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THE GAUSS-LUCAS THEOREM AND JENSEN POLYNOMIALS

BY

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ABSTRACT. A characterization is given of the sequences $\{\gamma_k\}_{k=0}^{\infty}$ with the property that, for any complex polynomial $f(z) = \sum a_k z^k$ and convex region K containing the origin and the zeros of f, the zeros of $\sum \gamma_k a_k z^k$ again lie in K. Many applications and related results are also given. This work leads to a study of the Taylor coefficients of entire functions of type I in the Laguerre-Pólya class. If the power series of such a function is given by $\sum \gamma_k z^k/k!$ and the sequence $\{\gamma_k\}$ is positive and increasing, then the sequence satisfies an infinite collection of strong conditions on the differences, namely $\Delta^n \gamma_k \geqslant 0$ for all n, k.

1. Introduction. This paper is concerned with functions of type I in the Laguerre-Pólya class; i.e. real entire functions which are the uniform limits, on compact subsets of the plane, of polynomials with only real zeros, all of which have the same sign. Let us represent such a function as a series $\Phi(z) = \sum_{k=0} \gamma_k z^k/k!$. Pólya and Schur **[PS]** gave two alternate characterizations of this class of entire functions.

(1.1) The series $\sum \gamma_k z^k/k!$ converges in the whole plane and the entire function $\Phi(z)$ or $\Phi(-z)$ can be represented in the form $ce^{\sigma z}z^m\prod_{n=1}^{\omega}(1 + z/z_n)$ where $\sigma \geqslant 0$, $z_n > 0$, $c \in \mathbf{R}$, $0 \leqslant \omega \leqslant \infty$, $\sum_{n=1}^{\infty} z_n^{-1} < \infty$ and m is a nonnegative integer.

(1.2) For each integer $n \geqslant 0$, the Jensen polynomial $g_n(x) = \sum_{k=0}^{n}\binom{n}{k}\gamma_k x^k$ has only real zeros, all of which have the same sign.

For any polynomial or entire function $f(x) = \sum a_k x^k$ we write $\Gamma[f(x)] = \sum a_k\gamma_k x^k$, whenever this series converges. Thus the polynomials g_n in (1.2) can be written $\Gamma[(1 + x)^n]$. Any sequence of real numbers $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ which satisfies (1.2) will be called a *multiplier sequence* (of the first kind). Pólya and Schur **[PS]** also showed that (1.2) is equivalent to the following much stronger property.

(1.3) For any polynomial f, all of whose zeros are real, the polynomial $\Gamma[f]$ again has only real zeros.

As an example, consider the sequence $\Gamma = \{0, 1, 2, 3, \ldots\}$ corresponding to the entire function ze^z. For any polynomial f, we have $\Gamma[f(x)] = xf'(x)$. For this reason, the operators Γ have been studied as a generalized form of differentiation (cf. **[CC2, CC3]**). The main theorem of §2 tells to what extent these operators Γ

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satisfy the Gauss-Lucas Theorem [**M1**, p. 22]. In particular, we prove the following result:

If Γ is an increasing nonnegative multiplier sequence, f is an arbitrary complex polynomial and K is a convex set containing the origin and the zeros of f, then all of the zeros of $\Gamma[f]$ lie in K.

In §3 we show that the above theorem remains valid for all nonnegative multiplier sequences if and only if K is an angle with vertex at the origin and angle opening at most π.

Finally, in §4 we give a classification of multiplier sequences in terms of intrinsic properties of the sequence (as opposed to zeros of polynomials or entire functions incorporating the sequence). In particular, we show that every positive sequence satisfies a convexity condition, $\gamma_k - \gamma_{k-1} \leqslant \gamma_{k+1} - \gamma_k$, for large k. If the sequence is increasing, this is true for all k and much stronger conditions also hold.

There are a number of easily proved facts about multiplier sequences which we shall need later (cf. [**CC1**]). Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence. Then:

(i) If $\gamma_j \gamma_l \neq 0$, then $\gamma_k \neq 0$ for $j \leqslant k \leqslant l$.

(ii) All nonzero terms of Γ have the same sign or they alternate in sign. (The former occurs when all the nonzero roots of g_n in (1.2) are negative; the latter occurs when all the nonzero roots are positive.)

(iii) For each positive integer k, the sequence satisfies $\gamma_k^2 \geqslant \gamma_{k-1}\gamma_{k+1}$. This is known as Turán's inequality. It is the only easily tested necessary condition for a multiplier sequence other than those given in §4.

Our theorems are generally stated only for nonnegative multiplier sequences. By (ii) above, if the terms of a sequence are not nonnegative, then multiplication of each term by either -1 or $(-1)^k$ (i.e. replacing $\Phi(x)$ by $\Phi(-x)$) yields a sequence all of whose terms are nonnegative. Thus there is very little loss of generality if we restrict our attention to nonnegative sequences. On the other hand, leading zeros in the sequence caused by a factor z^m in the product representation (1.1) are harder to handle because they also cause factorials to be introduced into the sequence. For this reason, these sequences require careful treatment in the proofs.

2. The Gauss-Lucas Theorem. We begin this section with a simple but very important lemma describing some restrictions on when two successive coefficients can be equal for an entire function Φ of type I. This result will be generalized in the final section. Note that the requirement that Φ be transcendental (i.e. not polynomial) is definitely required for our extension of the Gauss-Lucas Theorem.

LEMMA 2.1. *Let* $\Phi(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k!$, $0 \leqslant \gamma_0 \leqslant \gamma_1 \leqslant \cdots$, *be a transcendental entire function of type* I *in the Laguerre-Pólya class.*

(a) *If, for some nonnegative integer* p, $\gamma_p = \gamma_{p+1} \neq 0$, *then* $\gamma_0 = \gamma_1 = \cdots$ *and* $\Phi(x) = \gamma_0 e^x$.

(b) *If, for* $p > 0$, $\gamma_0 = \gamma_1 = \cdots = \gamma_{p-1} = 0$, *but* $\gamma_p \neq 0$, *then* $0 < \gamma_p < \gamma_{p+1} < \gamma_{p+2} < \cdots$.

