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# EMBEDDING $L^1$ IN $L^1/H^1$

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ABSTRACT. It is proved that $L^1$ is isomorphic to a subspace of $L^1/H^1$. More precisely, there exists a diffuse $\sigma$-algebra $\mathfrak{S}$ on the circle such that the corresponding expectation $\mathbf{E}$: $H^\infty \to L^\infty(\mathbf{C})$ is onto. The method consists in studying certain martingales on the product $\Pi^{\mathbf{N}}$.

**1. Introduction.** Let us start by fixing some terminology. As usual, $\Pi$ will denote the circle equipped with its Haar measure $m$, $H_0^1$ is the subspace of those $f \in L^1(\Pi)$ for which $\hat{f}(n) = 0$ for $n \leqslant 0$ and $q$: $L^1 \to L^1/H_0^1$ is the quotient map.

We are interested in the question whether or not there exists a linear embedding of the Banach space $L^1$ in the space $L^1/H_0^1$. We briefly indicate some motivation for this problem. First, it was (and still remains) an open question if the three-space-property holds for $L^1$-embedding, i.e. suppose $X$ a Banach space, $Y$ a subspace of $X$. Is it true that whenever $L^1$ embeds in $X$, it also has to embed in either $Y$ or $X/Y$?

The problem is also unsolved in the particular case $X = L^1$ and $Y$ isomorphic to a dual space. It is not hard to show that an embedding of $L^1$ in $X/Y$ is then equivalent to the existence of a subspace $S$ of $X$, $S$ isomorphic to $L^1$ so that the quotient map $X \to X/Y$ is an isomorphism when restricted to $S$.

In the special situation $X = L^1(\Pi)$ and $Y = H_0^1$, the answer was unknown for some time. There was hope that this may provide a counterexample in view of the following result, due to W. B. Johnson (see [**9**]).

PROPOSITION 1. *No complemented subspace of $L^1/H_0^1$ is isomorphic to $L^1$.*

This is a consequence of the fact that any operator $T$: $L^1/H^1 \to L^1$ maps weakly compact sets onto norm compact sets. Let us sketch the argument.

Consider the identity map $I$: $L^\infty/H^\infty \to L^1/H^1$. Then $(TI)^*$: $L^\infty \to H^\infty \to H^1$ is integral and therefore nuclear (since $H^1$ satisfies the Radon-Nikodym property). Consequently, also $TI$ is nuclear. Given now a weakly null sequence $(x_n)_{n=1,2,\dots}$ in $L^1/H^1$, it follows from the lifting property (see [**9**] for instance) that $x_n = q(f_n)$ where $\{f_n;\ n = 1,2,\dots\}$ is a relatively weakly compact set in $L^1(\Pi)$. Therefore, for each $\varepsilon > 0$, a truncation argument provides a bounded sequence $(g_n)$ in $L^\infty$ such that $\| f_n - g_n \|_1 < \varepsilon$ for each $n$. Thus

$$\|Tx_n - TI\tilde{g}_n\| \leqslant \|T\| \, \|x_n - I\tilde{g}_n\| < \varepsilon \|T\|.$$

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Because $TI$ is nuclear, the set $\{TI(\tilde{g}_n);\ n = 1, 2, \ldots\}$ is compact for each $\varepsilon > 0$. So we conclude that $\{Tx_n\}$ is compact, as announced.

Using Proposition 1, the following is proved in [**2**].

PROPOSITION 2. *There is no almost isometric embedding of the complex $L^1$ space in $L^1/H^1$.*

Thus $d(S, L^1) > \gamma > 1$ for each subspace $S$ of $L^1/H_0^1$, where $d$ is the Banach-Mazur distance (see [**8**, **9**] for definitions). This observation allows us to define a natural distortion of $L^1$, by taking

$$|||f||| = \|f\|_1 + \|q(f)\|, \qquad f \in L^1(\Pi).$$

Say that an operator $T\colon X \to Y$ is a semiembedding provided $T$ is one-one and maps the closed unit ball of $X$ on a norm-closed subset of $Y$. It can be shown that a semiembedding $T\colon L^1 \to L^1$ has to fix an $L^1$-copy (i.e. is an isomorphism when restricted to a subspace $S$ of $L^1$, $S$ isomorphic to $L^1$). On the other hand, (see [**3**]):

PROPOSITION 3. *The restriction of the quotient map $q\colon L^1 \to L^1/H_0^1$ to the subspace $L^1_{\mathbf{R}}$ of real functions in $L^1(\Pi)$ is a semiembedding.*

No example is known of a semiembedding of $L^1$ in a Banach space $X$ not containing $L^1$.

Our purpose is to prove the existence of a natural embedding of $L^1$ in $L^1/H_0^1$. There exists a diffuse $\sigma$-algebra $\mathfrak{S}$ on $\Pi$ so that the restriction of $q$ to the complex $L^1(\mathfrak{S})$-space is an isomorphism. More precisely:

THEOREM. *There exists an increasing sequence $(n_k)$ of positive integers, such that if $\mathfrak{S}$ is the $\sigma$-algebra on $\Pi$ generated by the functions $\sigma_k(\theta) = \operatorname{sign}\cos n_k\theta$, then the restriction of $q$ to $L^1(\mathfrak{S})$ is an isomorphism. Consequently, for this $\sigma$-algebra $\mathfrak{S}$, the expectation operator $\mathbf{E}\colon H^\infty \to L^\infty(\mathfrak{S})$ is onto.*

