in §2 of "partial" contexts which extend the concept of Morita contexts. We have found this idea to be a rather helpful crutch in the organization of the main results.

1. Preliminaries. It is important to stress that throughout this article *rings need not possess identity elements*. We will also be careful to write homomorphisms consistently on the side of a module opposite to that of the scalars.

Received by the editors June 7, 1982.

1980 *Mathematics Subject Classification*. Primary 16A42, 16A48, 16A64, 16A53, 16A65, 16A08.

While we will be dealing with maximal quotient rings, our treatment will be self-contained in the sense that any quotient rings that occur will actually be constructed. The reader unfamiliar with this topic may nonetheless wish to consult [**2**] as a reference.

Before we begin in earnest it may be worthwhile to review the information known prior to this article and relevant to the topic under discussion. The definitions of the terms used can be found in the references cited.

THEOREM A [**1**, **11**, **12**]. *For a ring R the following conditions are equivalent.*

(i) *R has a faithful critically compressible left ideal.*

(ii) *R is a prime ring with a nonsingular uniform left ideal.*

(iii) *There exists a Morita context* $({}_RM_S, {}_SN_R)$ *with S a left Ore domain,* ${}_SN$ *torsion-free,* ${}_RM$ *faithful, and such that* $(m, n) = 0$ *or* $(M, N)m = 0$ *for* $m \in M$ *and* $0 \neq n \in N$ *implies that* $m = 0$.

(iv) *R is isomorphic to a subring of* $\operatorname{End}_\Delta V$ *for some vector space* ${}_\Delta V$ *with the property that given* $\tau \in \operatorname{End}_\Delta V$ *and U a finite dimensional subspace of* ${}_\Delta V$ *there exist* $r, s \in R$ *with* $r\tau = s$, $Vr \subseteq U$ *and* $Ur = U$.

When R satisfies these conditions its maximal left quotient ring is a full linear ring.

THEOREM B [**6**, **9**]. *R has a maximal left quotient ring which is a full linear ring if and only if R is an irreducible nonsingular ring such that every nonzero left ideal of R contains a uniform left ideal.*

THEOREM C [**13**]. *A ring R has a faithful monoform module if and only if it is* (*isomorphic to*) *an "dense" ring of linear transformations on a right vector space* W_Δ (*i.e., given* $\tau \in \operatorname{End} W_\Delta$ *and U a finite dimensional subspace of* W_Δ *with* $\tau|_U \neq 0$ *there exist* $r, s \in R$ *with* $r\tau|_U = s|_U \neq 0$).

By a *partial homomorphism* f from a module ${}_RM$ to a module ${}_RN$ we mean an element of $\operatorname{Hom}_R(M_f, N)$ with M_f an R-submodule of M. One may form the direct limit $\varinjlim \operatorname{Hom}_R(M_0, N)$ over all essential submodules ${}_RM_0$ of ${}_RM$; we will denote this direct limit by $\operatorname{PHom}_R(M, N)$. Letting $[f]$ denote the class of $f \in \operatorname{Hom}_R(M_f, N)$ in $\operatorname{PHom}_R(M, N)$, we have $[f] = [g]$ if and only if $f = g$ on an essential submodule of $M_f \cap M_g$; addition of classes is defined by $[f] + [g] = [h]$ where $h = (f + g)|_{M_f \cap M_g}$. In practice we will not distinguish between an element f in $\operatorname{Hom}_R(M_f, N)$ and its class $[f]$ in $\operatorname{PHom}_R(M, N)$, and we will speak of an element f of $\operatorname{PHom}_R(M, N)$.

It is easy to check that $\operatorname{PHom}_R(M, M)$ becomes a ring when multiplication of $f \in \operatorname{Hom}_R(M_f, M)$ and $g \in \operatorname{Hom}_R(M_g, M)$ is defined as the restriction h of $f \circ g$ to $M_g f^{-1}$; that is, $[f][g] = [h]$. We refer to elements of $\operatorname{PHom}_R(M, M)$ as *partial endomorphisms* of ${}_RM$, and we set $\operatorname{PEnd}_R M = \operatorname{PHom}_R(M, M)$. ($\operatorname{PEnd}_R M$ is also called the *extended centralizer* of M; see [4].) If N is an R-S-bimodule then $\operatorname{PHom}_R(M, N)$ becomes a $\operatorname{PEnd}_R M$-S-bimodule under the action induced by defining $m(h \cdot f \cdot s) = ((mh)f)s$ for $s \in S$, $f \in \operatorname{Hom}_R(M_f, N)$, $h \in \operatorname{Hom}_R(M_h, M)$, and $m \in M_f h^{-1}$. The particular instance of this situation that we will need is the fact that $\operatorname{PHom}_R(M, R)$ is a $\operatorname{PEnd}_R M$-R-bimodule. We call elements of $\operatorname{PHom}_R(M, R)$ *partial linear functionals* of ${}_RM$.

A nonzero R-module M is *monoform* if all of its nonzero partial endomorphisms are monomorphisms. It is well known and quite easy to check that a monoform module is uniform (i.e., any two nonzero submodules have nonzero intersection), that a nonsingular uniform module is monoform, and that a module ${}_RM$ is uniform if and only if PEnd ${}_RM$ is a division ring [4]. Furthermore, when ${}_RM$ is faithful and monoform then nonzero partial linear functionals are also monomorphisms. To see this suppose that $0 \neq f \in \mathrm{PHom}_R(M, R)$ with domain $f = M_f$. Then $(M_f f)M \neq 0$ because ${}_RM$ is faithful, so there exists $m \in M$ with $(M_f f)m \neq 0$. Define $h \in \mathrm{Hom}_R(M_f, M)$ by $xh = (xf)m$ for $x \in M_f$. Then $\ker h \supseteq \ker f$ and so $0 = \ker h = \ker f$ because ${}_RM$ is monoform.

An overring Q of a ring R is called a *left quotient ring* of R if given any $x \in Q$ and $0 \neq y \in Q$ there exists $r \in R$ with $rx \in R$ and $ry \neq 0$. If Q is an overring of R maximal with respect to this property then Q is called a *maximal left quotient ring* of R. For rings with zero right annihilator a maximal left quotient ring always exists, and it is unique up to isomorphism over R [2, p. 68].

2. Nonsingular uniform modules. It will be convenient for our purposes to extend the concept of a Morita context as follows. A pair of (bi-)modules $({}_RM, {}_\Delta V_R)$ is called a *partial context* if there are functions $(\ ,\ )\colon X \to R$ and $[\ ,\ ]\colon V \times M \to \Delta$ with $X \subseteq M \times V$ such that whenever $m \in M$, $v, v' \in V$, and $(m, v) \in X$ then $v'(m, v) = [v', m]v$. We will then let (M, V) denote the ideal generated by the image of X in R, and it will be our custom to say that (m, v) *is defined* whenever the pair (m, v) is in X. (No confusion should arise from the fact that (m, v) denotes an element of both X and R.)