PROOF. (a) The function $\Phi^{(p)}(x)$ also belongs to the Laguerre-Pólya class. Thus, Turán's inequality, when applied to the first three Taylor coefficients of

$$\frac{1}{\gamma_p}\Phi^{(p)}(x) = 1 + x + \frac{\gamma_{p+2}}{\gamma_p}\frac{x^2}{2!} + \frac{\gamma_{p+3}}{\gamma_p}\frac{x^3}{3!} + \cdots,$$

shows that $\gamma_p \geq \gamma_{p+2}$. But by assumption $\gamma_p \leq \gamma_{p+1} \leq \gamma_{p+2}$, so that $\gamma_p = \gamma_{p+1} = \gamma_{p+2}$. From this it follows that $\Phi^{(p)}(x) = \gamma_p e^x$ [**CC3**, Proposition 4.5]. Thus $\Phi(x) = \gamma_p e^x + f(x)$ for some polynomial f. Since Φ has only real zeros, we must have that $f \equiv 0$ and $\Phi(x) = \gamma_p e^x = \gamma_0 e^x$.

(b) Suppose that $\gamma_0 = \gamma_1 = \cdots = \gamma_{p-1} = 0$ and $\gamma_p \neq 0$. If the sequence $\{\gamma_{k+p}\}_{k=0}^{\infty}$ is not strictly increasing, then there is an integer $m \geq 0$ such that $0 < \gamma_p \leq \gamma_{p+m} = \gamma_{p+m+1}$. But then by part (a), $\Phi(x) = \gamma_{p+m}e^x$. This contradicts the assumption that $\gamma_0 = 0$. Thus the sequence $\{\gamma_{k+p}\}_{k=0}^{\infty}$ is strictly increasing.

LEMMA 2.2. *Let $\Phi(x) = \sum_{k=0}^{\infty}\gamma_k x^k/k!$ be a transcendental entire function in the Laguerre-Pólya class. Suppose that the product representation of $\Phi(x)$ has the form*

$$\Phi(x) = cx^s e^{\sigma x}\prod_{n=1}^{\omega}(1 + x/x_n), \qquad 0 \leq \omega \leq \infty,$$

where $\sigma \geq 0$, $x_n > 0$, $c > 0$, $\sum x_n^{-1} < \infty$ and where s is a nonnegative integer. Then $\sigma \geq 1$ if and only if $0 \leq \gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \cdots$.

PROOF. We first note that if $\sigma \neq 0$, then any entire function Φ whose Hadamard factorization is of the above form is of order one and type $|\sigma|$ (see, for example, Boas [**B**, Lemma 2.10.13, p. 29 and Theorem 3.7.1, p. 51]).

Now suppose that $0 = \gamma_0 = \gamma_1 = \cdots = \gamma_{s-1} < \gamma_s \leq \gamma_{s+1} \leq \cdots$. Let $M(r, \Phi)$ denote the maximum modulus of Φ on the circle $|z| = r$. Then

$$\begin{aligned} M(r,\Phi) = \Phi(r) &= \gamma_s\frac{r^s}{s!} + \gamma_{s+1}\frac{r^{s+1}}{(s+1)!} + \cdots \\ &\geq \gamma_s\left[\frac{r^s}{s!} + \frac{r^{s+1}}{(s+1)!} + \cdots\right] = \gamma_s\left[e^r - \sum_{k=0}^{s-1}\frac{r^k}{k!}\right]. \end{aligned}$$

Thus it follows that the order of Φ is exactly one and that the type, σ, of Φ is at least one.

Conversely, suppose that $\sigma \geq 1$. Let

$$\phi(x) = \frac{\Phi(x)}{x^s} = \frac{\gamma_s}{s!} + \frac{\gamma_{s+1}x}{(s+1)!}\cdots.$$

Then $(\phi'/\phi)(x) = \sigma + \sum 1/(x + x_n)$ so that

(i) $\gamma_{s+1}/(s+1)\gamma_s = (\phi'/\phi)(0) = \sigma + \sum 1/x_n \geq \sigma$. Also,

(ii) $\Phi'(x) = c_1 x^{s_1}e^{\sigma_1 x}\prod(1 + x/x_{n,1})$, where $s_1 = \max(0, s-1)$, all the zeros $-x_{n,1}$ are those implied by Rolle's theorem or the multiple zeros of Φ, and $\sigma_1 \geq \sigma$. The monotonicity of the sequence $\{\gamma_k\}$ now follows from (i) and (ii) by induction. The same argument shows that $\{\gamma_k\}$ is strictly monotone unless $\Phi(x) = ce^x$. This completes the proof of Lemma 2.2.

REMARK. The authors wish to thank the referee for suggesting the preceding proof of Lemma 2.2.

THEOREM 2.3. *Let* $\Phi(x) = \sum \gamma_k x^k/k!$ *be a transcendental entire function of type* I *in the Laguerre-Pólya class. For each* $n = 1, 2, 3, \ldots$, *let* $g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$ *denote the Jensen polynomial associated with* $\Phi(x)$. *Then the zeros of* $g_n(x)$, $n = 1, 2, 3, \ldots$, *all lie in* $[-1, 0]$ *if and only if either* $0 \leqslant \gamma_0 \leqslant \gamma_1 \leqslant \cdots$ *or* $0 \geqslant \gamma_0 \geqslant \gamma_1 \geqslant \cdots$.

PROOF. Replacing $\Phi(x)$ by $\Phi(-x)$, if necessary, we need only to consider sequences $\{\gamma_k\}$ with nonnegative terms. We will first assume that $0 < \gamma_0 \leqslant \gamma_1 \leqslant \gamma_2 \leqslant \cdots$. If for some $m \geqslant 0$ we have $\gamma_m = \gamma_{m+1}$, then by Lemma 2.1(a) $\Phi(x) = \gamma_0 e^x$. Thus, in this case $g_n(x) = \gamma_0(1 + x)^n$. If $0 < \gamma_0 < \gamma_1 < \cdots$ and $\Phi(x)$ has no zeros, then it is easy to see that $g_n(x) = \gamma_0(1 + \sigma x)^n$, where, by Lemma 2.2, $\sigma \geqslant 1$. In the general case with $\gamma_0 > 0$, we consider

$$\Phi(x) = ce^{\sigma x} \prod_{n=1}^{\omega} \left(1 + \frac{x}{x_n}\right), \qquad 0 \leqslant \omega \leqslant \infty,$$

where $\sigma \geqslant 0$, c, $x_n > 0$ and $\sum x_n^{-1} < \infty$. Since $0 < \gamma_k < \gamma_{k+1}$, $k = 0, 1, 2, \ldots$, Lemma 2.2 implies that $\sigma \geqslant 1$. Now for each fixed $t > 0$, the function

$$\Phi(-xt) = ce^{-\sigma xt} \prod_{n=1}^{\omega} \left(1 - \frac{xt}{x_n}\right)$$

is a transcendental entire function of type I in the Laguerre-Pólya class. Since $\sigma \geqslant 1$, the function

$$\begin{aligned} \Phi_1(x) = e^x \Phi(-xt) &= ce^{(1-\sigma t)x} \prod_{n=1}^{\omega} \left(1 - \frac{xt}{x_n}\right) \\ &= \sum_{n=0}^{\infty} \frac{g_n(-t)}{n!} x^n \end{aligned} \tag{2.4}$$

also belongs to this class whenever $t \geqslant 1/\sigma$. In particular, $t \geqslant 1/\sigma$ if $t > 1$. Since $g_0(-t) = \gamma_0 \neq 0$, all of the coefficients $g_n(-t) \neq 0$ for $t > 1$ and $n = 1, 2, 3, \ldots$. That is, all the zeros of g_n lie in $[-1, 0)$.

