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COCYCLES AND LOCAL PRODUCT DECOMPOSITION

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Dedicated to Professor Haruo Sunouchi on his 60th birthday

ABSTRACT As an application of cocycles, we establish a relation between the classical Hardy spaces on the real line R and simply invariant subspaces on a quotient of the Bohr group. When this result is specialized suitably, it yields the well-known results concerning the elements of invariant subspaces. We also study, by using Gamelin's representation theorem, unitary functions which are the values of cocycles.

1. Preliminaries. Let K be a compact abelian group, not a circle, dual to a subgroup Γ of the discrete real line R_d. For each t in R, e_t is the element of K defined by $e_t(\lambda) = e^{i\lambda t}$ for all λ in Γ. Choose and fix a positive γ in Γ, and let K_γ be the compact subgroup consisting of all x in K such that $x(\gamma) = 1$. Then K may be identified measure-theoretically, and almost topologically, with $K_\gamma \times [0, 2\pi/\gamma)$ via the mapping $y + e_s$ to (y, s). We suppose for simplicity that 2π lies in Γ in this section, and §§2 and 4. Thus K may be regarded as $K_{2\pi} \times [0, 1)$. Let σ and σ_1 be the normalized Haar measures on K and $K_{2\pi}$, respectively. Then we may consider $d\sigma = d\sigma_1 \times dt$ on $K_{2\pi} \times [0, 1)$.

Our objective in this note, by using this local product decomposition, is to show the fact that a certain class of analytic functions on $K_{2\pi} \times R$ has a close connection to simply invariant subspaces on K. In the next section, our characterization of simply invariant subspaces, Theorem 2.1, is obtained. In §3 we investigate the values of cocycles and answer a question of Helson. We close with some remarks in §4.

For any simply invariant subspace $\mathfrak{M}$ of $L^2(\sigma)$, we define

$$(\mathfrak{M})_+ = \bigcap_{\lambda<0} \chi_\lambda \mathfrak{M} \quad \text{and} \quad (\mathfrak{M})_- = \text{the closure} \bigcup_{\lambda>0} \chi_\lambda \mathfrak{M},$$

where χ_λ denotes the character on K determined by λ in Γ. Then $\mathfrak{M}$ is called to be *normalized* if $\mathfrak{M} = (\mathfrak{M})_+$. Complex-valued functions of modulus one are said to be *unitary functions*. A *cocycle* is a unitary Borel function $A(x, t)$ on $K \times R$ which satisfies the cocycle identity

$$A(x, t + u) = A(x, t)A(x + e_t, u) \tag{1.1}$$

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for all x in K and s, t, u in R. A cocycle is *trivial* (resp. a *coboundary*) if it has the form $e^{irt}p(x)\overline{p(x+e_t)}$ (resp. $p(x)\overline{p(x+e_t)}$) for some r in R and some unitary function p on K. There exists a one-to-one correspondence between normalized simply invariant subspaces and cocycles [**4**, Chapter 2].

We denote by $H^p(\sigma)$ and $H^p(dt)$, $0 < p \leq \infty$, the usual Hardy spaces on K and R, respectively. It is known that $H^1(dt)$ is the space of all functions in $L^1(dt)$ whose Fourier transforms vanish on the negative real line. We let $H^\infty(dt/(1+t^2)) = H^\infty(dt)$, that is, the space of all the boundary functions of bounded analytic functions in the upper half-plane. The closure of $H^\infty(dt/(1+t^2))$ in $L^p(dt/(1+t^2))$ is denoted by $H^p(dt/(1+t^2))$, $0 < p < \infty$. Recall that the class of continuous function ϕ in $H^p(dt)$ with $|\phi(t)| = O(t^{-2})$ (as $|t| \to \infty$) is dense in $H^p(dt)$, $0 < p < \infty$ (cf. [**3**, Chapter II, §3]). Also recall that ϕ lies in $H^p(dt/(1+t^2))$ if and only if $\phi(t)(t+i)^{-2/p}$ lies in $H^p(dt)$.

We refer the reader to [**4** and **2**, Chapter VII] for further details of analyticity on compact abelian groups and to [**1** and **3**] for results about classical Hardy spaces.

The following lemma is a minor variation of [**4**, Theorem 17] so the proof will be omitted.