Given any module ${}_RM$ over a nonsingular ring we may always construct a *standard partial context* $({}_RM, {}_\Delta V_R)$ as follows. Take $V = \mathrm{PHom}_R(M, R)$, $\Delta = \mathrm{PEnd}_R M$, and $X = \{(m, f) \mid f \in \mathrm{PHom}_R(M, R),\ m \in \text{domain} f\}$; for (m, f) in X define $(m, f) = mf \in R$, and for $f \in \mathrm{PHom}_R(M, R)$ and $m \in M$ define $[f, m] \in \mathrm{PEnd}_R M$ by $x[f, m] = \{(x, f)m$ for any $x \in M_f = \text{domain} f\}$. It is evident that the functions $(\ ,\ )$ and $[\ ,\ ]$ are bilinear whenever possible for the standard partial context; but we will not need to require that this property be part of the definition of an arbitrary partial context.

Following [6] we will call a ring R *irreducible* if $A \cap r(A) = 0$ for A a nonzero ideal of R implies that $r(A) = 0$; here $r(A)$ denotes the right annihilator of A. We will often use without explicit mention the well-known fact that a nonzero homomorphism from a uniform module to a nonsingular (i.e., having zero singular submodule) module is a monomorphism; and we let $Z(M)$ or $Z_R(M)$ denote the singular submodule of ${}_RM$.

THEOREM 2.1. *For a ring R the following conditions are equivalent.*

(1) *There exists a faithful nonsingular uniform left R-module.*

(2) *R is a nonsingular irreducible ring such that each nonzero left ideal contains a uniform left ideal.*

(3) *There exists a partial context $({}_RM, {}_\Delta V_R)$ satisfying*: (i) *V_R is faithful*; (ii) *${}_\Delta V$ is a vector space*; *and* (iii) *given u, $0 \neq v \in V$ there exists $m \in M$ with (m, u) and (m, v) defined and $(m, v) \neq 0$.*

(4) *R is (isomorphic to) a subring of* End ${}_\Delta V$ *for some vector space* ${}_\Delta V$ *with the property that given* $\tau \in \operatorname{End}{}_\Delta V$ *and* ${}_\Delta U$ *a one-dimensional subspace of* ${}_\Delta V$ *there exist* $r, s \in R$ *with* $r\tau = s$ *and* $Vr = U$.

(5) *R has a (maximal) left quotient ring which is a full linear ring.*

(6) *R is (isomorphic to) a subring of* End ${}_\Delta V$ *for some vector space* ${}_\Delta V$ *with the property that given* $0 \neq \tau \in \operatorname{End}{}_\Delta V$ *there exist* $r, s \in R$ *with* $r\tau = s \neq 0$.

The equivalence of (2) and (5) is just Theorem B cited in the previous section; (2) is only included here for the sake of completeness. Conditions (4), (5) and (6) represent successively weaker ways of describing this class of rings. One would anticipate that the partial contexts introduced as an aid in proving this theorem will also be useful in other situations.

PROOF. (1) *implies* (2). Let M be a faithful nonsingular uniform left R-module. Since $Z(R)M \subseteq Z(M) = 0$, R is nonsingular.

Let I be any nonzero left ideal of R. Then $IM \neq 0$, so $Im \neq 0$ for some $0 \neq m \in M$. Since ${}_RM$ is nonsingular, $l_I(m) = \{r \in I \mid rm = 0\}$ is not essential in I, so there exists a left ideal $0 \neq J \subseteq I$ with $J \cap l_I(m) = 0$. It follows that $J \cong Jm$ under the multiplication homomorphism $r \to rm$, and this proves that every nonzero left ideal contains a uniform left ideal.

To see that R is irreducible, first observe that if $A \cap r(A) = 0$ for A an ideal of R then $I = r(A)$ is the unique left ideal maximal with respect to $A \cap I = 0$, and also $r(A)A \subseteq A \cap r(A) = 0$. It follows that $A + r(A)$ is an essential left ideal of R. Now $(A + r(A))(AM \cap r(A)M) \subseteq Ar(A)M + r(A)AM = 0$ so that $AM \cap r(A)M = 0$ because ${}_RM$ is nonsingular. Since ${}_RM$ is faithful and uniform this can only happen if $A = 0$ or $r(A) = 0$.

(2) *implies* (3). Let M be any nonzero uniform left ideal of R and consider the standard partial context $({}_RM, {}_\Delta V_R)$ derived from M. As was noted earlier, ${}_RM$ is a monoform module, Δ is a division ring, and V is a Δ-R-bimodule under the action defined by $x(dvr) = ((xd)v)r$ for $r \in R$, $v \in V = \operatorname{PHom}_R(M, R)$, $d \in \Delta = \operatorname{PEnd}_R M$, and $x \in M_v d^{-1}$. Furthermore, given $v, 0 \neq v' \in V$, (m, v) and (m, v') are defined for any $m \in M_v \cap M_{v'} \neq 0$; and if $m \neq 0$ then $(m, v') \neq 0$ because $(\ , v')\colon M_{v'} \to R$ is a monomorphism. It therefore remains only to check that V_R is faithful, and for this in turn it suffices to show that $r(M, V) = 0$.

Set $T = (M, V)$ and observe first that if U is a nonzero left ideal contained in T then U contains a left ideal of the form $M_f f$ for some $0 \neq f \in \operatorname{Hom}_R(M_f, R)$ and $M_f \subseteq M$. For $T = \sum M_f f$ where f ranges over $\operatorname{PHom}_R(M_f, R)$, so we may choose an element $0 \neq u = m_1 f_1 + \cdots + m_t f_t \in U$ with each $m_i \in M_{f_i}$ and t as small as possible. Because of this choice, for each $r \in R$, $rm_1 f_1 = 0$ implies that $ru = 0$. Hence the assignment $rm_1 f_1 \to ru$ for $r \in R$ yields an element of $\operatorname{Hom}_R(Rm_1 f_1, U)$ which we denote by g; and then $(Rm_1)f_1 g$ is the desired left ideal contained in U.

Now suppose to the contrary that $r(T) \neq 0$. Then because R is irreducible, $T \cap r(T) \neq 0$. By the preceding paragraph, we may assume that $T \cap r(T) \supseteq M_f f$ for some $f \in \operatorname{Hom}_R(M_f, R)$ and $M_f \subseteq M$. Since $l_R(M_f)$ is not an essential left ideal of R we may apply the hypothesis to choose a uniform left ideal U_0 with $U_0 M_f \neq 0$. It follows that $U_0 m \neq 0$ for some $m \in M_f$, and so the rule $u \to um$ defines a monomorphism of U_0 into M. Setting $M_0 = U_0 m \subseteq M$ and defining $f_0 \in \operatorname{Hom}_R(M_0, R)$ by $(um)f_0 = u$ for $u \in U_0$, we see that $U_0 = M_0 f_0 \subseteq T$. Hence $TM_f \supseteq U_0 M_f \neq 0$, and it follows that $T(M_f f) = (TM_f)f \neq 0$ contradicting the choice of $M_f f \subseteq T \cap r(T)$. We conclude therefore that $r(T) = 0$.