We next consider the case when $0 \leqslant \gamma_0 \leqslant \gamma_1 \leqslant \cdots$ and $\Phi(x)$ has a zero of order s at the origin. That is, we assume that $0 = \gamma_0 = \cdots = \gamma_{s-1}$ and $0 < \gamma_s \leqslant \gamma_{s+1} \leqslant \cdots$. Now for a fixed $\varepsilon > 0$ we set

$$\Phi_\varepsilon(x) = \Phi(x + \varepsilon) = \sum_{k=0}^{\infty} \frac{\gamma_k(\varepsilon)}{k!} x^k$$

and observe that (i) $\lim_{\varepsilon \to 0} \gamma_k(\varepsilon) = \gamma_k$ and (ii) $0 < \gamma_k(\varepsilon) < \gamma_{k+1}(\varepsilon)$ for $k = 0, 1, 2, \ldots$. But then property (ii), in conjunction with the proof given above, implies that for each $\varepsilon > 0$ the Jensen polynomial $\sum_{k=0}^{n} \binom{n}{k} \gamma_k(\varepsilon) x^k$, associated with $\Phi_\varepsilon(x)$, has all its zeros in the interval $[-1, 0)$. Finally, using property (i) we see that all the zeros of $g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$ lie in the closed interval $[-1, 0]$.

Conversely, suppose that all the zeros of $g_n(x)$, $n = 1, 2, 3, \ldots$, lie in $[-1, 0]$. If m denotes the smallest nonnegative integer such that $\gamma_m \neq 0$, then $\gamma_{m+k} \neq 0$ for $k = 0, 1, 2, \ldots$, since $\Phi(x)$ is by assumption a transcendental function of type I in the Laguerre-Pólya class. Thus by hypothesis, the polynomial

$$g_{m+1}(x) = x^m[(m+1)\gamma_m + \gamma_{m+1}x]$$

has all its zeros in $[-1, 0]$. Consequently, $\gamma_m < \gamma_{m+1}/(m+1)$. Also, for $n \geqslant m+2$, $d^{n-1}g_n(x)/dx^{n-1} = n!(\gamma_{n-1} + \gamma_n x)$. Thus, Rolle's theorem, together with a straightforward induction argument, implies that $\gamma_n \leqslant \gamma_{n+1}$ for $n = 0, 1, 2, \ldots$. This completes the proof of the theorem.

REMARK 2.5. If $\gamma_0 \neq 0$ and either Φ has an infinite number of zeros or Φ has a finite number of zeros but $\sigma > 1$, then it follows from (2.4) that Φ_1 is also a transcendental entire function (of type I). Consequently, for such functions Φ, we have $g_n(-t) \neq 0$ for $t \geqslant 1$; i.e. the zeros of each g_n all lie in the open interval $(-1, 0)$.

If Φ has a finite number of zeros, $\Phi(0) > 0$, and $\sigma = 1$, then

$$\Phi(x) = e^x h(x),$$

where the polynomial $h(x) = \sum_{k=0}^{N} b_k x^k/k!$ has only real negative zeros. Thus in this case for $t > 0$,

$$e^x\Phi(-xt) = e^x e^{-xt}h(-xt) = \sum \frac{g_n(-t)}{n!}x^n.$$

Hence for $t = 1$, we obtain $h(-x) = \sum g_n(-1)x^n/n!$. Thus, comparing coefficients yields

$$b_n = (-1)^n g_n(-1) \quad \text{for } n = 0, 1, 2, \ldots, N$$

and so

$$g_n(-1) = 0 \quad \text{if and only if } n > N.$$

That is, in general the zeros of g_n all lie in $[-1, 0)$.

In order to facilitate our extension of the Gauss-Lucas Theorem, we introduce the following definition.

DEFINITION. A sequence $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ of real numbers is said to possess the *Gauss-Lucas property* if it satisfies the following condition. Let $f(z) = \sum_{k=0}^{n} a_k z^k$, $a_k \in \mathbf{C}$, be an arbitrary complex polynomial. If K is a convex region containing the origin and all the zeros of $f(z)$, then the zeros of the polynomial $\Gamma[f(z)] = \sum_{k=0}^{n} a_k\gamma_k z^k$ also lie in K.

Next we give a complete characterization of sequences which possess the Gauss-Lucas property. In addition to the foregoing results, our proof requires the following classical composition theorem due to Schur [S] and Szegö [Sz] (see also Obreschkoff [**O**, p. 26]).

THEOREM 2.6 (THE SCHUR-SZEGÖ COMPOSITION THEOREM). *If the zeros of the polynomial*

$$A(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k$$

lie in a convex region K containing the origin and if the zeros of

$$B(x) = \sum_{k=0}^{n} \binom{n}{k} b_k x^k$$

all lie in the open interval $(-1, 0)$, *then all the zeros of the composite polynomial*

$$C(x) = \sum_{k=0}^{n} \binom{n}{k} a_k b_k x^k$$

also lie in K.

REMARK 2.7. In the preceding theorem we may assume K is closed since it must contain the convex hull of the origin and the zeros of $A(x)$. Since the roots of any polynomial vary continuously with the coefficients, the theorem remains true for polynomials $B(x)$ whose roots lie in the closed interval $[-1, 0]$.

THEOREM 2.8. *Let $\Gamma = \{\gamma_k\}$ be a nonzero sequence of real numbers. Then Γ possesses the Gauss-Lucas property if and only if Γ is a multiplier sequence of the first kind and either* $0 \leqslant \gamma_n \leqslant \gamma_{n+1}$ *for* $n = 0, 1, 2, \ldots,$ *or* $0 \geqslant \gamma_n \geqslant \gamma_{n+1}$ *for* $n = 0, 1, 2, \ldots$.