LEMMA 1.1. *Let $\mathcal{M}_A$ be the normalized simply invariant subspace of $L^2(\sigma)$ associated with a cocycle A. Then for any f in $L^\infty(\sigma)$, the following are equivalent*:

(i) *f lies in $\mathcal{M}_A$*;

(ii) *the function of t, $A(y,t)f(y+e_t)$, lies in $H^\infty(dt/(1+t^2))$ for σ_1-a.a. y in $K_{2\pi}$; and*

(iii) *the function of t, $\overline{A(y,t)}\,\overline{f(y+e_t)}$, is orthogonal to $H^1(dt)$ for σ_1-a.a. y in $K_{2\pi}$.*

We next consider certain spaces of analytic functions on $K_{2\pi} \times R$. Let $\mathcal{H}$ be the space of all bounded Borel functions $f(y,t)$ on $K_{2\pi} \times R$ which satisfy

(1.2) the function of t, $f(y,t)$, belongs to $H^1(dt)$ for σ_1-a.a. y in $K_{2\pi}$, and

(1.3) $\text{ess.sup}\{|f(y,t)|;\ (y,t) \text{ in } K_{2\pi} \times [n, n+1)\} = O(n^{-2})$.

We denote by $\mathcal{H}^p$, $0 < p < \infty$, the closure of $\mathcal{H} \cap L^p(d\sigma_1 \times dt)$ in $L^p(d\sigma_1 \times dt)$, where we use the ordinary metric on $L^p(d\sigma_1 \times dt)$ when $0 < p < 1$. Let $B(y,t)$ be a unitary function on $K_{2\pi} \times R$. Then for each f in $\mathcal{H}$, we define a bounded Borel function $\Phi_B(f)$ on K by

$$\Phi_B(f)(y,s) = \sum_{n=-\infty}^{\infty} \overline{B(y-e_n, s+n)}\, f(y-e_n, s+n) \tag{1.4}$$

for each (y,s) in $K_{2\pi} \times [0,1)$. Then Φ_B is a linear mapping of $\mathcal{H}$ into $L^\infty(\sigma)$. Moreover, for any p, $0 < p \leq 1$, it can be easily seen that the restriction of Φ_B to $\mathcal{H} \cap L^p(d\sigma_1 \times dt)$ may be extended to a bounded linear mapping of $\mathcal{H}^p$ into $L^p(\sigma)$ (cf. [**7**, Lemma 1]).

2. Cocycles and the space $\mathcal{H}$. We may now state our main result.

THEOREM 2.1. *Let $\mathcal{M}_A$ be the simply invariant subspace of $L^2(\sigma)$ associated with a cocycle A. Then $\Phi_A(\mathcal{H})$ is dense in $(\mathcal{M}_A)_-$.*

PROOF. We first note that, for each $y = (y,0)$ in $K_{2\pi} \times [0,1)$, $y + e_t = (y,0) + e_t = (y + e_{[t]}, t - [t])$, where $[t]$ is the largest integer not exceeding t. It follows from the cocycle identity (1.1) that

$$A(y,t)\overline{A(y - e_{n-[t]}, n + t - [t])} = A(y,t)\overline{A(y - e_{n-[t]}, n - [t])}\ \overline{A(y,t)} = \overline{A(y - e_{n-[t]}, n - [t])}$$

for each (y, t) in $K_{2\pi} \times R$. Hence if $f(y,t)$ lies in $\mathcal{H}$, then we obtain

$$A(y,t)\Phi_A(f)(y + e_t) = A(y,t)\Phi_A(f)(y + e_{[t]}, t - [t]) = \sum_{n=-\infty}^{\infty} \overline{A(y - e_{n-[t]}, n - [t])}\, f(y - e_{n-[t]}, n + t - [t])$$

by (1.4). Let ϕ be any function in $H^1(dt)$. Then the property (1.3) assures that