The argument presented here is rather delicate. In order to give the reader an idea how it is organised, we briefly outline the proof. We have to introduce the $\sigma$-algebra $\mathfrak{S}$ such that the inequality

$$(*) \qquad \|h - \mathbf{E}_{\mathfrak{S}}[h]\|_1 \geqslant \delta\|h\|_1$$

holds for each $h \in H_0^1$. But choosing the sequence $(n_k)$ sufficiently lacunary, it is enough to verify $(*)$ for functions $h$ with spectrum contained in a set of the form

$$E = \{\Sigma' \nu_k n_k;\ |\nu_k| \leqslant a_k \text{ for each } k\}$$

where $(a_k)$ is a sequence of positive integers and $(n_k)$, $(a_k)$ satisfy the transference property. Thus the $n_k$-frequencies can be replaced by independent variables. The space $H_0^1 \cap L_E^1$ identifies with a subspace of the space $\mathcal{H} \subset L^1(\Pi^{\mathbf{N}})$ of those functions $h = \Sigma h_k$ on $\Pi^{\mathbf{N}}$ such that each increment $h_k = h_k(x_1, \ldots, x_k)$ is an $H_0^1$-function in $x_k$. The required inequality now becomes

$$(**) \qquad \|h - \mathbf{E}_{\mathfrak{F}}[h]\|_1 \geqslant \|h\|_1$$

for $h \in \mathcal{H}$, where $\mathfrak{F}$ is a natural diadic product $\sigma$-algebra on $\Pi^{\mathbf{N}}$ (generated by the functions $\sigma_k(x) = \operatorname{sign}\cos x_k$).

This reduction of the problem is worked out in §4. Its purpose is to approach the problem with martingale techniques. The martingale prerequisites are given in §2. To obtain (**) we first prove $L^1$-estimations for certain square functions related to $h$ (see Lemma 4). These are derived using a "step-by-step" method (explained at the beginning of §5) and an examination of what happens at each increment. More precisely, we have to consider at this point functions of the form $a + h - b\sigma$, where $a$, $b$ are scalars, $h \in H_0^1$ and $\sigma = \operatorname{sign}\cos$.

Minorations of the $L^{-1}$-norm of such expressions are given in Propositions 8 and 9 below. It is only at this place that some complex function theory will be involved.

**2. Martingale preliminaries.** Let $(\mathfrak{F}_k)_{k=0,1,2,\ldots}$ be an increasing sequence of $\sigma$-algebras on a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ assuming $\mathfrak{F} = \bigvee_{k=1}^{\infty} \mathfrak{F}_k$. Denote by $\mathbf{E}_k$ the expectation with respect to $\mathfrak{F}_k$. For $f \in L^1(\mathfrak{F})$ let

$$f^* = \sup_k |\mathbf{E}_k[f]| \quad \text{and} \quad S(f) = \left[ |\mathbf{E}_0[f]|^2 + \sum_{k=1}^{\infty} |\mathbf{E}_k[f] - \mathbf{E}_{k-1}[f]|^2 \right]^{1/2}.$$

We will use the notation $C$ to indicate a numerical constant. Let us recall the following result, due to D. Davis (see [7]).

PROPOSITION 4. $C^{-1}\|S(f)\|_1 \leqslant \|f^*\|_1 \leqslant C\|S(f)\|_1$.

The next inequality is probably known, but we include its proof here for the sake of completeness.

PROPOSITION 5. *Let $(v_k)$ be an adapted sequence of functions; thus $v_k$ is $\mathfrak{F}_k$-measurable for each $k$. Then*

$$\left\| \left[ \sum |\mathbf{E}_{k-1}[|v_k|]|^2 \right]^{1/2} \right\|_1 \leqslant C \left\| \left[ \sum |v_k|^2 \right]^{1/2} \right\|_1.$$

PROOF. It is no restriction to assume the $\mathfrak{F}_k$ finite algebras. Moreover, since one may always tensor the $v_k$ against a Rademacher sequence, we can assume $\mathbf{E}_{k-1}[v_k] = 0$ and thus $(v_k)$ is an adapted martingale difference sequence. Since, then

$$\left\| \left[ \sum |v_k|^2 \right]^{1/2} \right\|_1 = \left\| \sum v_k \right\|_{H^1(\mathfrak{F}_k)},$$

it follows from the atomic decomposition property for $H^1$-functions (see for instance [7, Chapter I]) and convexity, that we may take for $\sum v_k$ a function of the form (for some positive integer $j$)

$$a = \frac{1}{|A|}(\varphi - \mathbf{E}_{j-1}[\varphi])$$

where $A$ is an $\mathfrak{F}_j$-atom, $\operatorname{supp}\varphi \subset A$ and $\|\varphi\|_\infty \leqslant 1$. In this case

$$\begin{aligned} v_k &= \mathbf{E}_k[a] - \mathbf{E}_{k-1}[a] = 0 \quad \text{for } k < j, \\ &= \frac{1}{|A|}(\mathbf{E}_k[\varphi] - \mathbf{E}_{k-1}[\varphi]) \quad \text{for } k \geqslant j. \end{aligned}$$

Also, $\mathbf{E}_k[\varphi]$ is supported by $A$ for $k \geqslant j$ and hence $v_k$ for $k > j$. Thus the left side in Proposition 5 is dominated by

$$\|v_j\|_1 + \left\| \left( \sum_{k>j} \mathbf{E}_{k-1}\left[|v_k|^2\right] \right)^{1/2} \right\|_1$$

$$\leqslant 2 + \int_A \left( \sum_{k>j} \mathbf{E}_{k-1}\left[|v_k|^2\right] \right)^{1/2} \quad \text{(by Cauchy-Schwarz)}$$

$$\leqslant 2 + |A|^{1/2} \left( \int \sum_{k>j} |v_k|^2 \right)^{1/2}$$

$$\leqslant 2 + |A|^{1/2} \|a\|_2 \leqslant 3,$$

proving the result.

PROPOSITION 6. *For $f \in H^1(\mathfrak{F}_k)$, one has an inequality*

$$\left( \sum \|(\mathbf{E}_k - \mathbf{E}_{k-1})[f]\|_1^2 \right)^{1/2} \leqslant C \|f\|_1^{1/2} \|f\|_{H^1}^{1/2}.$$

To prove this, we will first deal with the special case of the Rademacher projection on the Cantor group (in fact, only this will be used later on).