PROOF. Suppose that the real sequence $\Gamma = \{\gamma_k\}$ has the Gauss-Lucas property. For a nonnegative integer n let $f(x) = (1 + x)^n$ and set $K = [-1, 0]$. Then by hypothesis all the zeros of $\Gamma[f(x)] = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$ lie in K and a fortiori Γ is a multiplier sequence of the first kind. An application of Theorem 2.3 now provides the desired conclusion.

Conversely, suppose $\Gamma = \{\gamma_k\}$ is a multiplier sequence of the first kind. As usual we may assume all $\gamma_k \geqslant 0$ and $\gamma_n \leqslant \gamma_{n+1}$ for $n = 0, 1, 2, \ldots$. Thus the function

$$\Phi(x) = \Gamma[e^x] = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k = c x^s e^{\sigma x} \prod_{n=1}^{\omega} \left(1 + \frac{x}{x_n}\right), \qquad 0 \leqslant \omega \leqslant \infty,$$

where $c > 0$, $\sigma \geqslant 0$, $x_n > 0$, $\sum x_n^{-1} < \infty$ and s is a nonnegative integer, is an entire function of type I in the Laguerre-Pólya class. By Theorem 2.3, the zeros of the Jensen polynomials $g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$ all lie in $[-1, 0]$. Now let $f(z) = \sum_{k=0}^{n} a_k z^k$, $a_k \in \mathbf{C}$, be an arbitrary complex polynomial. Let K be a convex region containing the origin and all the zeros of $f(z)$. Then the Schur-Szegö Composition Theorem together with Remark 2.7 shows that the roots of $\Gamma[f(z)] = \sum_{k=0}^{n} a_k \gamma_k z^k$ also lie in the convex region K. This completes the proof of the theorem.

REMARK 2.9. As noted in the introduction, the classical example of this theorem is its application to the sequence $\Gamma = \{0, 1, 2, 3, \ldots\}$ since $\Gamma[f(x)] = xf'(x)$. For this reason we have had to include the origin as a point of the set K containing the roots of f. If one does not require the origin to be in K and considers the class of linear operators which are translation invariant, then differentiation is essentially the only example. I. Raitchinov [**R**] has shown that all such linear operators have the form $f(z) \mapsto cf^{(s)}(z)$ for some constant c.

3. Related results. In this section we look at several directions for extending the results of the previous section. We consider the extent to which we can replace the polynomial f by a transcendental entire function. Another direction is to introduce the sequence $\{\gamma_k\}$ in other ways such as in the sum $\sum \gamma_k z^k f^{(k)}(z)/k!$ in Corollary 3.7. In Theorem 3.3 we give a complete characterization of the sets K which satisfy a Gauss-Lucas type of theorem for all nonnegative multiplier sequences. We begin by examining the scope of Theorem 2.8 in relation to some known results. Let α and β be two nonnegative real numbers, $0 \leqslant \beta - \alpha \leqslant \pi$, and let K denote the angle

$$K = \{z \in \mathbf{C} \mid \alpha \leqslant \arg z \leqslant \beta\},$$

where this set is assumed to contain the origin. Let $p(z)$ be any polynomial all of whose zeros lie in K. Then it is known [**L**, p. 342] that for any multiplier sequence $\Gamma = \{\gamma_k\}$, $\gamma_k \geqslant 0$, the zeros of the polynomial $\Gamma[p(z)]$ also belong to K. In fact, this assertion remains valid for certain entire functions $f(z)$ all of whose zeros lie in the angle K (see, for example, [**L**, Theorem 8, p. 343]).

For general unbounded convex regions the situation is more complicated. However, for entire functions of genus zero, we obtain as a consequence of Theorem 2.8 the following corollary (cf. Marden [**M2**] and Porter [**P1**, **P2**]).

COROLLARY 3.1. *Let $\Gamma = \{\gamma_k\}$, $0 \leqslant \gamma_0 \leqslant \gamma_1 \leqslant \cdots$, be a multiplier sequence of the first kind. Let K be a closed unbounded convex region which contains the origin and all the zeros of the entire function*

$$\begin{aligned} f(z) &= \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right), \qquad |z_1| \leqslant |z_2| \leqslant \cdots, \\ &= \sum_{k=0}^{\infty} a_k z^k, \end{aligned}$$

where $\sum |z_n|^{-1} < \infty$. Then the zeros of the entire function $\Gamma[f(z)] = \sum_{k=0}^{\infty} a_k \gamma_k z^k$ also lie in K.

PROOF. By hypothesis $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0$ and by Lemma 2.2 we have $\lim_{n\to\infty} \sqrt[n]{\gamma_n} = \sigma$, $1 \leqslant \sigma < \infty$, so that $\Gamma[f]$ is an entire function of type σ (see, for example, [**B**, p. 11]). If

$$f_n(z) = \prod_{k=1}^{n}\left(1 - \frac{z}{z_k}\right), \qquad n = 1, 2, \ldots,$$

then by Theorem 2.8 the zeros of the polynomial $\Gamma[f_n]$, $n = 1, 2, \ldots,$ also lie in K. Now standard arguments show that the sequence $\{f_n(z)\}$ of polynomials converges uniformly, on compact subsets of $\mathbf{C}$, to $f(z)$. But then it is easy to infer that the sequence $\{\Gamma[f_n]\}$ of polynomials also converges uniformly, on compact subsets, to $\Gamma[f(z)]$. Since the convex region K is closed, Hurwitz's theorem implies that the zeros of $\Gamma[f(z)]$ also lie in K. This completes the proof of the corollary.

REMARK 3.2. The above corollary can also be extended to certain classes, $C(R)$, of R-functions introduced by Korevaar (see [**K**] and the references contained therein). An R-function is defined as follows. Let R be an unbounded closed set in the complex plane. Let $p(z)$ be an *R-polynomial*, that is, $p(z)$ is a polynomial all of whose zeros lie in R. Then an entire function f, $f \not\equiv 0$, is an *R-function* ($f \in C(R)$) if f is the uniform limit, on compact subsets of $\mathbf{C}$, of a sequence of R-polynomials.

Thus, for example, if R is the nonpositive portion of the real axis, then $C(R)$ is precisely the class of entire functions of type I, with nonnegative Taylor coefficients, in the Laguerre-Pólya class. We also note that Korevaar [**K**] obtained characterizations of $C(R)$ in terms of the geometrical properties of the set R.