(3) *implies* (4). Let $({}_RM, {}_\Delta V_R)$ be the partial context given by hypothesis. Since V_R is faithful we may assume that $R \subseteq \operatorname{End}_\Delta V$. Let $\tau \in \operatorname{End}_\Delta V$ and ${}_\Delta U$ a one-dimensional subspace of ${}_\Delta V$ be given. Let v be an arbitrary nonzero element of U, and use the hypothesis to choose $m \in M$ with $(m, v\tau)$ and (m, v) defined and $(m, v) \neq 0$. Set $r = (m, v)$ and $s = (m, v\tau)$. Then for any $w \in V$ we have

$$w(r\tau - s) = (w(m, v))\tau - w(m, v\tau) = ([w, m]v)\tau - [w, m](v\tau) = 0$$

because $\tau \in \operatorname{End}_\Delta V$; and also $wr = w(m, v) = [w, m]v \in U$. Hence $r\tau = s$ and $Vr \subseteq U$. Finally, $Vr \neq 0$ because V_R is faithful, and thus $Vr = U$.

(4) *implies* (5). Assume that R is a subring of $\operatorname{End}_\Delta V$ with the stated property. Then given $\tau, 0 \neq \tau' \in \operatorname{End}_\Delta V$, choose any one-dimensional subspace U of ${}_\Delta V$ with $U\tau' \neq 0$. By hypothesis there exist $r, s \in R$ with $r\tau = s$ and $Vr = U$. Then $Vr\tau' = U\tau' \neq 0$ so $r\tau \in R$ and $r\tau' \neq 0$. This proves that $\operatorname{End}_\Delta V$ is a left quotient ring of R. It is actually a maximal left quotient ring of R because of the well-known fact that $\operatorname{End}_\Delta V$ is a left selfinjective ring.

(5) *implies* (6). This is obvious.

(6) *implies* (1). We may assume that $R \subseteq \operatorname{End}_\Delta V$ with the property stated in (6). Set $M = \operatorname{Hom}_\Delta(V, \Delta)$; M has a natural structure of R-module given by $(v)rm = (vr)m$ for any $v \in V$, $r \in R$, $m \in M$. We claim that M is a faithful nonsingular uniform left R-module.

If $0 \neq r \in R$ then $Vr \neq 0$, and hence $V(rM) = (Vr)M \neq 0$ because $M = \operatorname{Hom}_\Delta(V, \Delta)$. Thus $rM \neq 0$ proving that ${}_RM$ is faithful.

To show that ${}_RM$ is nonsingular we proceed as follows. Given any $0 \neq m \in M$ set $l_R(m) = \{r \in R \mid rm = 0\}$. Then $Vl_R(m) \subseteq \operatorname{kernel} m \subset V$. Choose a nonzero subspace ${}_\Delta U$ of ${}_\Delta V$ with $U \oplus \operatorname{kernel} m = V$, and define $\tau \in \operatorname{End}_\Delta V$ by requiring that $\tau|_U = 1|_U$ and $\tau|_{\operatorname{kernel} m} = 0$. Then use the hypothesis to choose $r, s \in R$ with $r\tau = s \neq 0$. Observe that $Vs \cap \operatorname{kernel} m \subseteq V\tau \cap \operatorname{kernel} m = U \cap \operatorname{kernel} m = 0$. Now for any $\sigma \in \operatorname{End}_\Delta V$, $\sigma sm = 0$ if and only if $V\sigma sm = 0$ if and only if $V\sigma s \subseteq Vs \cap \operatorname{kernel} m = 0$. Hence $\sigma sm = 0$ if and only if $\sigma s = 0$, and this implies in particular that $l_R(m) \cap R^1 s = 0$ where $R^1 s$ is the left ideal of R generated by s. Thus $l_R(m)$ is not an essential left ideal of R, and since $0 \neq m \in M$ was arbitrary, ${}_RM$ is nonsingular.

Next suppose $0 \neq m$, $0 \neq n \in M$ are given. Choose $u, v \in V$ with $um \neq 0$ and $vn \neq 0$ in Δ, and write $um = d(vn)$ for some $d \in \Delta$. Then $V = \Delta u \oplus \operatorname{kernel} m$ and so we may define $\tau \in \operatorname{End}_\Delta V$ by $u\tau = dv$, $(\operatorname{kernel} m)\tau = 0$. Now $um = d(vn) = (dv)n = (u\tau)n = u(\tau n)$ and $(\operatorname{kernel} m)m = 0 = (\operatorname{kernel} m)\tau n$, so $m = \tau n$. We may use the hypothesis to choose $r, s \in R$ with $r\tau = s \neq 0$. Then $rm = r\tau n = sn$ and $sn \neq 0$ because $Vsn = Vr\tau n = \Delta vn \neq 0$. This shows that ${}_RM$ is uniform, and the proof is now complete. $\square$

REMARKS. It is worth recording several observations produced in the course of the proof. First of all, there is the fact that if M is a nonzero uniform left ideal of R or a faithful nonsingular uniform module then $r(M, V) = 0$ where $V = \mathrm{PHom}_R(M, R)$; this is equivalent to (M, V) being essential as a left ideal of R. Furthermore, regarding R as embedded in $\mathrm{End}\, {}_\Delta V$ where $\Delta = \mathrm{PEnd}\, {}_R M$, it is evident that the ideal (M, V) consists of linear transformations of finite rank on ${}_\Delta V$. Finally, the maximal left quotient ring of R was actually constructed as $\mathrm{End}\, {}_\Delta V$ where V is as above.

We also have the following consequence of the preceding theorem.

COROLLARY 2.2. *Suppose that R has a left quotient ring which is of the form* $\mathrm{End}\, W_\Delta$ *for some right vector space W_Δ. Then the maximal left quotient ring of R is of the form* $\mathrm{End}\, {}_\Delta V$ *for some left vector space ${}_\Delta V$.*

PROOF. In view of the previous remarks it will suffice to show that if W is as above then ${}_R W$ is a faithful nonsingular uniform module and $\mathrm{PEnd}\, {}_R W \cong \Delta$.

We may assume that $R \subseteq \mathrm{End}\, W_\Delta$ and therefore ${}_R W$ is faithful. Set $T = \mathrm{End}\, W_\Delta$ and observe that W is a nonsingular left T-module. For if L is a nonzero left ideal of T with $r_W(L) = \{w \in W \mid Lw = 0\} \neq 0$ then $r_W(L) \neq W$ and we may write $W = r_W(L) \oplus W'$ for some subspace W' of W_Δ. Define $\tau \in T$ by requiring that $\tau = 1$ on $r_W(L)$ and $\tau = 0$ on W'. Then for any $\sigma \in T$, $\sigma\tau \in L$ implies that $\sigma\tau r_W(L) = 0$, whence $\sigma\tau = 0$. Thus $T\tau \cap L = 0$ which proves that L is not an essential left ideal of T. This shows that ${}_T W$ is nonsingular.