PROPOSITION 7. *If $D = \{1, -1\}^{\mathbf{N}}$ is the Cantor-group and $f \in H^1(D)$, then*

$$\left( \sum |\hat{f}(k)|^2 \right)^{1/2} \leqslant C \|f\|_1^{1/2} \|f\|_{H^1}^{1/2}$$

*where $\hat{f}(k) = \int f(\varepsilon)\varepsilon_k$.*

PROOF. We will use the theorem of [6] on the BMO-distance of a BMO-function to $L^\infty$ (in the diadic setting). The result asserts, in particular, that for $\varphi \in \mathrm{BMO}(D)$, $\mathrm{dist}_{\mathrm{BMO}}(\varphi, L^\infty) = 0$ and $\varepsilon > 0$, there exists a decomposition $\varphi = \alpha + \beta$ such that

$$\|\alpha\|_{\mathrm{BMO}} \leqslant C_1 \varepsilon \quad \text{and} \quad \|\beta\|_\infty \leqslant C_2 \max(\varepsilon, \lambda_0(\varepsilon))$$

where $\lambda_0 = \lambda_0(\varepsilon)$ has to satisfy

$$\sup_I \frac{1}{|I|} \left| \{ x \in I, |\varphi(x) - \varphi_I| > \lambda \} \right| \leqslant e^{-\lambda/\varepsilon}$$

whenever $\lambda > \lambda_0$ ($\varphi_I = |I|^{-1} \int_I \varphi$).

Now take $\varphi = \sum a_k \varepsilon_k$ with $\sum |a_k|^2 = 1$. It follows from the distribution property of Rademacher that for each diadic interval $I$,

$$\left| \{ \alpha \in I; |\varphi(x) - \varphi_I| > \lambda \} \right| \leqslant C e^{-c\lambda^2} |I|,$$

for numerical constants $c > 0$, $C < \infty$. Hence $\mathrm{dist}_{\mathrm{BMO}}(\varphi, L^\infty) = 0$ and $\lambda_0(\varepsilon) \sim 1/\varepsilon$.

Decomposing $\varphi = \alpha + \beta$ as above, we get

$$|\langle f, \varphi \rangle| \leqslant |\langle f, \alpha \rangle| + |\langle f, \beta \rangle| \leqslant C_1' \varepsilon \|f\|_{H^1} + C_2' \frac{1}{\varepsilon} \|f\|_1.$$

Taking supremum over $\varphi$ and choosing $\varepsilon = \|f\|_1^{1/2} \|f\|_{H^1}^{-1/2}$, the inequality follows.

PROOF OF PROPOSITION 6. Assume $f$ real and estimate

$$\left(\sum_{k=1}^{K} \|(\mathbf{E}_k - \mathbf{E}_{k-1})[f]\|_1^2\right)^{1/2}.$$

Define for each $k$,

$$\sigma_k = \operatorname{sign} \Delta f_k \quad \text{and} \quad b_k = \tfrac{1}{2}(\sigma_k - \mathbf{E}_{k-1}[\sigma_k]).$$

Then

$$\|f\|_1 \geqslant \iint |f| \prod_{k=1}^{K} (1 + \varepsilon_k b_k)\, d\varepsilon \mathbf{P}(d\omega) \geqslant \frac{1}{2} \int \left| \sum_{k=1}^{K} \varepsilon_k \Phi_k(\varepsilon) \right| d\varepsilon$$

where

$$\Phi_k(\varepsilon) = \int \prod_{j=1}^{k-1} (1 + \varepsilon_j b_j) |\Delta f_k| \, d\omega.$$

Application of Proposition 7 to the function $\sum \varepsilon_k \Phi_k(\varepsilon)$ then gives

$$\begin{aligned}\left(\sum_{k=1}^{K} \|\Delta f_k\|_1^2\right)^{1/2} &\leqslant C\|f\|_1^{-1/2}\left[\int \left(\sum |\phi_k(\varepsilon)|^2\right)^{1/2} d\varepsilon\right]^{1/2} \\ &\leqslant C\|f\|_1^{1/2}\left[\iint S(f) \prod (1 + \varepsilon_j b_j)\, d\omega\, d\varepsilon\right]^{1/2} \\ &= C\|f\|_1^{1/2} \|f\|_{H^1}^{1/2}\end{aligned}$$

as announced.

REMARK. The author is grateful to P. W. Jones for outlining a more explicit procedure to obtain the decomposition used in the proof of Proposition 7.

**3. Some inequalities involving $H_0^1$-functions.** The purpose of this section is to prove the following results.

PROPOSITION 8. *For $a \in \mathbf{C}$ and $h \in H_0^1$, one has*

$$\|a + h\|_1 \geqslant \left\| \left(|a|^2 + \delta^2 |h|^2\right)^{1/2} \right\|_1$$

*where $\delta > 0$ is a fixed constant.*

PROPOSITION 9. *There exists $\delta > 0$ such that for $a \in \mathbf{C}$, $b \in \mathbf{C}$ and $h \in H_0^1$,*

(i) $$\|a + h - b\sigma\|_1 \geqslant \left\{ |a|^2 + \delta^2 \left[ \frac{\operatorname{Re}\left(\overline{\langle h, \sigma \rangle}\,(\langle h, \sigma \rangle - b)\right)}{|\langle h, \sigma \rangle| + |b|} \right]^2 \right\}^{1/2},$$

(ii) $$\|a + h - \langle h, \sigma \rangle \sigma\|_1 \geqslant \left\| \left\{ |a|^2 + \delta^2 |h_e - \langle h, \sigma \rangle \sigma|^2 \right\}^{1/2} \right\|_1$$

*where $\sigma = \operatorname{sign} \cos$ and $h_e(\theta) = \sum_{n=1}^{\infty} \hat{h}(n) \cos n\theta$.*

It is clear that it suffices to prove Propositions 8 and 9, with $a = 1$.