Finally we remark that the question of extension of our results to real entire functions of less restricted growth but all whose zeros are real is still open (cf. Hellerstein and Korevaar [**HK**]).

THEOREM 3.3. *Let R be a closed subset of the complex plane. Consider the property*

(3.4) *For any nonnegative multiplier sequence Γ, if $f \in C(R)$, then $\Gamma[f] \in C(R)$.*

Then R satisfies (3.4) *if and only if R is an angle of the form $\{z \in \mathbf{C} \mid \alpha \leqslant \arg z \leqslant \beta\}$, where $0 \leqslant \beta - \alpha \leqslant \pi$.*

PROOF. Assume first that R is an angle as specified. As mentioned above, property (3.4) holds for polynomials in $C(R)$. A limiting argument as in Corollary 3.1 establishes (3.4) for entire functions in $C(R)$. Conversely, assume R satisfies (3.4). The fact that R must be an angle follows from the following two properties: (i) If $z_0 \in R$, then R contains the ray from the origin through z_0; (ii) R is convex. To establish property (i), we use the sequence $\Gamma = \{1, r, r^2, r^3, \ldots\}$, $r > 0$, corresponding to the entire function e^{rx}. Apply Γ to the polynomial $x - z_0$ to get $rx - z_0$ with root z_0/r which ranges over all points of the ray except the origin. To get the origin apply $\Gamma = \{0, 1, 2, \ldots\}$, the sequence corresponding to differentiation. This establishes (i). To prove (ii), let $z_1, z_2 \in R$, and let $0 < t < 1$. By (i) we know that the points $2tz_1$, and $2(1-t)z_2$ are in R, so the polynomial $(x - 2tz_1)(x - 2(1-t)z_2)$ belongs to $C(R)$. Again we use the sequence $\{0, 1, 2, 3, \ldots\}$ to conclude that $tz_1 + (1-t)z_2$ lies in R, finishing the proof of the theorem.

We conclude this section with some applications and extensions of the foregoing results. Our next theorem in this direction provides a simple but useful tool for constructing examples of sequences which satisfy the Gauss-Lucas property.

THEOREM 3.4. *Let $h(x) = \sum_{k=0}^{n} b_k x^k$ be a real polynomial with only real negative zeros. Let $\tilde{h}(x)$ denote the polynomial*

$$\tilde{h}(x) = \sum_{k=0}^{n} b_k x(x-1) \cdots (x-k+1). \tag{3.5}$$

Let $f(z) = \sum_{k=0}^{m} a_k z^k$, $a_k \in \mathbf{C}$, be an arbitrary complex polynomial and let K denote a convex region containing the origin and all the zeros of $f(z)$. Then all the zeros of the polynomials (a) *$A(z) = \sum_{k=0}^{m} a_k h(k) z^k$ and* (b) *$B(z) = \sum_{k=0}^{m} a_k \tilde{h}(k) z^k$ also lie in K.*

PROOF. (a) By a well-known theorem of Laguerre [**O**, p. 6] the sequence $\{h(k)\}_{k=0}^{\infty}$ is a multiplier sequence of the first kind. Since $b_k > 0$, $k = 0, 1, \ldots, n$, the sequence $\{h(k)\}_{k=0}^{\infty}$ is strictly increasing and consequently by Theorem 2.8 all the zeros of $A(z)$ also lie in the convex region K.

(b) A computation shows that $\Phi(x) = \sum_{k=0}^{\infty} \tilde{h}(k) x^k / k! = h(x) e^x$ and thus $\Phi(x)$ is a function of type I in the Laguerre-Pólya class (cf. [**CC2**, Corollary 11]). Hence $\{\tilde{h}(k)\}_{k=0}^{\infty}$ is a multiplier sequence of the first kind. By Lemma 2.2 the sequence

$\{\tilde{h}(k)\}_{k=0}^{\infty}$ is strictly increasing and consequently another appeal to Theorem 2.8 proves that all the zeros of $B(z)$ also lie in K.

REMARK 3.6. Let $\theta = z(d/dz)$. Then it is easy to verify that the polynomials $A(z)$ and $B(z)$ of Theorem 3.4 are given by $A(z) = h(\theta)f(z)$ and $B(z) = \tilde{h}(\theta)f(z)$. It is noteworthy that, while with the aid of the differential operator $\tilde{h}(\theta)$ we can extend the Gauss-Lucas theorem, the polynomial $\tilde{h}(x)$ defined by (3.5) need not have, in general, any real zeros (consider, for example, $h(x) = 1 + 2x + x^2$).

COROLLARY 3.7. *Let $\Phi(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k!$, $\gamma_k > 0$, be a function of type* I *in the Laguerre-Pólya class and let $f(z)$ be an arbitrary complex polynomial of degree m. If K is a closed convex region containing the origin and the zeros of $f(z)$, then all the zeros of the polynomial*

$$P(z) = \sum_{k=0}^{m} \frac{\gamma_k}{k!} z^k f^{(k)}(z)$$

also lie in K.

PROOF. Let $n > m$ and set $h_n(x) = g_n(x/n)$, where $g_n(x)$ is the Jensen polynomial associated with $\Phi(x)$. Then by Theorem 3.4(b) and Remark 3.6 the zeros of the polynomial $\tilde{h}_n(\theta)f(z)$ all lie in the convex region K. In order to obtain an explicit expression for the polynomial $\tilde{h}_n(\theta)f(z)$, we make use of the following known operational formula (see, for example, Riordan [**Ri**]):

$$\theta(\theta - 1) \cdots (\theta - k + 1) = \sum_{j=1}^{k} s(k, j)\theta^j = x^k \frac{d^k}{dx^k}, \tag{3.8}$$

where $\theta = x(d/dx)$ and where $s(k, j)$ denotes a Stirling number of the first kind. Thus with the aid of (3.8) we obtain that

$$\tilde{h}_n(\theta)f(z) = \sum_{k=0}^{m} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \frac{\gamma_k}{k!} z^k f^{(k)}(z).$$

Finally, letting n tend to infinity, we conclude that all the zeros of $P(z)$ also lie in the closed convex region K.

PROPOSITION 3.9. *Let $h(x) = \sum_{j=0}^{n} b_j x^j$ be a real polynomial with only real negative zeros. If the zeros of the real polynomial $f(x) = \sum_{k=0}^{m} a_k x^k$ all lie in the strip*

$$K = \{z \in \mathbf{C} \,|\, |\operatorname{Im} z| \leqslant M\}, \qquad M \geqslant 0,$$

then all the zeros of the polynomial

$$C(x) = \sum \frac{x^j}{j!} h^{(j)}(x) f^{(j)}(x)$$

also lie in K.