Next, for any $w \in W$, if $l_R(w) = \{r \in R \mid rw = 0\}$ were an essential left ideal of R then $Tl_R(w)$ would be an essential left ideal of T because ${}_R T$ is an essential extension of ${}_R R$. Since $Tl_R(w)w = 0$ and ${}_T W$ is nonsingular this implies that $w = 0$. Thus ${}_R W$ is nonsingular.

To see that ${}_R W$ is uniform proceed as follows. Given $u \neq 0$, $w \neq 0$ in W, choose $\tau \in \mathrm{End}\, W_\Delta$ with $\tau u = w$ and $\tau U = 0$ where $W = U \oplus u\Delta$. Then use the hypothesis to pick $r, s \in R$ with $r\tau = s \neq 0$. Now $su = r\tau u = rw$ and $su \neq 0$ because $s \neq 0$ and $sU = r\tau U = 0$. Hence ${}_R W$ is uniform.

To prove that $\mathrm{PEnd}\, {}_R W \cong \Delta$, suppose $0 \neq f \in \mathrm{Hom}_R(W_f, W)$ is given with W_f an R-submodule of W. By Proposition 2.4 of **[13]**, ${}_R W$ is a quasi-injective R-module and $\mathrm{End}\, {}_R W \cong \Delta$ acting as right multiplications. Hence f extends uniquely to an element of $\mathrm{End}\, {}_R W \cong \Delta$, which is what was desired. □

For any module ${}_R M$ there are natural isomorphisms $\mathrm{PEnd}\, {}_R M \cong \mathrm{PEnd}\, {}_R \overline{M}$ and $\mathrm{PHom}_R(M, R) \cong \mathrm{PHom}_R(\overline{M}, R)$ where $\overline{M}$ denotes the quasi-injective hull of ${}_R M$. Thus the standard partial context for ${}_R M$ extends to a standard partial context for ${}_R \overline{M}$, $({}_R \overline{M}, {}_\Delta V_R)$, where we set $V = \mathrm{PHom}_R(\overline{M}, R)$ and $\Delta = \mathrm{PEnd}\, {}_R \overline{M}$. When ${}_R M$ is nonsingular, we also have $\mathrm{PEnd}\, {}_R \overline{M} \cong \mathrm{End}\, {}_R \overline{M}$, and then $\dim {}_R M = \dim {}_\Delta \Delta$ where dim denotes uniform dimension. There are other less obvious relationships between module dimensions which we indicate below.

PROPOSITION 2.3. *Let ${}_R M$ be a nonsingular module such that* $V = \mathrm{PHom}_R(M, R)$ *is a faithful right R-module. Then* $\dim \overline{M}_\Delta \leqslant \dim {}_R R \leqslant \dim {}_\Delta V$ *where* $\Delta = \mathrm{End}\, {}_R \overline{M}$. *In particular, if M is a faithful nonsingular uniform module over a ring R which satisfies Theorem* 2.1, *then* $\dim \overline{M}_\Delta \leqslant \dim {}_R R = \dim {}_\Delta V$.

PROOF. We first show that dim $\overline{M}_\Delta \leqslant$ dim $_RR$. Suppose that $m_1, \ldots, m_k \in \overline{M}$ are linearly independent over Δ. Set $A_i = \bigcap_{j \neq i} l_R(m_j)$ where $l_R(m_j) = \{r \in R \mid rm_j = 0\}$. By a well-known result [7, Theorem 2.3], $A_i \not\subseteq l_R(m_i)$, and it follows that $A_i \cap l_R(m_i)$ is not essential in A_i. Hence there exists a nonzero left ideal $B_i \subseteq A_i$ with $B_i \cap l_R(m_i) = 0$. It is then easy to check that $\Sigma_{i=1}^k B_i$ is a direct sum. For if $\Sigma_{i=1}^k b_i = 0$ with each $b_i \in B_i$ then $b_i m_i = (\Sigma_{j \neq i} - b_j)m_i = 0$, so each $b_i = 0$ since $B_i \cap l_R(m_i) = 0$. Hence dim $_RR \geqslant k$.

Next, we show that dim $_RR \leqslant$ dim $_\Delta V$. For suppose that $\Sigma_{i=1}^k A_i$ is a direct sum of nonzero left ideals of R. We claim that $\Sigma_{i=1}^k VA_i$ is then a direct sum of nonzero subspaces of $_\Delta V$. For suppose that $\Sigma_{i=1}^k v_i = 0$ where each $v_i \in VA_i$ and $v_i \in$ Hom $_R(M_i, R)$ for some essential submodule of M_i of $_R\overline{M}$. Set $M_0 = \bigcap_{i=1}^k M_i$; M_0 is an essential submodule of $_R\overline{M}$, and for every $m \in M_0$, $mv_i \in A_i$ because $v_i \in VA_i$. Thus $\Sigma_{i=1}^k mv_i = 0$ implies that $mv_i = 0$ for each i. Since M_0 is essential in $_R\overline{M}$ this implies that each $v_i = 0$. Hence dim $_RR \leqslant$ dim $_\Delta V$.

For the reverse inequality, suppose that M is a faithful nonsingular uniform module and that $v_1, \ldots, v_k \in V$ are linearly independent over Δ. By Theorem 2.1(4), $A_i = \{r \in R \mid Vr \subseteq \Delta v_i\}$ is a nonzero left ideal of R. We claim that $\Sigma_{i=1}^k A_i$ is a direct sum. For if $r \in A_i \cap \Sigma_{j \neq i} A_j$ then $Vr \subseteq \Delta v_i \cap \Sigma_{j \neq i} \Delta v_j = 0$, and so $r = 0$. Hence dim $_\Delta V \leqslant$ dim $_RR$ and the proof is complete. □

For M a nonsingular uniform left ideal of a prime ring R it was shown in [**1**, Theorem 10C] that dim $R_R =$ dim $M_S \geqslant$ dim $_SW \geqslant$ dim $_RR$ where $S =$ End $_RM$ and $W =$ Hom $_R(M, R)$. When R fails to be prime this relationship breaks down as is illustrated by the following elementary example. Set $R = \left(\begin{smallmatrix} \Delta & 0 \\ \Delta & \Delta \end{smallmatrix}\right)$ and $M = \left(\begin{smallmatrix} 0 & 0 \\ \Delta & 0 \end{smallmatrix}\right)$ where Δ is an arbitrary division ring. Then $S =$ End $_RM \cong \Delta$ acting as right multiplications, while $W =$ Hom $_R(M, R) \cong \Delta \oplus \Delta$. Hence dim $R_R = 2 =$ dim $_SW$ while dim $M_S = 1$. When M is a faithful nonsingular uniform left ideal of a ring R it can still be seen that dim $R_R =$ dim M_S, but it is an open question as to whether an analogue of the inequality dim $M_S \geqslant$ dim $_SW$ holds in this case.

By constrast we are able to provide a sufficient condition for equality to hold in Proposition 2.3.