PROOF OF PROPOSITION 8. Factoring $1 + h$ gives $1 + h = (1 + g_1)(1 + g_2)$ where $g_1, g_2 \in H_0^2$ and

$$\|1 + h\|_1 = \left(1 + \|g_1\|_2^2\right)^{1/2}\left(1 + \|g_2\|_2^2\right)^{1/2}.$$

Since $|h| \leqslant |g_1| + |g_2| + |g_1||g_2|$ the result follows from the majorations

$$\left\|\left(1 + |g_i|^2\right)^{1/2}\right\|_1 \leqslant \left\|\left(1 + |g_i|^2\right)^{1/2}\right\|_2 = \left(1 + \|g_i\|_2^2\right)^{1/2} \leqslant \|1 + h\|_1 \qquad (i = 1, 2)$$

and

$$\left\|\left(1 + |g_1|^2|g_2|^2\right)^{1/2}\right\|_1 \leqslant 1 + \|g_1 g_2\|_1 \leqslant 1 + \|g_1\|_2\|g_2\|_2 \leqslant \|1 + h\|_1.$$

Also to obtain Proposition 9, we will use the $L^2$-theory. Our argument here is, however, more complicated. This is the only point where explicit constructions of $H^\infty$-functions appear.

LEMMA 1. *Given a measurable subset $A$ of $\Pi$, there exist $H^\infty$-functions $\varphi$ and $\psi$ satisfying the following conditions*:

(i) $|\varphi| + |\psi| \leqslant 1$,

(ii) $\operatorname{Re}\psi$ *is an even function on* $\Pi$,

(iii) $|\varphi - 1/8| < 1/100$ *on the set* $A$,

(iv) $\|\varphi\|_1 \leqslant C|A|$,

(v) $\|\operatorname{Re}\psi - 1\|_1 \leqslant C|A|$.

PROOF. Fix some (large) $M > 0$ and define the following $H^\infty$-functions:

$$\tau(z) = -M\int_A \frac{e^{i\theta} + z}{e^{i\theta} - z} m(d\theta), \quad \varphi = \frac{1}{8}(1 - e^\tau)^2,$$

$$\psi(z) = \exp\left\{\int_\Pi \log(1 - \alpha(\theta))\frac{e^{i\theta} + z}{e^{i\theta} - z} m(d\theta)\right\}$$

where $\alpha(\theta) = |\varphi(e^{i\theta})| \vee |\varphi(e^{-i\theta})|$.

Notice that this makes sense, because $e^\tau$ has boundary value $e^{-M(\chi_A + i\mathcal{H}(\chi_A))}$ ($\mathcal{H}$ = Hilbert-transform) and therefore $\|\alpha\|_\infty \leqslant \frac{1}{2}$.

(i) is obvious. On $\Pi$, we have $\operatorname{Re}\psi = (1 - \alpha)\cos\mathcal{H}(\log(1 - \alpha))$ and thus an even function. Since $|\varphi - \frac{1}{8}| \leqslant \frac{3}{8}|e^\tau|$ and thus $|\varphi - \frac{1}{8}| < e^{-M}$ on $A$ (iii) holds for $M$ large enough. Because on $\Pi$

$$8|\varphi| \leqslant \chi_A + M^2|\mathcal{H}(\chi_A)|^2,$$

(iv) follows. Finally,

$$|1 - \operatorname{Re}\psi| \leqslant |\alpha| + \tfrac{1}{2}|\mathcal{H}(\log(1 - \alpha))|^2, \quad \|1 - \operatorname{Re}\psi\|_1 \leqslant 4\|\varphi\|_1$$

and hence (v).

We refer the reader to [**4**, Proposition 1.6] for the following Marcinkiewicz type decomposition.

LEMMA 2. *There is a constant $C < \infty$ such that for given $h \in H_0^1$ and $\lambda > 0$, there exists $h_\lambda \in H_0^\infty$ satisfying*:

(i) $|h_\lambda| \leqslant C|h|$,

(ii) $\|h_\lambda\|_\infty \leqslant C\lambda$,

(iii) $\|h - h_\lambda\|_1 \leqslant C\int_{[|h|>\lambda]}|h|$.

Let $h$ be as in Proposition 9. For $\lambda > 0$, define $A_\lambda = [|h| > \lambda]$. Application of Lemma 1 to the set $A_\lambda$ provides $H^\infty$-functions $\varphi_\lambda$, $\psi_\lambda$. We are now ready to prove

LEMMA 3. $\|1 + h - b\sigma\|_1 \geqslant 1 + c\int_{A_\lambda}|h| + c\lambda^{-2}\|\mathrm{Im}(h_\lambda - b\sigma)\|_2^2$ *if* $\lambda > K$ *and* $|b| < \lambda/K$ ($c > 0$ *and* $K < \infty$ *being numerical constants*).

PROOF. First, since $1 - b\sigma$ is even and $\mathrm{Im}\,\psi_\lambda$ odd, we find

$$\begin{aligned}\|1 + h - b\sigma\|_1 &\geqslant \|(1 + h - b\sigma)\varphi_\lambda\|_1 + \left|\int(1 - b\sigma)\psi_\lambda\right| \\ &\geqslant \frac{1}{9}\int_{A_\lambda}\big||h| - (1 + |b|)\big| + \left|\int(1 - b\sigma)\,\mathrm{Re}\,\psi_\lambda\right| \\ &\geqslant \frac{1}{9}\int_{A_\lambda}|h| - \frac{1}{9}(1 + |b|)|A_\lambda| + 1 - (1 + |b|)\|1 - \mathrm{Re}\,\psi_\lambda\|_1 \\ &\geqslant \frac{1}{9}\int_{A_\lambda}|h| - C(1 + |b|)|A_\lambda| + 1\end{aligned}$$

for some constant $C$. Thus, choosing $K$ large enough, we get

$$(*) \qquad \|1 + h - b\sigma\|_1 \geqslant 1 + \frac{1}{10}\int_{A_\lambda}|h|.$$

Fix some small constant $\delta > 0$. Since we always have

$$\|1 + af\|_1 \leqslant \|1 + f\|_1 \quad \text{for } 0 \leqslant a \leqslant 1 \text{ and } f \text{ of mean } 0,$$

it follows that

$$\|1 + h - b\sigma\|_1 \geqslant \|1 + \delta\lambda^{-1}(h - b\sigma)\|_1 \geqslant \|1 + \delta\lambda^{-1}(h_\lambda - b\sigma)\|_1 - \delta\lambda^{-1}\|h - h_\lambda\|_1.$$