$$\begin{aligned}
&\int_{-\infty}^{\infty} A(y,t)\Phi_A(f)(y + e_t)\phi(t)\,dt \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_k^{k+1} \overline{A(y - e_{n-[t]}, n - [t])}\, f(y - e_{n-[t]}, n + t - [t])\phi(t)\,dt \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_k^{k+1} \overline{A(y - e_{n-k}, n - k)}\, f(y - e_{n-k}, t + n - k)\phi(t)\,dt \\
&= \sum_{m=-\infty}^{\infty} \sum_{n-k=m} \int_k^{k+1} \overline{A(y - e_{n-k}, n - k)}\, f(y - e_{n-k}, t + n - k)\phi(t)\,dt \\
&= \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_k^{k+1} \overline{A(y - e_m, m)}\, f(y - e_m, t + m)\phi(t)\,dt \\
&= \sum_{m=-\infty}^{\infty} \overline{A(y - e_m, m)} \int_{-\infty}^{\infty} f(y - e_m, t + m)\phi(t)\,dt = 0
\end{aligned}$$

for σ_1-a.a. y in $K_{2\pi}$. Therefore, by Lemma 1.1, $\Phi_A(f)$ belongs to $\mathfrak{M}_A$. Let p be a function in $\mathfrak{M}_A$ which is orthogonal to $\Phi_A(\mathcal{H})$. We set $g(y,t) = p(y + e_t)$. Then it can be seen that

$$\int_K \overline{p(x)}\,\Phi_A(f)(x)\,d\sigma(x) = \iint_{K_{2\pi}\times R} \overline{g(y,t)}\ \overline{A(y,t)}\, f(y,t)\,d\sigma_1(y) \times dt = 0$$

for each f in $\mathcal{H}$. This implies that the function of t, $\overline{A(y,t)}\,\overline{p(y + e_t)}$, lies in $H^2(dt/(1 + t^2))$ for σ_1-a.a. y in $K_{2\pi}$. On the other hand, since the function of t, $A(y,t)p(y + e_t)$, lies in $H^2(dt/(1 + t^2))$, it must be constant. Hence $|p|$ is constant on K. From this, we may assume that p is a unitary function on K. Thus we have $A(x,t) = p(x)\overline{p(x + e_t)}$ and $\mathfrak{M}_A = pH^2(\sigma)$. This completes the proof.

We collect some corollaries following from Theorem 2.1. Recall that a cocycle $A(x,t)$ is *continuous* if $A(y,t)$ is continuous on $K_{2\pi} \times R$ as a function of (y,t) [**4**, Chapter 5]. Let $C_0(K_{2\pi} \times R)$ denote the space of all continuous functions on $K_{2\pi} \times R$ which vanish at infinity. We notice that $\mathcal{H} \cap C_0(K_2 \times R)$ is dense in $\mathcal{H}$ as a subspace of $L^1(d\sigma_1 \times dt)$, and that $\Phi_A(f)$ lies in $C(K)$ for any f in

$$\mathcal{H} \cap C_0(K_{2\pi} \times R).$$

These facts easily imply the following.

COROLLARY 2.2. *Let A and $\mathfrak{M}_A$ be as in Theorem* 2.1. *If A is continuous, then $(\mathfrak{M}_A)_- \cap C(K)$ is dense in $(\mathfrak{M}_A)_-$.*

We next give another proof of Helson's existence theorem [**4**, Theorem 16; **8**, **10**].

COROLLARY 2.3. *Let A and $\mathfrak{M}_A$ be as in Theorem* 2.1. *Then $\mathfrak{M}_A$ contains a unitary function.*

PROOF. Define a function $w(t)$ in $L^1(dt)$ by

$$w(t) = \begin{cases} n^{-2} & \text{on } [n, n+1), |n| \geq 1, \\ 3\sum_{j=1}^{\infty} j^{-2} & \text{on } [0,1). \end{cases}$$

It is easy to see that $\log w(t)$ belongs to $L^1(dt/(1+t^2))$. Hence there is a function ϕ in $H^1(dt)$ such that $|\phi(t)| = w(t)$ (cf. [**3**, Chapter II, Theorem 4.4]). If we set $f(y,t) = \phi(t)$, then $\Phi_A(f)$ lies in $\mathfrak{M}_A$ and $|\Phi_A(f)| \geq \sum_{j=1}^{\infty} j^{-2}$ on K. Thus it follows from Szegö's theorem that $\mathfrak{M}_A$ contains a unitary function.