PROOF. Let $\Phi(x) = \Gamma[e^x]$, where $\Gamma = \{\tilde{h}(k)\}_{k=0}^{\infty}$ and $\tilde{h}(x) = \sum_{j=0}^{n} b_j x(x-1) \cdots (x - j + 1)$. Then, as in the proof of Theorem 3.4(b), we conclude that Γ is a strictly increasing multiplier sequence of the first kind. For each positive integer N

consider the polynomial

$$P_N(x) = \Gamma\left[\left(1+\frac{x}{N}\right)^N f(x)\right].$$

By Theorem 2.8 the zeros of $P_N(x)$ all lie in the strip K. Consequently, all the zeros of the entire function

$$\lim_{N\to\infty} P_N(x) = \Gamma[e^x f(x)] \tag{3.10}$$

also lie in K. Now in order to simplify the subsequent calculations we note that if $\tilde{f}(x) = \sum_{k=0}^m a_k x(x-1)\cdots(x-k+1)$, then $\tilde{f}(\theta)e^x = f(x)e^x$. But then a computation shows that

$$\begin{aligned}
\Gamma[e^x f(x)] &= \tilde{f}(\theta)\tilde{h}(\theta)e^x = \tilde{f}(\theta)\Phi(x) = \sum_{k=0}^m a_k x^k \Phi^{(k)}(x) \\
&= \sum_{k=0}^m a_k x^k \sum_{j=0}^k \binom{k}{j} h^{(j)}(x)e^x \\
&= e^x \sum_{j=0}^m \frac{h^{(j)}(x)}{j!} \sum_{k=j}^m a_k x^k \frac{k!}{(k-j)!} \\
&= e^x \sum_{j=0}^m \frac{x^j}{j!} h^{(j)}(x) f^{(j)}(x).
\end{aligned}$$

Thus by (3.10) all the zeros of the polynomial

$$e^{-x}\Gamma[e^x f(x)] = \sum \frac{x^j}{j!} h^{(j)}(x) f^{(j)}(x)$$

also lie in K. This completes the proof of the proposition.

Proposition 3.9 seems to be new even in the special case when all the zeros of $f(x)$ are real.

THEOREM 3.11. *Let $\Phi(x) = \sum_{k=0}^\infty \gamma_k x^k/k!$, $\gamma_k > 0$, be a function of type* I *in the Laguerre-Pólya class. Let*

$$\Psi(x) = ce^{-\alpha x^2 + \beta x} x^m \prod_{n=1}^\infty \left(1 - \frac{x}{x_n}\right) e^{x/x_n},$$

where $\alpha \geqslant 0$, c, β and x_n are real, $\sum_{n=1}^\infty x_n^{-2} < \infty$ and m is a nonnegative integer. Let $\Psi_1(x) = \Psi(x)f(x)$, where $f(x)$ is a real polynomial all of whose zeros lie in the strip

$$K = \{z \in \mathbf{C} \,|\, |\operatorname{Im} z| \leqslant M\}, \qquad M \geqslant 0.$$

Then all the zeros of the entire function

$$\sum_{k=0}^\infty \frac{\gamma_k}{k!} x^k \Psi_1^{(k)}(x)$$

also lie in K.

The proof of Theorem 3.11 is left to the reader, since it involves only minor modification of the methods used above.

4. Properties of multiplier sequences. In §2 we saw that increasing multiplier sequences are in fact very special sequences. In this section we examine these sequences more closely and expand the results as far as possible to all multiplier sequences.

REMARK 4.1. In the sequel we will need the following facts concerning entire functions $\Phi(x)$ of type I in the Laguerre-Pólya class. If $\Phi(x) = cx^s e^{\sigma x}(1 + x/x_n)$ is a function of type I in the Laguerre-Pólya class, then the derivative $\Phi'(x) = cx^{s-1}e^{\sigma' x}\prod(1 + x/x_n')$ belongs to the same class (see, for example, [**PS**]). Moreover, it is known that $\sigma = \sigma'$ [**HW**, see footnote on p. 107].

DEFINITION. For any real sequence $\{\gamma_k\}_{k=0}^{\infty}$ we define $\Delta^0\gamma_p = \gamma_p$, $\Delta\gamma_p = \gamma_{p+1} - \gamma_p$ and $\Delta^n\gamma_p = \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\gamma_{p+j}$ for n, $p = 0, 1, 2, \ldots$.

PROPOSITION 4.2. *Let* $\Phi(x) = \sum_{k=0}^{\infty}\gamma_k x^k/k!$, $0 \leqslant \gamma_0 \leqslant \gamma_1 \leqslant \cdots$, *be a transcendental entire function of type* I *in the Laguerre-Pólya class. Then*

$$\Delta^n\gamma_p \geqslant 0, \qquad n, p = 0, 1, 2, \ldots.$$

PROOF. Since $0 \leqslant \gamma_0 \leqslant \gamma_1 \leqslant \cdots$, it follows from Remark 4.1 and Lemma 2.2 that for each nonnegative integer p, the type of $\Phi^{(p)}(x)$ is $\sigma_p \geqslant 1$. Hence, for each $t \geqslant -1$ the function

$$e^{xt}\Phi^{(p)}(x) = \sum_{n=0}^{\infty}\frac{g_{n,p}^*(t)}{n!}x^n,$$

where

$$g_{n,p}^*(t) = \sum_{k=0}^{n}\binom{n}{k}\gamma_{k+p}t^{n-k},$$

is a function of type I in the Laguerre-Pólya class. Moreover, for each fixed t, $t \geqslant -1$, the zeros of $e^{xt}\Phi^{(p)}(x)$ are all real and nonpositive. Consequently, the inequalities $g_{n,p}^*(t) \geqslant 0$ hold for $t \geqslant -1$ and n, $p = 0, 1, 2, \ldots$. In particular, for $t = -1$ we obtain

$$\Delta^n\gamma_p = g_{n,p}^*(-1) \geqslant 0, \qquad n, p = 0, 1, 2, \ldots.$$

REMARK 4.3. Let $\Phi(x) = \sum\gamma_k x^k/k!$, $0 \leqslant \gamma_0 \leqslant \gamma_1 \leqslant \cdots$, be a transcendental entire function of type I in the Laguerre-Pólya class. Then by Proposition 4.2 the function

$$\Phi_p(x) = \sum_{n=0}^{\infty}\Delta^n\gamma_p\frac{x^n}{n!}, \qquad p = 0, 1, 2, \ldots,$$

is also a member of this class. We remark that by Lemma 2.2 the functions $\sum_{p=0}^{\infty}\Delta^n\gamma_p x^p/p! = e^x d^n(e^x\Phi(x))/dx^n$, where $n = 0, 1, 2, \ldots$, are also entire functions of type I in the Laguerre-Pólya class. Moreover, it follows from Proposition 4.2 that the sequence $\{\Delta^n\gamma_p\}_{p=0}^{\infty}$ is an increasing sequence for each fixed nonnegative integer n.