Because $\delta\lambda^{-1}|h_\lambda - b\sigma| \ll 1$ the inequality

$$(1 + t)^{1/2} \geqslant 1 + t/3 \quad \text{for } 0 \leqslant t \leqslant 1$$

yields

$$|1 + \delta\lambda^{-1}(h_\lambda - b\sigma)| \geqslant [1 + \delta\lambda^{-1}\,\mathrm{Re}(h_\lambda - b\sigma)]\left[1 + \tfrac{1}{12}\delta^2\lambda^{-2}(\mathrm{Im}(h_\lambda - b\sigma))^2\right].$$

Therefore, also

$$(**) \qquad \|1 + h - b\sigma\|_1 \geqslant 1 + \frac{1}{20}\delta^2\lambda^{-2}\int\mathrm{Im}^2(h_\lambda - b\sigma) - c\delta\lambda^{-1}\int_{A_\lambda}|h|.$$

The required minoration clearly follows combining $(*)$ and $(**)$.

PROOF OF PROPOSITION 9. *First*

$$\|1 + h - b\sigma\|_1 \geqslant d(b\sigma, H^1) \geqslant \frac{|b|}{2\pi}\left|\int_{-\pi}^{\pi}\sigma(\theta)e^{i\theta}\,d\theta\right| = \frac{2}{\pi}|b|$$

and hence, also,

$$\|1 + h - b\sigma\|_1 \geq \tfrac{1}{3}\|1 + h\|_1 \geq \tfrac{1}{3}\|h\|_1.$$

Notice that the right member of (i), (ii) is bounded by $1 + 2\delta\|h\|_1$. Since now $\|1 + h - b\sigma\|_1 \geq \frac{1}{6}\|h\|_1 + \frac{1}{6}|b|$, it follows that (i) (resp. (ii)) are satisfied for $|b| \geq 6$ (resp. $|\langle h, \sigma\rangle| \geq 6$). Hence, we may assume $|b| \leq M$ in (i), $|\langle h, \sigma\rangle| \leq M$ in (ii) where $M$ is some numerical constant.

Fix a constant $\lambda > KM$ and put $k = h_\lambda$ for simplicity. Using again Lemma 2(iii), the right member of (i) can be majorized by

$$\left[1 + 2\delta^2\left(|\mathrm{Re}\langle h, \sigma\rangle|^2 + |\mathrm{Im}(\langle h, \sigma\rangle - b)|^2\right)\right]^{1/2}$$

$$\leq \left[1 + 2\delta^2\left(|\mathrm{Re}\langle k, \sigma\rangle|^2 + |\mathrm{Im}(\langle k, \sigma\rangle - b)|^2\right)\right]^{1/2} + 2\delta C\int_{A_\lambda} |h|.$$

Taking Lemma 3 into account, we see that it suffices to check the inequality

$$|\mathrm{Re}\langle k, \sigma\rangle|^2 + |\mathrm{Im}(\langle k, \sigma\rangle - b)|^2 \leq \|\mathrm{Im}(k - b\sigma)\|_2^2$$

which is straightforward:

$$\|\mathrm{Im}(k - b\sigma)\|_2^2 = \frac{1}{2}\sum_{n>0} |\mathrm{Im}\,\hat{k}(n) - 2\,\mathrm{Im}\,b\hat{\sigma}(n)|^2 + \frac{1}{2}\sum_{n>0} |\mathrm{Re}\,\hat{k}(n)|^2$$

while

$$|\mathrm{Re}\langle k, \sigma\rangle| \leq \sum_{n>0} |\mathrm{Re}\,\hat{k}(n)|\hat{\sigma}(n) \leq \frac{1}{\sqrt{2}}\left(\sum |\mathrm{Re}\,\hat{k}(n)|^2\right)^{1/2},$$

$$|\mathrm{Im}(\langle k, \sigma\rangle - b)| \leq \sum_{n>0} |\mathrm{Im}\,\hat{k}(n) - 2\,\mathrm{Im}\,b\hat{\sigma}(n)|\hat{\sigma}(n)$$

$$\leq \frac{1}{\sqrt{2}}\left(\sum_{n>0} |\mathrm{Im}\,\hat{k}(n) - 2\,\mathrm{Im}\,b\hat{\sigma}(n)|^2\right)^{1/2}.$$

For the right member of (ii), a similar reasoning reduces the question to the verification of

$$\int |k_e - \langle k, \sigma\rangle\sigma|^2 \leq \|\mathrm{Im}(k - b\sigma)\|_2^2,$$

which the reader will easily do.

**4. Reduction of the problem.** In this section, we will reduce the problem of proving that certain elements of $L^1(\Pi)$ normed by the quotient norm $L^1/H^1$ to the verification of an inequality for certain functions in $L^1(\Pi^{\mathbf{N}})$, where $\Pi^{\mathbf{N}} = \Pi \times \Pi \times \cdots$ is the product group. Denote by $\mathbf{E}_k$ $(k = 1, 2, \ldots)$ the expectation with respect to the $k$ first variables $(x_1, x_2, \ldots, x_k)$, where $x = (x_1, x_2, \ldots)$ is the product variable.

We consider the subspace $\mathcal{H}$ of $L^1(\Pi^{\mathbf{N}})$ of those functions $h$ such that for each $k$ the difference $\mathbf{E}_k[h] - \mathbf{E}_{k-1}[h]$ is an $H_0^1$-function with respect to $x_k$. Thus $h$ is of the form

$$h = \sum h_k \quad \text{where } h_k = \sum_{n>0} \hat{h}_k(n) e^{inx_k}$$

and the $\hat{h}_k(n)$ are functions of $x_1, \ldots, x_{k-1}$.

Again let $\sigma = \operatorname{sign}\cos$ and $\sigma_k(x) = \sigma(x_k)$ for each $k$. Let $\mathfrak{F}$ be the $\sigma$-algebra on $\Pi^{\mathbf{N}}$ generated by the $\sigma_k$. In the next section, we show the following

PROPOSITION 10. *There is a constant* $c > 0$ *s.t.* $\|h - \mathbf{E}_{\mathfrak{F}}[h]\|_1 \geq c\|h\|_1$ *for all* $h \in \mathcal{H}$.