COROLLARY 2.4. *In the setting of the second sentence of Proposition* 2.3, dim $_RR < \infty$ *if and only if* dim $\overline{M}_\Delta < \infty$; *and when this is the case*, dim $_RR =$ dim $\overline{M}_\Delta$.

PROOF. We first observe that dim $\overline{M}_\Delta < \infty$ if and only if the maximal left quotient ring of R is simple artinian. This is shown in Theorem 3.1 of [**13**] wherein it is also demonstrated that End $\overline{M}_\Delta$ is then a maximal left quotient ring of R. Since dim(End $\overline{M}_\Delta$) = dim $\overline{M}_\Delta$ when dim $\overline{M}_\Delta < \infty$, and since dim $_RR$ equals the dimension of a maximal left quotient ring of R, the corollary is proved. □

3. Faithful left ideals. It is instructive to consider the following examples. Let V be a left vector space over a division ring Δ with basis $\{v_i \mid 1 \leqslant i < k\}$ with $k > 2$, and set $R_1 = \{r \in \text{End}\,_\Delta V \mid v_i r = 0 \text{ for } i \geqslant 2\}$ and $R_2 = \{r \in \text{End}\,_\Delta V \mid v_i r \in \Sigma_{1 \leqslant j \leqslant i} \Delta v_j$ for each $i \geqslant 1\}$. Then End $_\Delta V$ is a maximal left quotient ring of both R_1 and R_2. Observe also that R_1 has no faithful left ideals because $l(R_1) \neq 0$. By contrast, $M = \{r \in \text{End}\,_\Delta V \mid v_i r \in \Delta v_1$ for each $i \geqslant 1\}$ is a faithful uniform left ideal of R_2.

The general situation when a ring contains a faithful nonsingular uniform left ideal is actually typified by R_2 as the next result shows.

THEOREM 3.1. *For a ring R the following conditions are equivalent.*

(i) *R has a faithful nonsingular uniform left ideal.*

(ii) *The maximal left quotient ring of R is of the form* End $_{\Delta}V$ *for some vector space $_{\Delta}V$ and there is an element v in V such that given any $0 \neq u \in V$ there exists $r \in R$ with $Vr = \Delta v$ and $ur \neq 0$.*

PROOF. (i) *implies* (ii). Let M be a faithful nonsingular uniform left ideal of R. We already know that the maximal left quotient ring of R is End $_{\Delta}V$ where $V =$ PHom $_R(M, R)$ and $\Delta =$ PEnd $_R M$. Since M is a left ideal, we may assume that $0 \neq \mathrm{Hom}_R(M, R) \subseteq V$. Fix an element $0 \neq v \in \mathrm{Hom}_R(M, R)$ and suppose that $0 \neq u \in V$ is given with $u \in \mathrm{Hom}_R(M_u, R)$. Then $(M_u u)M \neq 0$, so $0 \neq [u, M] \subseteq \Delta$. Hence $0 \neq [u, M]v = u(M, v) \subseteq \Delta v \cap uR$. For any $r \in (M, v)$ with $ur \neq 0$ we have $Vr \subseteq V(M, v) = [V, M]v = \Delta v$. Thus $Vr = \Delta v$ as we were to prove.

(ii) *implies* (i). We already know from Theorem 2.1 that R is nonsingular. Choose $0 \neq v \in V$ as in (ii) and fix an element $t \in R$ satisfying $Vt = \Delta v$ and $vt \neq 0$. We claim that Rt is a faithful uniform left ideal of R.

If $0 \neq s \in R$ then $Vs \neq 0$. Choose $0 \neq u \in Vs$ and apply (ii) to pick $r \in R$ with $Vr = \Delta v$ and $ur \neq 0$. Write $ur = dv$ with $0 \neq d \in \Delta$. Then $urt = dvt \neq 0$ and $urt \in Vsrt$, so $0 \neq srt \in sRt$. Since s was arbitrary in R this shows that Rt is a faithful left ideal.

We can actually show that $M = \{r \in R \mid Vr \subseteq \Delta v\} \supseteq Rt$ is a uniform left ideal. For let $0 \neq t_1$ and $0 \neq t_2$ be given elements of M. Then $Vt_1 = Vt_2 = \Delta v$ so there exist $u, w \in V$ with $ut_1 = wt_2 \neq 0$. Then $V = \Delta u \oplus \text{kernel } t_1 = \Delta w \oplus \text{kernel } t_2$ and we may choose $\tau \in$ End $_{\Delta}V$ with $w\tau = u$ and $(\text{kernel } t_2)\tau = 0$. Since End $_{\Delta}V$ is the maximal left quotient ring of R there exist $r, s \in R$ with $r\tau = s$ and $rt_2 \neq 0$. Then $(w)\tau t_1 = ut_1 = wt_2$ and $(\text{kernel } t_2)\tau t_1 = 0 = (\text{kernel } t_2)t_2$ so that $\tau t_1 = t_2$. Hence $st_1 = r\tau t_1 = rt_2 \neq 0$ and therefore M is a uniform left ideal of R. $\square$

It should be noted that we have actually proved that a ring satisfying Theorem 3.1 contains a faithful nonsingular uniform left ideal which is principal. It is an open question as to whether a ring satisfying Theorem 2.1 possesses a faithful nonsingular uniform module which is cyclic.

4. Related rings. In this section we will demonstrate that the property of having a faithful nonsingular uniform module or left ideal survives passage to polynomial rings and matrix rings. One key step is the following result whose present formulation was suggested by K. Nicholson.

PROPOSITION 4.1. *Suppose that $_RM$ is a (faithful) nonsingular uniform module and that $(_RP_S, {}_SQ_R)$ is a Morita context satisfying the following conditions*: (i) $S \neq 0$; (ii) $(P, sQ) \neq 0$ *whenever* $0 \neq s \in S$; *and* (iii) (P, Q) *is an essential left ideal of R. Set $N = Q \otimes_R M/K$ where*

$$K = \left\{ \sum_{i=1}^{k} q_i \otimes m_i \in Q \otimes_R M \mid \sum_{i=1}^{k} (p, q_i)m_i = 0 \text{ for all } p \in P \right\}.$$

Then N is a (faithful) nonsingular uniform left S-module.

PROOF. For each $p \in P$ there is a group homomorphism $\tilde{p} \in \operatorname{Hom}_{\mathbf{Z}}(Q \otimes_R M, M)$ defined on generators $q \otimes m \in Q \otimes_R M$ by $\tilde{p}(q \otimes m) = (p, q)m$. Note that $K = \bigcap_{p \in P} \operatorname{kernel} \tilde{p}$ and so we may also regard $\tilde{p}$ as an element of $\operatorname{Hom}_{\mathbf{Z}}(N, M)$. Furthermore, for every $p \in P$, $q \in Q$, and $x \in Q \otimes_R M$, $q \otimes \tilde{p}(x) = [q, p]x$.