PROPOSITION 4.4. *Let $\Gamma = \{\gamma_k\}$ be a multiplier sequence. Then one of the following holds*:

(i) $\{|\gamma_k|\}$ *is monotone increasing.*

(ii) *There exists an $m \geqslant 0$ such that $|\gamma_0|, |\gamma_1|, \ldots, |\gamma_m|$ is monotone increasing and $|\gamma_{m+1}|, |\gamma_{m+2}|, \ldots$ is monotone decreasing.*

PROOF. Without loss of generality, we may assume all $\gamma_k \geqslant 0$. Assume that $\gamma_m > \gamma_{m+1}$. From Turán's inequality, $\gamma_{m+1}^2 \geqslant \gamma_m\gamma_{m+2}$, we obtain $\gamma_{m+2} \leqslant \gamma_{m+1}^2/\gamma_m < \gamma_{m+1}$ unless $\gamma_{m+1} = \gamma_{m+2} = 0$. Thus once a sequence begins to decrease, it must continue to do so. The proposition follows.

LEMMA 4.5. *Assume $\{\gamma_k\}$ is a positive multiplier sequence. For each k, let $r_k = \gamma_k/\gamma_{k-1}$. Then the sequence $r_1, r_2, \ldots,$ is a decreasing sequence.*

PROOF. By Turán's inequality, $\gamma_{k+1} \leqslant \gamma_k^2/\gamma_{k-1} = r_k^2\gamma_{k-1}^2/\gamma_{k-1} = r_k\gamma_k$ which implies that $r_k \geqslant r_{k+1}$.

We shall call a sequence $\{\gamma_k\}$ *convex* if the function $f(k) = \gamma_k$ is a convex function; i.e. $\gamma_{k+1} - \gamma_k \geqslant \gamma_k - \gamma_{k-1}$ for all k.

THEOREM 4.6. *Let $\Gamma = \{\gamma_k\}$ be a multiplier sequence. Then there exists $m \geqslant 0$ such that $\{|\gamma_{k+m}|\}$ is convex. If $\{|\gamma_k|\}$ is monotone increasing, we may take $m = 0$.*

PROOF. Again we may assume that Γ is a nonnegative sequence. Assume first that we are in case (ii) of the previous proposition. We may assume that Γ is monotone decreasing since we are only concerned with the tail end of the sequence. If $\gamma_k = 0$ for all large k, we are done; so assume all $\gamma_k > 0$. Let $r_k = \gamma_k/\gamma_{k-1}$ as in the previous lemma. Let $s = \lim r_k$, which exists since the numbers r_k are decreasing and positive. Since Γ is decreasing, we have $0 \leqslant s < 1$. Since $(s+1)/2 > s$, we can find n such that for $k > n$, we have $r_k < (s+1)/2$. Letting $0 < \varepsilon < (1-s)^2/2$, there exists an $m > n$ such that for $k > m$, $|r_k - s| < \varepsilon$. In particular, we have $\gamma_k - s\gamma_{k-1} < \varepsilon\gamma_{k-1}$ and $s\gamma_k - \gamma_{k+1} \leqslant 0$. Adding and rearranging we obtain

$$\begin{aligned}\gamma_k - \gamma_{k+1} &< s(\gamma_{k-1} - \gamma_k) + \gamma_{k-1}(1-s)^2/2 \\ &< \gamma_{k-1}\left[s(1 - \gamma_k/\gamma_{k-1}) + (1-s)(1 - \gamma_k/\gamma_{k-1})\right] \\ &= \gamma_{k-1} - \gamma_k,\end{aligned}$$

since $r_k = \gamma_k/\gamma_{k-1} < (s+1)/2$ implies $(1-s)/2 < 1 - \gamma_k/\gamma_{k-1}$. Thus the sequence $\{\gamma_{k+m}\}$ is convex.

Now assume that we are in case (i). By Proposition 4.2, we have $\Delta^2\gamma_k \geqslant 0$ for $k = 0, 1, 2, \ldots$. But this is precisely the statement that Γ is a convex sequence.

LEMMA 4.7. *Let $\Phi(x) = \sum_{k=0}^{\infty}\gamma_k x^k/k!$ be an entire function of type* I *in the Laguerre-Pólya class where each $\gamma_k \geqslant 0$ and $\Phi(x)$ does not have the form $ce^{\sigma x}$ with $\sigma \leqslant 1$. Then, for sufficiently large n, the associated Jensen polynomial $g_n(x)$ has a root in $(-1, 0]$.*

PROOF. By hypothesis, either $\Phi(x) = ce^{\sigma x}$ with $\sigma > 1$ or Φ has a zero r in $(-\infty, 0]$. In the former case, $g_n(x) = c(1 + \sigma x)^n$ and the conclusion holds. Assume we are in the latter case. Since the polynomials $g_n(x/n)$ converge uniformly to Φ on compact subsets, for all sufficiently large n, the polynomial $g_n(x/n)$ has a nonpositive root r_n such that $|r_n - r| < 1$. But then r_n/n is a root of g_n and, if we also have $n > \max(2, 2|r|)$, then r_n/n lies in $(-1, 0]$.

We are now ready to prove the main theorem of this section. It classifies multiplier sequences into one of four categories. For clarity we have stated it only for nonnegative sequences, assuming that the relatively simple changes needed for nonpositive and alternating sequences can be easily supplied by the reader.

THEOREM 4.8. *Let* $\Gamma = \{\gamma_k\}$ *be a multiplier sequence of nonnegative elements,* $\Phi(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k!$ *and* $g_n(x) = \sum \binom{n}{k}\gamma_k x^k$ *for* $n = 0, 1, 2, \ldots$. *Then exactly one of the following conditions holds*:

(i) *There exists* $r \geqslant 0$ *such that* $0 = \gamma_0 = \gamma_1 = \cdots = \gamma_{r-1} < \gamma_r$ *and* $\gamma_r, \gamma_{r+1}, \ldots$ *is a strictly increasing convex sequence. In this case each* g_n *has all of its roots in* $(-1, 0]$. *If* $r = 0$, *the roots lie in* $(-1, 0)$.