This fact obviously implies

COROLLARY 11. $\inf_{h\in\mathcal{H}}\|f - h\|_1 \geq c'\|f\|_1$ *for all* $f \in L^1(\mathfrak{F})$.

For $a$, $n$ positive integers, $\mathfrak{F}_a$ will be the Fejér kernel

$$F_a(\theta) = \sum_{|j|\leq a} \frac{a+1-|j|}{a+1} e^{ij\theta}$$

and $F_{a,n}(\theta) = F_a(n\theta)$.

We consider sequences of positive integers $(n_k)$, $(a_k)$ satisfying the following conditions: ($\mathfrak{S}$ denotes again the $\sigma$-algebra on $\Pi$ generated by the functions $\sigma(n_k\theta)$.)

(i) The transference property, i.e. let $E = \{\Sigma' \nu_k n_k;\ (\nu_k) \in F\}$ where $F$ is the subset $\{(\nu_k), |\nu_k| \leq a_k\}$ of the dual group of $\Pi^{\mathbf{N}}$. Then the operator

$$T\colon L^1_E(\Pi) \to L^1_F(\Pi^n), \quad T(f)(x) = \sum_{(\nu_k)\in F} \hat{f}\left(\sum \nu_k n_k\right) e^{i(\Sigma \nu_k x_k)}$$

satisfies

$$\tfrac{1}{2}\|f\|_1 \leq \|T(f)\|_1 \leq 2\|f\|_1.$$

Moreover, $T(f) \in \mathcal{H}$ for $f \in L^1_E \cap H^1_0$.

(ii) Defining for each $k$,

$$\xi_k = \sigma * F_{a_k}, \quad K = \prod_k F_{a_k,n_k},$$
$$R(\theta,\psi) = \prod\left[1 + \xi_k(n_k\theta)\sigma(n_k\psi)\right],$$

one has

($\alpha$) $\sum\|\xi_k - \sigma\|_1 \leq \varepsilon$,

($\beta$) $\|K\|_1 = 1$.

For $f \in L^1(\mathfrak{S})$,

($\gamma$) $\|f - f * K\|_1 < \varepsilon\|f\|_1$,

($\delta$) $\|f - R(f)\|_1 < \varepsilon\|f\|_1$ where $R(f)(\theta) = \int f(\psi)R(\theta,\psi)m(d\psi)$ (where $\varepsilon > 0$ is a small constant).

The reader will easily convince himself that the realisation of these conditions is straightforward. Details on the transference property can be found in [**1**].

Let us now show that the sequence $(n_k)$ satisfies the Theorem. Thus, fix $f \in L^1(\mathfrak{S})$ and $h \in H^1_0$. We get, by (ii),

$$\|f - h\|_1 \geq \|f * K - h * K\|_1 \geq \|R(f) - h * K\|_1 - 2\varepsilon\|f\|_1.$$

Notice that $R(f) \in L^1_E$. By (i),

$$\|R(f) - h * K\|_1 \geq \tfrac{1}{2}\|T(R(f)) - h_1\|_1$$

where $h_1 = T(h * K) \in \mathcal{H}$.

Now

$$T(R(f))(x)=\int f(\psi)\prod\left[1+\xi_k(x_k)\sigma(n_k\psi)\right]m(d\psi).$$

By (ii)($\alpha$), we see that for any ($\pm 1$)-sequence $(\tau_k)$

$$\left\|\bigotimes(1+\tau_k\xi_k)-\prod(1+\tau_k\sigma_k)\right\|_1<\varepsilon$$

implying that

$$\|T(R(f))-f_1\|\leqslant 2\varepsilon\|f\| \quad \text{where } f_1=\mathbf{E}[T(R(f))].$$

It follows then from Corollary 11 that

$$\begin{aligned}\|f-h\|_1 &\geqslant \frac{1}{2}\|f_1-h_1\|_1-3\varepsilon\|f\|_1\geqslant\frac{c}{2}\|f_1\|_1-3\varepsilon\|f\|_1\\ &\geqslant\frac{c}{2}\|T(R(f))\|_1-4\varepsilon\|f\|_1\geqslant\frac{c}{4}\|f\|_1-5\varepsilon\|f\|_1\geqslant c'\|f\|_1\end{aligned}$$

taking $\varepsilon>0$ small enough.

**5. Proof of the Theorem.** It remains to prove Proposition 10. So fix $h=\sum h_k\in\mathcal{H}$ where

$$h_k=\sum_{n>0}\hat{h}_k(n)(x_1,\dots,x_{k-1})e^{inx_k}.$$

We also define

$$\begin{aligned}[h_k]_e&=\sum\hat{h}_k(n)\cos nx_k,\\ [h_k]_0&=\sum\hat{h}_k(n)\sin nx_k,\\ \langle h_k,\sigma_k\rangle&=\sum\hat{h}_k(n)\hat{\sigma}(\eta)\end{aligned}$$

(which is thus a function of $x_1,\dots,x_{k-1}$). If $f=\mathbf{E}_{\mathcal{F}}[h]$, then $f=\sum b_k\cdot\sigma_k$, where $b_k=b_k(x_1,\dots,x_{k-1})=\mathbf{E}_{\mathcal{F}}[\langle h_k,\sigma_k\rangle]$.

Using E. Stein's theorem on $H^1$-multipliers (see **[11]**), it is easily seen that $\|h\|_1\sim\|S(h)\|_1$ ($S=$ square function with respect to the natural decomposition).

We give a direct proof of this fact, based on Proposition 8.