It is evident that K is an S-submodule of $Q \otimes_R M$ and so N is a left S-module. Suppose $sN = 0$ with $s \in S$. Then $\tilde{p}(sQ \otimes_R M) = 0$ for every $p \in P$ so that $(P, sQ)M = 0$. If ${}_RM$ is faithful, $(P, sQ) = 0$; and then $s = 0$ by hypothesis (ii). Thus ${}_SN$ is faithful if ${}_RM$ is.

To see that ${}_SN$ is uniform, let $n_1 = x_1 + K$ and $n_2 = x_2 + K$ be arbitrary nonzero elements of N. There then exist p_1 and p_2 in P with $\tilde{p}_1(x_1) \neq 0$ and $\tilde{p}_2(x_2) \neq 0$ in M. Since ${}_RM$ is uniform we may choose r_1 and r_2 in R with $r_1\tilde{p}_1(x_1) = r_2\tilde{p}_2(x_2) \neq 0$. Set $m = r_1\tilde{p}_1(x_1)$. Since ${}_RM$ is nonsingular and (P, Q) is an essential left ideal of R, $(P, Q)m \neq 0$. Hence there exists $q \in Q$ with $q \otimes m \notin K$; that is, $q \otimes r_1\tilde{p}_1(x_1) = q \otimes r_2\tilde{p}_2(x_2) \notin K$. Thus $[qr_1, p_1]n_1 = [qr_2, p_2]n_2 \neq 0$ in N, and this proves that ${}_SN$ is uniform.

Next, let n be an arbitrary nonzero element of N; we will show that $l_S(n)$ is not an essential left ideal of S. Choose $p \in P$ with $\tilde{p}(n) \neq 0$ in M. Then $l_R(\tilde{p}(n))$ is not an essential left ideal of R because ${}_RM$ is nonsingular, so there exists a nonzero left ideal L of R with $L \cap l_R(\tilde{p}(n)) = 0$.

First, observe that $[QL, p] \neq 0$. For if $[QL, p] = 0$ then $(P, Q)L\tilde{p}(n) = \tilde{P}(QL \otimes \tilde{p}(n)) = \tilde{P}([QL, p]n) = 0$, and so $L\tilde{p}(n) = 0$ because (P, Q) is an essential left ideal of R. But this would contradict the fact that $L \cap l_R(\tilde{p}(n)) = 0$.

Finally, we can show that $[QL, p] \cap l_S(n) = 0$. For if $q \in QL$ and $[q, p] \in l_S(n)$ then $(P, q)p \otimes n = P \otimes [q, p]n = 0$, and it follows that $(P, q) \in L \cap l_R(\tilde{p}(n)) = 0$. By hypothesis (ii) this yields $[q, p] = 0$. Hence $l_S(n)$ is not an essential left ideal of S and thus ${}_SN$ is a nonsingular module. □

PROPOSITION 4.2. *Suppose that M is a left ideal of R and that $({}_RP_S, {}_SQ_R)$ is a Morita context satisfying* (iv) $(P, q) \neq 0$ *for every* $0 \neq q \in Q$. *Then $N = Q \otimes_R M/K$ and QM are isomorphic left S-modules.*

PROOF. Consider the S-homomorphism ϕ: $Q \otimes_R M \to QM \subseteq Q$ defined on generators $q \otimes m \in Q \otimes M$ by $(q \otimes m)\phi = qm$. First note that $\Sigma_{i=1}^k q_i \otimes m_i \in \operatorname{kernel} \phi$ implies that $\Sigma_{i=1}^k (p, q_i)m_i = (p, \Sigma_{i=1}^k q_i m_i) = 0$ for every $p \in P$, so kernel $\phi \subseteq K$. On the other hand, if $\Sigma_{i=1}^k q_i \otimes m_i \in K$ then $(p, \Sigma_{i=1}^k q_i m_i) = \Sigma_{i=1}^k (p, q_i)m_i = 0$ for every $p \in P$, so that $(P, \Sigma_{i=1}^k q_i m_i) = 0$. By hypothesis (iv) it follows that $\Sigma_{i=1}^k q_i m_i = 0$. So ϕ induces an isomorphism of N onto QM as S-modules. □

COROLLARY 4.3. *If R has a faithful nonsingular uniform module or left ideal then the same is true for R_n, the ring of $n \times n$ matrices over R.*

PROOF. This follows from a consideration of the Morita context $({}_RR_{(n)R_n}, {}_{R_n}Q_R)$ where $R_{(n)}$ and $R^{(n)}$ denote, respectively, spaces of row and column vectors, R' is the usual ring with identity element containing R, and $Q = R'^{(n)}/(r_{R'}(R))^{(n)}$. This context satisfies conditions (i), (ii), (iii), and (iv) of the previous two propositions. Furthermore, $QM \cong M^{(n)}$ which is certainly isomorphic to a left ideal of R_n when M is a left ideal of R. □

We remark that the same result holds by virtue of the same argument for any subring of row-finite matrices over R which contains all matrices of bounded size.

COROLLARY 4.4. *Suppose that P is a torsionless left R-module such that PQ is an essential left ideal of R where $Q = \mathrm{Hom}_R(P, R)$. Then* $\mathrm{End}_R P$ *has a (faithful) nonsingular uniform module if R does. Furthermore, if additionally M is a faithful nonsingular uniform left ideal of R with $MP \neq 0$ then $[QM, p]$ is a faithful nonsingular uniform left ideal of* $\mathrm{End}_R P$ *for some $p \in P$.*

PROOF. Consider the standard Morita context for ${}_RP$. □

By taking $P = Re$ in the previous corollary we have the following consequence.

COROLLARY 4.5. *If $e = e^2 \in R$ and ReR is an essential left ideal of R then eRe has a faithful nonsingular uniform module whenever R does.*

The case of polynomial rings is treated by the next series of results.

LEMMA 4.6. *Suppose that m is an element of an R-module M such that $l_R(m) \cap L = 0$ for some left ideal L of R. Then $l_{R[X]}(m) \cap L[X] = 0$ for any set X of indeterminates.*

PROOF. It clearly suffices to treat the case $X = \{x_1, \ldots, x_n\}$, and for this we use induction on the number of indeterminates. Suppose $f \in l_{R[x_1,\ldots,x_n]}(m) \cap L[x_1, \ldots, x_n]$, and write $f = \sum_{i=0}^k f_i x_n^i$ where each $f_i \in L[x_1, \ldots, x_{n-1}]$. Then $fm = 0$ implies that each $f_i m = 0$, so each $f_i \in l_{R[x_1,\ldots,x_{n-1}]}(m) \cap L[x_1, \ldots, x_{n-1}] = 0$. □

PROPOSITION 4.7. *If M is a nonsingular left R-module then $M[X]$ is a nonsingular left $R[X]$-module for any set X of indeterminates.*

PROOF. We first treat the case of one variable, say $X = \{x\}$. Suppose $Z_{R[x]}(M[x]) \neq 0$ and choose $0 \neq m(x) \in Z_{R[x]}(M[x])$ of least possible degree d; write $m(x) = \sum_{i=0}^d m_i x^i$ with each $m_i \in M$. For $h(x) \in R[x]$ it is easy to check that because of the minimal choice of d, $h(x)m(x) = 0$ if and only if $h(x)m_d = 0$. Hence $l_{R[x]}(m(x)) = l_{R[x]}(m_d)$. By applying the lemma and using the fact that ${}_RM$ is nonsingular we learn that $l_{R[x]}(m(x))$ is not an essential left ideal of $R[x]$, and this contradicts our assumption that $Z_{R[x]}(M[x]) \neq 0$. Hence $Z_{R[x]}(M[x]) = 0$.