COROLLARY 2.4. *Let A and $\mathfrak{M}_A$ be as in Theorem* 2.1. *Then there exists a function f in $\mathfrak{M}_A$ satisyfing*:

(i) *the function of t, $f(y+e_t)$, can be extended analytically to $\{z; \operatorname{Im} z > -\sqrt{3}/2\}$ for each σ_1-a.a. y in $K_{2\pi}$*;

(ii) $\log|f(x)|$ *belongs to $L^1(\sigma)$; and*

(iii) *for any positive λ in Γ, f does not lie in $\chi_\lambda \mathfrak{M}_A$.*

PROOF. Let $h(y,t) = 4/(2t - 1 + \sqrt{3}\,i)^2$. Then $|h(y,t)| \geq 1$ on $K_{2\pi} \times [0,1]$, and $|h(y,t)| < 1$ otherwise. So by Theorem 2.1 we may choose an integer m such that the function $g_1 = \Phi_A(h^m)$ in $\mathfrak{M}_A$ satisfies (i) and $|g_1(y,s)| > 1$ on $K_{2\pi} \times [\frac{1}{6}, \frac{5}{6})$. It may be assumed that g_1 has property (iii). Similarly, we can construct a function g_2 in $\mathfrak{M}_A$ which satisfies (i) and $|g_2(y,s)| > 1$ on $K_{2\pi} \times \{[0, \frac{2}{6}) \cup (\frac{4}{6}, 1)\}$. It follows from Jensen's inequality and Fubini's theorem that

$$\frac{1}{2\pi}\int_K \int_0^{2\pi} \log|g_1(x) + e^{i\theta} g_2(x)|\, d\sigma(x)\, d\theta$$
$$\geq \int_K \max(\log|g_1(x)|, \log|g_2(x)|)\, d\sigma(x) \geq 0.$$

Thus there exists a θ in $[0, 2\pi)$ for which $f = g_1 + e^{i\theta} g_2$ has the desired properties.

The above corollary is the main step in the proof of [**4**, Theorem 26].

We may easily choose a unitary function B on $K_{2\pi} \times R$ such that the closure $[\Phi_B(\mathcal{H})]_2$ of $\Phi_B(\mathcal{H})$ in $L^2(\sigma)$ is a doubly invariant subspace, so it is worthwhile to note a condition under which $[\Phi_B(\mathcal{H})]_2$ is simply invariant.

THEOREM 2.5. *Let B be a unitary function on $K_{2\pi} \times R$. Then $[\Phi_B(\mathcal{H})]_2$ is simply invariant if and only if there exists a cocycle $A(x, t)$ for which*

(2.1) *the function of t, $A(y, t)\overline{B(y, t)}$, belongs to $H^\infty(dt/(1 + t^2))$ for σ_1-a.a. y in $K_{2\pi}$.*

PROOF. Suppose that there exists a cocycle A with property (2.1). Then $A\bar{B}f$ lies in $\mathcal{H}$ for each f in $\mathcal{H}$. Since $\Phi_B(f) = \Phi_A(A\bar{B}f)$, it follows from Theorem 2.1 that $[\Phi_B(\mathcal{H})]_2$ is a simply invariant subspace. Conversely, suppose that $[\Phi_B(\mathcal{H})]_2$ is simply invariant, and let A be the cocycle of $([\Phi_B(\mathcal{H})]_2)_+$. Let ϕh be a function in $\mathcal{H}$ which is the product of a function ϕ in $H^1(dt)$ times a function h in $C(K_{2\pi})$. We notice that

$$A(y, t) = A(y, [t] - n)A(y + e_{[t]-n}, t - [t] + n)$$

and

$$\begin{aligned}\Phi_B(\phi h)(y + e_t) &= \Phi_B(\phi h)(y + e_{[t]}, t - [t]) \\ &= \sum_{n=-\infty}^{\infty} \overline{B(y + e_{[t]-n}, t - [t] + n)}\,\phi(t - [t] + n)h(y + e_{[t]-n}).\end{aligned}$$

It follows from Lemma 1.1 and an argument similar to the proof of Theorem 2.1 that

$$\begin{aligned}&\int_{-\infty}^{\infty} A(y, t)\Phi_B(\phi h)(y + e_t)\psi(t)\,dt \\ &\quad = \sum_{m=-\infty}^{\infty} A(y, -m)h(y - e_m) \\ &\qquad \times \int_{-\infty}^{\infty} A(y - e_m, t + m)\,\overline{B(y - e_m, t + m)}\,\phi(t + m)\psi(t)\,dt \\ &\quad = 0\end{aligned}$$

for each ψ in $H^1(dt)$. Since h is arbitrary in $C(K_{2\pi})$, we have

$$\int_{-\infty}^{\infty} A(y - e_m, t + m)\,\overline{B(y - e_m, t + m)}\,\phi(t + m)\psi(t)\,dt = 0$$

for each integer m. This implies that A satisfies (2.1).