(ii) Γ *is a constant sequence. In this case* $g_n(x) = \gamma_0(x + 1)^n$.

(iii) *There exist integers* r, s, t *with* $0 \leqslant r \leqslant s \leqslant t$ *such that* $0 = \gamma_0 = \gamma_1 = \cdots = \gamma_{r-1} < \gamma_r$, *the sequence* $\gamma_r, \gamma_{r+1}, \ldots, \gamma_{s-1}$ *is strictly increasing, the sequence* $\gamma_s, \gamma_{s+1}, \ldots$ *is strictly decreasing and the sequence* $\gamma_t, \gamma_{t+1}, \ldots$ *is convex. For sufficiently large* n, *the polynomials* g_n *have at least one root in* $(-\infty, -1)$. *If* $\Phi(x) \neq ce^{\sigma x}$ $(\sigma < 1)$, *then* g_n *also has at least one root in* $(-1, 0]$.

(iv) *There exist integers* r, s, t *with* $0 \leqslant r \leqslant s \leqslant t$ *such that* $0 = \gamma_0 = \gamma_1 = \cdots = \gamma_{r-1} < \gamma_r$, *the sequence* $\gamma_r, \gamma_{r+1}, \ldots, \gamma_{s-1}$ *is strictly increasing, the sequence* $\gamma_s, \gamma_{s+1}, \ldots, \gamma_t$ *is strictly decreasing and* $\gamma_{t+1} = \gamma_{t+2} = \cdots = 0$.

PROOF. If Γ is constant, it is clear that g_n has the form $\gamma_0(x + 1)^n$. Assume Γ is not constant. By Proposition 4.4, there exist integers r, s, t with $0 \leqslant r \leqslant s \leqslant t$ such that $0 = \gamma_0 = \cdots = \gamma_{r-1} < \gamma_r$, the sequence $\gamma_r, \gamma_{r+1}, \ldots, \gamma_{s-1}$ is monotone increasing and $\gamma_s, \gamma_{s+1}, \ldots$ is either monotone increasing or decreasing. If Γ is increasing, then, being nonconstant, it is strictly increasing from the rth term onward by Lemma 2.1. In this case the roots of the polynomials g_n lie in the right place by Theorem 2.3 and Remark 2.5. We are left with the cases in which Γ eventually decreases or becomes zero. The sequence cannot have three successive equal terms by **[CC3**, Proposition 4.5] (cf. proof of Lemma 2.1(a) above). If two successive terms are equal, say $\gamma_k = \gamma_{k+1}$, then Lemma 4.5 implies $\gamma_{k+2} < \gamma_{k+1}$ and the sequence must strictly decrease beyond this point unless it reaches zero. Thus we are either in case (iii) or (iv). The fact that the roots in case (iii) are as stated follows from Theorem 2.3 and Lemma 4.7.

EXAMPLE 4.9. In the previous section we found that certain theorems are true for entire functions of the form $\prod_{n=1}^{\infty}(1 + z/z_n)$ because they are limits of polynomials with zeros in the convex hull of the z_n's (in fact equal to the z_n's). On the other hand, all of the entire functions we have been considering are limits of associated Jensen polynomials $g_n(x/n)$. Entire functions of the above form correspond to multiplier sequences in case (iii) of Theorem 4.8 and for large n the polynomial g_n has a root r_n in $(-\infty, -1)$. That is nr_n is a root of $g_n(x/n)$ which runs to infinity as $n \to \infty$ rather than converging to a zero of the entire function.

REMARK 4.10. Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be an increasing sequence. In Remark 4.3 we noted that for each n, the sequence $\{\Delta^n \gamma_k\}_{k=0}^{\infty}$ is again an increasing multiplier sequence. Since, for $n = 2$, this says $\{\gamma_k\}$ is convex, this can be viewed as a very strong convexity property. On the other hand, if $\{\gamma_k\}$ is a decreasing positive sequence, then it eventually becomes convex and a similar argument works for the differences. In general, one must move out farther and farther in the difference sequences to find convexity, however. For example, consider $\{(1 + 2k)2^{-k}\}_{k=0}^{\infty}$ corresponding to the entire function $e^{x/2}(1 + x)$. To obtain multiplier sequences, we must drop one term each time we take differences: $\{\Delta^n \gamma_k\}_{k=n}^{\infty}$ is a multiplier sequence for each $n = 0, 1, 2, \ldots$, and fails to be a multiplier sequence if we include $\Delta^n \gamma_{n-1}$ as the leading term. (Note also that these alternate between positive and negative sequences.)

We thus have somewhat weaker conditions on decreasing sequences. On the other hand, any multiplier sequence can be made increasing as follows: given $\Phi(x) = \sum \gamma_k x^k / k!$ with $\gamma_k \geqslant 0$, form $e^x \Phi(x) = \sum g_n(1) x^n / n!$. This entire function has $\sigma \geqslant 1$, hence corresponds to an increasing sequence by Lemma 2.2. The new coefficients are $g_n(1) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k$.

It would be extremely valuable to have a complete characterization of functions in the Laguerre-Pólya class in terms of their coefficients. As a step in this direction, one can ask about the converse to Proposition 4.2. That is, if $\Gamma = \{\gamma_k\}$ is a sequence satisfying $\Delta^n \gamma_p \geqslant 0$ for all $n, p \geqslant 0$, is Γ a multiplier sequence? It is not hard to construct counterexamples to this. In fact, even if we add the conditions that all of the sequences of differences satisfy Turán's inequality, i.e. $(\Delta^n \gamma_{k+1})^2 \geqslant \Delta^n \gamma_k \Delta^n_{k+2}$ for all $n, k \geqslant 0$, then we still do not necessarily have a multiplier sequence. As an example, consider the sequence $\{6 + 5k/2 + 25k^2/2\}_{k=0}^{\infty}$. Since this corresponds to the entire function $(6 + 15x + 25x^2/2)e^x$, which has two nonreal roots, it cannot be a multiplier sequence. In fact, if one applies the first five terms to $(x + 1)^4$, the resulting polynomial has only real roots; but applying the first six terms to $(x + 1)^5$ gives a polynomial with two nonreal roots. On the other hand, $\Delta^3 \gamma_k = 0$ for all k, so it is easy to check that all differences are nonnegative and Turán's inequality is satisfied.

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