Fix $1>\varepsilon>0$ and a positive sequence $(s_k)_{k=1,2,\dots}$ in $L^\infty(\Pi^{\mathbf{N}})$ satifying $\|(\sum s_k^2)^{1/2}\|_\infty\leqslant\varepsilon$. Fixing a positive integer $K$, we get, using Proposition 8,

$$\begin{aligned}\|\mathbf{E}_K[h]\|_1&=\|\mathbf{E}_{K-1}[h]+h_K\|_1\\ &\geqslant\left\|\left(|\mathbf{E}_{K-1}[h]|^2+\delta^2|h_K|^2\right)^{1/2}\right\|_1\\ &\geqslant\left\||\mathbf{E}_{K-1}[h]|\left(1-s_K^2\right)^{1/2}\right\|_1+\delta\left\||h_K|s_K\right\|_1\\ &\geqslant\|\mathbf{E}_{K-1}[h]\|_1+\delta\left\||h_K|s_K\right\|_1-\|\mathbf{E}_{K-1}[h]s_K^2\|_1.\end{aligned}$$

Iterating,

$$\begin{aligned}\|h\|_1&\geqslant\delta\sum\left\||h_k|s_k\right\|_1-\sum\|\mathbf{E}_{k-1}[h]s_k^2\|_1\\ &\geqslant\delta\sum\left\||h_k|s_k\right\|_1-\varepsilon^2\left\|\max_k|\mathbf{E}_k[h]|\right\|_1.\end{aligned}$$

Taking supremum over the sequences $(s_k)$, it follows that

$$\|h\|_1 \geqslant \delta\varepsilon\|S(h)\|_1 - \varepsilon^2\Big\|\max_k |\mathbf{E}_k[h]|\Big\|_1$$

and choosing

$$\varepsilon^2 = \frac{\|h\|_1}{\|\max|\mathbf{E}_k[h]|\|_1},$$

we get

$$\|S(h)\|_1 \leqslant \delta^{-1}\|h\|_1^{1/2}\|\max|\mathbf{E}_k[h]|\|_1^{1/2}.$$

Hence, by Proposition 4, $\|S(h)\|_1 \leqslant \delta^{-2}\|h\|_1$ as required.

Before continuing, notice that since $\mathfrak{F}$-expectation is a contraction, $\|S(f)\|_1 \leqslant \|S(h)\|_1$. Since for each $k$, $|\cdots|\langle h_k, \sigma_k\rangle| \leqslant \mathbf{E}_{k-1}[|h_k|]$, application of Proposition 5 yields

$$\Big\|\Big(\sum |\langle h_k, \sigma_k\rangle|^2\Big)^{1/2}\Big\|_1 \leqslant C\|h\|_1.$$

If we now apply the previous procedure using Proposition 9, the following inequalities are derived.

LEMMA 4.

(I) $$\left\|\left\{\sum_k \left|\frac{\operatorname{Re}\overline{\langle h_k, \sigma_k\rangle}(\langle h_k, \sigma_k\rangle - b_k)}{|\langle h_k, \sigma_k\rangle| + |b_k|}\right|^2\right\}^{1/2}\right\|_1 \leqslant C\|h - f\|_1^{1/2}\|h\|_1^{1/2},$$

(II) $$\left\|\left\{\sum_k |[h_k]_8 - \langle h_k, \sigma_k\rangle\sigma_k|^2\right\}^{1/2}\right\|_1 \leqslant C\Big\|h - \sum\langle h_k, \sigma_k\rangle\sigma_k\Big\|_1^{1/2}\|h\|_1^{1/2}.$$

PROOF. Let us show how (I) follows from Proposition 9(i). The argument for (II) is analogous. Fix $0 < \varepsilon < 1$ and a sequence $(s_k)_{k=1,2,\ldots}$ of positive $L^\infty$-functions on $\Pi^{\mathbf{N}}$ satisfying $\|(\sum s_k^2)^{1/2}\|_\infty \leqslant \varepsilon$. Fix an integer $k$ and apply Proposition 9(i) in the variable $x_k$. We get

$$\begin{aligned}\|\mathbf{E}_k[h-f]\|_1 &= \|\mathbf{E}_{k-1}[h-f] + h_k - b_k\sigma_k\|_1 \\ &\geqslant \left\|\left\{|\mathbf{E}_{k-1}[h-f]|^2 + \delta^2\left[\frac{\operatorname{Re}\overline{\langle h_k, \sigma_k\rangle}(\langle h_k, \sigma_k\rangle - b_k)}{|\langle h_k, \sigma_k\rangle| + |b_k|}\right]^2\right\}^{1/2}\right\|_1 \\ &\geqslant \|\mathbf{E}_{k-1}[h-f]\|_1 + \delta\left\|\frac{\operatorname{Re}\overline{\langle h_k, \sigma_k\rangle}(\langle h_k, \sigma_k\rangle - b_k)}{|\langle h_k, \sigma_k\rangle| + |b_k|}s_k\right\|_1 \\ &\quad - \|\mathbf{E}_{k-1}[h-f]s_k^2\|_1.\end{aligned}$$

Iterating and using the same considerations as in the beginning of this section it follows that the left member of (I) is dominated by

$$\delta^{-1}\varepsilon^{-1}\|h-f\|_1 + \text{const.}\ \varepsilon\|S(h-f)\|_1,$$

and hence, choosing $\varepsilon$ appropriately, by the right member of (I). We first make use of (I) to show

LEMMA 5. $\|[\Sigma|\langle h_k, \sigma_k\rangle - b_k|^2]^{1/2}\|_1 \leq C\|h-f\|_1^{1/4}\|h\|_1^{3/4}$.