3. Cocycles and unitary functions. Let $A(x, t)$ be a cocycle on K. In [**5**, §4], Helson has shown that if the function of x, $A_u(x) = A(x, u)$, lies in $H^2(\sigma)$, then it must be constant. This odd result grew out of a basic problem concerning spectrum of cocycles. We provide, by using Gamelin's representation theorem, some remarks on this theorem.

In this section we do not assume 2π belongs to Γ. Let K_γ be as in §1 for a positive γ in Γ, and put $u = 2\pi/\gamma$. We denote by $\mathcal{U}(K_\gamma)$ and $\mathcal{U}(K)$ the classes of all unitary functions on K_γ and K, respectively. We first recall the definition of cocycles introduced by Gamelin (see [**2**, Chapter VII, §11]). For any β in $\mathcal{U}(K_\gamma)$ the cocycle

$B_\beta(x, t)$ is given explicitly for positive t by

$$(3.1)\qquad B_\beta(y,t) = \begin{cases} 1 & \text{on } K_\gamma \times [0,u), \\ \prod_{j=0}^{m-1} \beta(y + e_{ju}) & \text{on } K_\gamma \times [mu, (m+1)u) \end{cases}$$

for each positive integer m, and $B_\beta(y + e_s, t) = B_\beta(y, s + t)$ for s in $[0, u)$. Then B_β is trivial if and only if there is an f in $\mathfrak{U}(K_\gamma)$ and some r in R for which β can be expressed in the form $\beta(y) = e^{ir} f(y)\overline{f(y + e_u)}$ for a.a. y in K_γ [**4**, Chapter 4, §9]. Gamelin's representation theorem [**2**, Chapter VII, Theorem 11.1] asserts that every cocycle A on K has the factorization $A = B_\beta C$, where β is a function in $\mathfrak{U}(K_\gamma)$, and C is a coboundary.

The following theorem shows vaguely which unitary functions on K are the values of cocycles and settles a question posed by Helson [**5**, §1]: *Is the class of all A_1 on K different from the class of all A_2?*

THEOREM 3.1. *For any positive u in R, let $\gamma = 2\pi/u$, and let $\{A_u\}$ denote the class of all the values $A_u(x) = A(x, u)$ of cocycles A. Then we obtain the following properties*:

(i) *if γ belongs to Γ, then every A_u in $\{A_u\}$ has the form*

$$A_u(x) = \beta(y)q(x)\overline{q(x + e_u)}$$

for σ-a.a. $x = (y, s)$ in $K_\gamma \times [0, u)$, where β is a function in $\mathfrak{U}(K_\gamma)$ and q is a function in $\mathfrak{U}(K)$;

(ii) *for each positive integer m, $\{A_u\}$ contains $\{A_{mu}\}$; and*

(iii) *if γ belongs to Γ, then for any v in $(0, u)$, there exists an A_v in $\{A_v\}$ which does not lie in $\{A_u\}$.*

PROOF. (i) is a direct consequence of Gamelin's representation theorem, so it is enough to show (ii) and (iii). We notice that if $A(x, t)$ is a cocycle, then so is $A(x + x_0, t)$ for any fixed x_0 in K, and the product of two cocycles is also a cocycle. For any positive integer m, we set

$$B(x,t) = A(x,t)A(x + e_u, t) \cdots A\big(x + e_{(m-1)u}, t\big).$$

Then it follows from the cocycle identity (1.1) that $B(x, t)$ is a cocycle which satisfies $B(x, u) = A(x, mu)$. Thus we have (ii). On the other hand, by Gamelin's representation theorem, we may choose a function β in $\mathfrak{U}(K_\gamma)$ for which B_β is a nontrivial cocycle. We now show that for each v in $(0, u)$, $B_\beta(x, v)$ cannot belong to $\{A_u\}$. By Definition (3.1), it can be seen that

$$(3.2)\qquad B_\beta(x,v) = \begin{cases} 1 & \text{for } x = (y,s) \text{ in } K_\gamma \times [0, u - v), \\ \beta(y) & \text{for } x = (y,s) \text{ in } K_\gamma \times [u - v, u). \end{cases}$$