Now suppose that X is an arbitrary set of indeterminates and suppose $0 \neq m(X) \in Z_{R[X]}(M[X])$. Then by renaming indeterminates if necessary we may assume that $m(X) = m(x_1, \ldots, x_n)$. By an obvious induction on the result of the preceding paragraph we know that there exists a nonzero left ideal L of $R[x_1, \ldots, x_n]$ with $l_{R[x_1,\ldots,x_n]}(m(X)) \cap L = 0$. Applying the lemma to the element $m(X)$ we have that $l_{R[X]}(m(X)) \cap L[X] = 0$. Since $m(X)$ was an arbitrary element of $M[X]$ this proves that $Z_{R[X]}(M[X]) = 0$. □

COROLLARY 4.8. *If M is a faithful nonsingular uniform left R-module then the same is true of $M[X]$ as an $R[X]$-module for any set X of indeterminates.*

PROOF. It remains only to show that $M[X]$ is faithful and uniform over $R[X]$. This can be found in [**3**, p. 88 or **10**]. □

5. Direct products of full linear rings. With but little additional effort we can also treat rings whose maximal left quotient rings are direct products or finite direct sums of full linear rings.

It is customary to say that a module M *has enough uniforms* if every nonzero submodule of M contains a uniform submodule. For a nonsingular module there are well known rephrasings of this property.

PROPOSITION 5.1. *For a nonsingular module M the following conditions are equivalent.*

(i) *M has enough uniforms.*

(ii) *M contains an essential sum of uniform submodules.*

(iii) *M contains an essential direct sum of uniform submodules.*

PROOF. We will show that (ii) *implies* (iii), the implications (i) *implies* (ii) and (iii) *implies* (i) being standard. So suppose that $\Sigma_{i\in I} M_i$ is an essential submodule of ${}_RM$ with each ${}_RM_i$ uniform. Apply Zorn's lemma to choose a maximal direct sum of the form $M_0 = \bigoplus_{j\in J} M_j$ with $J \subseteq I$. We claim that M_0 is essential in $\Sigma_{i\in I} M_i$.

It will suffice to establish that for each finite subset $\{i_1,\ldots,i_n\} \subseteq I$, $M_0 \cap \Sigma_{k=1}^n M_{i_k}$ is essential in $\Sigma_{k=1}^n M_{i_k}$. First note that for each k, $M_0 \cap M_{i_k}$ is essential in M_{i_k} because of the maximal choice of M_0 and the fact that M_{i_k} is uniform. Let m be an arbitrary nonzero element of $\Sigma_{k=1}^n M_{i_k}$. Write $m = \Sigma_{k=1}^n m_{i_k}$ with each $m_{i_k} \in M_{i_k}$ and set $I = \bigcap_{k=1}^n \{r \in R \mid rm_{i_k} \in M_0\}$. Then I is an essential left ideal of R and therefore $0 \neq Im \subseteq \Sigma_{k=1}^n Im_{i_k} \subseteq M_0 \cap \Sigma_{k=1}^n M_{i_k}$. We conclude that M_0 is essential in $\Sigma_{i\in I} M_i$, and consequently M_0 is essential in M. $\square$

Let Δ_i be a division ring and V_i a left Δ_i-vector space for each $i \in I$. Set $\Delta = \bigoplus_{i\in I}\Delta_i$, $V = \bigoplus_{i\in I} V_i$, and let π_i denote the projection map of V onto V_i. If U is a Δ-submodule of V we define $\dim_\Delta U = \Sigma_{i\in I} \dim_{\Delta_i} U\pi_i$. Observe that $U\pi_i = \Delta_i U = U \cap V_i$ so that $U = \bigoplus_{i\in I}(U \cap V_i)$. Since $\operatorname{Hom}_\Delta(V_i, V_j) = 0$ for $i \neq j$, we can and will identify $\operatorname{End}_\Delta V$ with $\Pi_{i\in I} \operatorname{End}_{\Delta_i} V_i$. Similarly, we identify $\operatorname{Hom}_\Delta(V, \Delta)$ with $\Pi_{i\in I} \operatorname{Hom}_{\Delta_i}(V_i, \Delta_i)$. With these conventions we can extend Theorem 2.1 as follows.

THEOREM 5.2. *For a ring R the following conditions are equivalent.*

(1) *There exists a faithful nonsingular left R-module which has enough uniforms.*

(2) *R is a nonsingular ring which has enough uniforms.*

(3) *There exists a partial context $({}_RM, {}_\Delta V_R)$ satisfying*: (i) *V_R is faithful*; (ii) *Δ is a direct sum of division rings Δ_i and V is a direct sum of Δ_i-vector spaces*; *and* (iii) *given u, $0 \neq v$ in V there exists $m \in M$ with (m, u) and (m, v) defined and $(m, v) \neq 0$.*

(4) *R is (isomorphic to) a subring of a direct product of full linear rings $\Pi_{i\in I} \operatorname{End}_{\Delta_i} V_i$ with the property that given $\tau \in \Pi_{i\in I} \operatorname{End}_{\Delta_i} V_i$ and ${}_{\Delta_i}U$ a one-dimensional subspace of some ${}_{\Delta_i}V_i$ there exist $r, s \in R$ with $r\tau = s$ and $Vr = U$.*

(5) *R has a (maximal) left quotient ring isomorphic to a direct proauct of full linear rings.*

(6) *R is (isomorphic to) a subring of a direct product of full linear rings $\Pi_{i\in I} \operatorname{End}_{\Delta_i} V_i$ with the property that given $0 \neq \tau \in \Pi_{i\in I} \operatorname{End}_{\Delta_i} V_i$ there exist $r, s \in R$ with $r\tau = s \neq 0$.*

PROOF. We can follow almost verbatim the proof of Theorem 2.1, and we will therefore highlight only the portions which require modification.

(1) *implies* (2). We must show that R has enough uniforms when ${}_RM$ is a faithful nonsingular module with enough uniforms. To see this let I be any nonzero left ideal of R. $IM \neq 0$ because M is faithful, so $Im \neq 0$ for some $m \in M$. Since $l_R(m) \cap I$ is not essential in I, there exists a left ideal $0 \neq J \subseteq I$ with $J \cap l_R(m) = 0$. Hence $J \cong Jm$. Since M has enough uniforms there exists a uniform submodule of the form Km with K a nonzero left ideal contained in J. Hence K is a uniform left ideal contained in I.