Suppose to the contrary that $B_\beta(x, v)$ belongs to $\{A_u\}$. Then by (i) there are functions α in $\mathcal{U}(K_\gamma)$ and p in $\mathcal{U}(K)$ such that

$$B_\beta(x, v) = \alpha(y)p(x)\overline{p(x + e_u)}$$

for σ-a.a. $x = (y, s)$ in $K_\gamma \times [0, u)$. Therefore it follows from (3.2) and Fubini's theorem that there is an s in $[0, u - v)$ such that $\alpha(y) = \overline{p((y, s))}p((y + e_u, s))$ for a.a. y in K_γ. From this fact, we can easily see that $\beta(y) = \delta(y)\overline{\delta(y + e_u)}$ for some δ in $\mathcal{U}(K_\gamma)$, so B_β must be trivial. Thus we have a contradiction, and this completes the proof.

From (ii) and (iii) of Theorem 3.1 we have

COROLLARY 3.2. *If π belongs to Γ, then $\{A_1\}$ contains strictly $\{A_2\}$.*

4. Remarks. We recall that a Borel function f on $K_{2\pi} \times R$ is *automorphic* if $f(y, s + 1) = f(y + e_1, s)$ a.a. on $K_{2\pi} \times R$, and any Borel function on $K = K_{2\pi} \times [0, 1)$ can be extended uniquely to be automorphic on $K_{2\pi} \times R$ [**2**, Chapter VII, §6].

(a) Let $\tilde{H}^\infty(\sigma)$ be the space of all automorphic extensions of functions in $H^\infty(\sigma)$. The following question is interesting and probably difficult:

For any cocycle A, does there exist an f in $\mathcal{H}^1$ for which $(f + \Phi_A^{-1}(\{0\}))\cdot\tilde{H}^\infty(\sigma)$ is dense in $\mathcal{H}^1$?

This is related to the old problem of whether every simply invariant subspace is generated by one of its elements. Indeed, if we could choose such a function f, then $\Phi_A(f)$ would be a single generator of $(\mathfrak{M}_A)_-$.

(b) We know that the dual space of $H^p(\sigma)$, $0 < p < 1$, has dimension one [**7**, **9**]. By the argument of [**7**], Theorem 2.1 provides an extension of this result:

Let $\mathfrak{M}$ be a simply invariant subspace of $L^p(\sigma)$, $0 < p < 1$. Then the dual space of $\mathfrak{M}$ has at most dimension one.

(c) Let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$. For any f in $\mathfrak{M}$, let $\tilde{f}$ denote the automorphic extension of f to $K_{2\pi} \times R$. Then we may easily verify that there exists a unitary function q on $K_{2\pi} \times R$ such that the closure $[\tilde{f}\mathcal{H}]_1$ of $\tilde{f}\mathcal{H}$ in $L^1(d\sigma_1 \times dt)$ coincides with $q\mathcal{H}^1$. Let $p(x) = p(y, s)$ be the restriction of q to $K_{2\pi} \times [0, 1)$, and set $\beta(y, s) = q(y, s + 1)\overline{q(y + e_1, s)}$. Since $\tilde{f}$ is automorphic, β defines a unitary function on $K_{2\pi}$. We denote by C_β the cocycle defined by (3.1). Then it can be seen that the cocycle $C_\beta(x, t)p(x)\overline{p(x + e_t)}$ corresponds to the simply invariant subspace generated by f (cf. [**5**, §3]). Similarly, let $\tilde{\mathfrak{M}}$ denote the space of all automorphic extensions of functions in $\mathfrak{M}$. Then it is not hard to see that $[\tilde{\mathfrak{M}}\cdot\mathcal{H}]_1 = q\mathcal{H}^1$ for some unitary function q on $K_{2\pi} \times R$. Thus in the same manner, we may find the cocycle associated with $\mathfrak{M}$. This provides another naive definition of cocycles (cf. [**4**, Chapter 2]).

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