(2) *implies* (3). Choose a direct sum $M = \oplus_{i \in I} M_i \subseteq R$ of uniform left ideals maximal with respect to the property that no two distinct M_i have nonzero isomorphic submodules (i.e., no two M_i are *subisomorphic*). Set $V = \oplus_{i \in I} V_i$ and $\Delta = \oplus_{i \in I} \Delta_i$ where $V_i = \mathrm{PHom}\,_R(M_i, R)$ and $\Delta_i = \mathrm{PEnd}_R M_i$. Then we construct the partial context $({}_RM, {}_\Delta V_R)$ where the context mappings are defined componentwise. It is evident that given any $u, 0 \neq v \in V$ there exists $m \in M$ with (m, u) and (m, v) defined and $(m, v) \neq 0$. The only point that requires checking is that V is a faithful right R-module; or, equivalently, that $r_R(M, V) = 0$.

First observe that if U is an arbitrary uniform left ideal of R then either (i) $U \cap \oplus_{i \in I} M_i = 0$ whence by the maximal choice of $\oplus_{i \in I} M_i$, U is subisomorphic to some M_i; or (ii) $U \cap \oplus_{i \in I} M_i \neq 0$, whence for some $j \in I$ the projection homomorphism $U \cap \oplus_{i \in I} M_i \to M_j$ must be an isomorphism of a submodule of U with a left ideal contained in M_j. Thus, in any event, U is subisomorphic to some M_i, $i \in I$.

Next, choose $f \in \mathrm{PHom}\,_R(M_i, R)$ with $0 \neq M_f f \subseteq U$. Since $l_R(M_f)$ is not an essential left ideal of R there exists a uniform left ideal U_0 with $U_0 M_f \neq 0$. Hence $U_0 m \neq 0$ for some $m \in M_f$ and therefore U_0 is isomorphic to a submodule of M_f. It follows that $U_0 \subseteq (M, V)$ and so $(M, V)U \supseteq U_0 U \supseteq (U_0 M_f) f \neq 0$. Hence $U \not\subseteq r_R(M, V)$ and since U was an arbitrary uniform left ideal, $r_R(M, V) = 0$.

The implications (3) *implies* (4), (4) *implies* (5), and (5) *implies* (6) require no additional comment.

(6) *implies* (1). Set $\Delta = \oplus_{i \in I} \Delta_i$ and $V = \oplus_{i \in I} V_i$. We can assume that $R \subseteq \mathrm{End}\,_\Delta V = \Pi_{i \in I} \mathrm{End}\,_{\Delta_i} V_i$ with the property stated in (6). Set $M = \mathrm{Hom}\,_\Delta(V, \Delta) = \Pi_{i \in I} \mathrm{Hom}\,_{\Delta_i}(V_i, \Delta_i)$. Then as before we can see that ${}_RM$ is faithful and nonsingular. We need only show that ${}_RM$ has enough uniforms.

By the argument given in the proof of Theorem 2.1 we know that each $M_i = \mathrm{Hom}\,_{\Delta_i}(V_i, \Delta_i)$ is a uniform left R-module. Furthermore, we claim that $\oplus_{i \in I} M_i$ is an essential R-submodule of M.

To see this, let $0 \neq \{m_i\}_{i \in I} \in M$ be arbitrary with each $m_i \in M_i$ and $m_j \neq 0$ for some $j \in I$. Write $V_j = \Delta v_j \oplus \text{kernel } m_j$ and choose $\tau \in \mathrm{End}\,_\Delta V$ with $V\tau = \Delta v_j$. Next, choose $r, s \in R$ with $r\tau = s \neq 0$. Then $Vsm_j = \Delta v_j m_j = \Delta_j$ and $Vsm_i = \Delta v_j m_i = 0$ for $i \in I$, $i \neq j$. Hence $0 \neq s\{m_i\}_{i \in I} = \{n_i\}_{i \in I} \in \oplus_{i \in I} M_i$ where $n_j = sm_j$ and $n_i = 0$ for $i \in I$, $i \neq j$. This establishes the claim.

Finally, we apply Proposition 5.1 to conclude that ${}_RM$ has enough uniforms. $\square$

The equivalence of conditions (2) and (5) in the above theorem was first established in [**6**]. Also, it is easy to check that condition (3)(ii) in the theorem can be replaced by (ii)′ Δ is a direct product of division rings.

COROLLARY 5.3. *R has a (maximal) left quotient ring which is a finite direct sum of full linear rings if and only if R has a faithful finite dimensional nonsingular module.*

COROLLARY 5.4. *The maximal left quotient ring of R is semisimple artinian if and only if R has a finite dimensional nonsingular module M such that $l_R(m_1,\ldots,m_k) = 0$ for some $m_1,\ldots,m_k \in M$.*

PROOF. Apply Theorem 2.3 of [**7**] to the previous corollary. □

These results are illustrative of the technique one employs in order to generalize results presented earlier in this article. In the interest of concluding at a reasonable point, we make the following blanket assertion, offered without proof. All properties established in the earlier sections of this paper have analogues in the more general situations in which the maximal left quotient ring is a direct product or finite direct sum of full linear rings.

REFERENCES

1. S. A. Amitsur, *Rings of quotients and Morita contexts*, J. Algebra **17** (1971), 273–298.

2. C. Faith, *Lectures on injective modules and quotient rings*, Lecture Notes in Math., vol. 49, Springer-Verlag, Berlin and New York, 1967.

3. K. R. Goodearl, *Ring theory: Nonsingular rings and modules*, Marcel Dekker, New York, 1976.

4. R. E. Johnson, *The extended centralizer of a ring over a module*, Proc. Amer. Math. Soc. **2** (1951), 891–895.

5. ______, *Representations of prime rings*, Trans. Amer. Math. Soc. **74** (1953), 351–357.

6. ______, *Quotient rings of rings with zero singular ideal*, Pacific J. Math **11** (1961), 1385–1392.

7. R. E. Johnson and E. T. Wong, *Quasi-injective modules and irreducible rings*, J. London Math. Soc. **36** (1961), 260–268.

8. K. Koh and A. C. Mewborn, *Prime rings with maximal annihilator and maximal complement right ideals*, Proc. Amer. Math. Soc. **16** (1965), 1073–1076.

9. J. Lambek and G. O. Michler, *On products of full linear rings*, Publ. Math. Debrecen **24** (1977), 123–127.

10. W. K. Nicholson, J. F. Watters and J. M. Zelmanowitz, *On extensions of weakly primitive rings*, Canad. J. Math. **32** (1980), 937–944.

11. J. M. Zelmanowitz, *Semiprime modules with maximum conditions*, J. Algebra **25** (1973), 554–574.

12. ______, *Weakly primitive rings*, Comm. Algebra **9** (1981), 23–45.

13. ______, *Representations of rings with faithful monoform modules*.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106.

Current address: Department of Mathematics, McGill University, Montreal, Canada H3A 2K6