PROOF. Write

$$2\frac{\operatorname{Re}\overline{\langle h_k,\sigma_k\rangle}\left(\langle h_k,\sigma_k\rangle - h_k\right)}{|\langle h_k,\sigma_k\rangle| + |b_k|} = \xi_k - |b_k|$$

where

$$\xi_k = \frac{|\langle h_k,\sigma_k\rangle - b_k|^2}{|\langle h_k,\sigma_k\rangle| + |b_k|} + |\langle h_k,\sigma_k\rangle|.$$

By the triangle inequality, the left side of (I) dominates

$$\left\|\left(\Sigma\,|\xi_k|^2\right)^{1/2}\right\|_1 - \left\|\left(\Sigma\,|b_k|^2\right)^{1/2}\right\|_1.$$

Also, since $b_k = \mathbf{E}[\langle h_k,\sigma_k\rangle]$,

$$\left\|\left(\Sigma\,|b_k|^2\right)^{1/2}\right\|_1 \leq \left\|\left(\Sigma\,|\langle h_k,\sigma_k\rangle|^2\right)^{1/2}\right\|_1.$$

Write

$$\left[\Sigma\left(\xi_k^2 - |\langle h_k,\sigma_k\rangle|^2\right)\right]^{1/2}$$
$$= \left[\left(\Sigma\xi_k^2\right)^{1/2} + \left(\Sigma\,|\langle h_k,\sigma_k\rangle|^2\right)^{1/2}\right]^{1/2}\left[\left(\Sigma\xi_k^2\right)^{1/2} - \left(\Sigma\,|\langle h_k,\sigma_k\rangle|^2\right)^{1/2}\right]^{1/2}$$

and apply Cauchy-Schwarz. From (I) and previous observations

$$\left\|\left[\Sigma\left(\xi_k^2 - |\langle h_k,\sigma_k\rangle|^2\right)\right]^{1/2}\right\|_1 \leq C\|h\|_1^{1/2}\|h-f\|_1^{1/4}\|h\|_1^{1/4} = C\|h-f\|_1^{1/4}\|h\|_1^{3/4}.$$

Since for each $k$,

$$\xi_k^2 - |\langle h_k,\sigma_k\rangle|^2 = \left(\xi_k + |\langle h_k,\sigma_k\rangle|\right)\frac{|\langle h_k,\sigma_k\rangle - b_k|^2}{|\langle h_k,\sigma_k\rangle| + |b_k|} \geq C|\langle h_k,\sigma_k\rangle - b_k|^2.$$

Lemma 5 is proved.

The left side of Lemma 5 dominates $\|f - \Sigma\langle h_k,\sigma_k\rangle\sigma_k\|_1$.

LEMMA 6. $\|\Sigma[h_k]_0\|_1$ *and* $\|[\Sigma|[h_k]_e - b_k\sigma_k|^2]^{1/2}\|_1 \leq C\|h-f\|_1^{1/8}\|h\|_1^{7/8}$.

PROOF. Since $\Sigma[h_k]_0 = h - \Sigma[h_k]_e$, the first inequality is a consequence of the second. Write

$$\left\|\left[\Sigma\,|[h_k]_e - b_k\sigma_k|^2\right]^{1/2}\right\|_1 \leq \left\|\left[\Sigma\,|[h_k]_e - \langle h_k,\sigma_k\rangle\sigma_k|^2\right]^{1/2}\right\|_1 + \left\|\left[\Sigma\,|\langle h_k,\sigma_k\rangle - b_k|^2\right]^{1/2}\right\|_1,$$

which by Lemmas 4(II) and 5 is estimated by

$$C\left\|h - \sum \langle h_k, \sigma_k \rangle \sigma_k\right\|_1^{1/2} \|h\|_1^{1/2} + C\|h - f\|_1^{1/4}\|h\|_1^{3/4} \leqslant C\|h - f\|_1^{1/8}\|h\|_1^{7/8}.$$

Define for $u \in L^1(\Pi^{\mathbf{N}})$,

$$(u)_e(x) = \int_D u(\varepsilon_1 x_1, \varepsilon_2 x_2, \dots)\, d\varepsilon$$

(= the natural projection on the even part in $x_1, x_2, \dots$).

LEMMA 7. $\|[\Sigma_k |(\hat{h}_k(1))_e|^2]^{1/2}\|_1 \leqslant C\|h - f\|_1^{11/16}\|h\|_1^{15/16}$.

PROOF. At this point, we will make use of Proposition 7. Fix $x \in \Pi^{\mathbf{N}}$ and remark that the sequence of functions in $\varepsilon \in D$,

$$[h_k]_0(\varepsilon_1 x_1, \varepsilon_2 x_2, \dots),$$

is a martingale difference sequence.

Moreover, the $k$th Rademacher coefficient is clearly given by

$$\sum_{n>0} (\hat{h}_k(n))_e(x) \sin nx_k$$

and Proposition 7 yields

$$\left[\sum_k \left|\sum_{n>0} (\hat{h}_k(n))_e(x) \sin nx_k\right|^2\right]^{1/2}$$
$$\leqslant C\left[\int \left|\sum [h_k]_0(\varepsilon \cdot x)\right| d\varepsilon\right]^{1/2}\left[\int \left[\sum |[h_k]_0(\varepsilon \cdot x)|^2\right]^{1/2} d\varepsilon\right]^{1/2}.$$

Integration in $x$, application of Cauchy-Schwarz and Lemma 6, gives

$$(+)\qquad \left\|\left[\sum_k \left|\sum_{n>0} (\hat{h}_k(n))_e \sin nx_k\right|^2\right]^{1/2}\right\| \leqslant C\|h - f\|_1^{1/16}\|h\|_1^{7/16}\left\|\left[\sum |[h_k]_0|^2\right]^{1/2}\right\|_1^{1/2}.$$

Also

$$\left\|\left[\sum |[h_k]_0|^2\right]^{1/2}\right\|_1 \leqslant C\|h\|_1.$$

On the other hand, we can multiply the $k$th increment in the left member of (+) by $\sin x_k$ and then take $\mathbf{E}_{k-1}$-expectation. Proposition 5 shows that

$$\left\|\left[\sum_k |(\hat{h}_k(1))_e|^2\right]^{1/2}\right\|_1 \leqslant C\|h - f\|_1^{1/16}\|h\|_1^{15/16},$$

proving Lemma 7.

Now, rewriting

$$\left[\sum |[h_k]_e - b_k\sigma_k|^2\right]^{1/2} = \left[\sum_k \left|\sum_{n>0} \hat{h}_k(n) \cos nx_k - b_k\sigma_k\right|^2\right]^{1/2}$$

multiplication of the $k$th increment by $\cos x_k$ and taking $\mathbf{E}_{k-1}$-expectation yields (by Proposition 5 and Lemma 6)

$$\left\| \left[ \sum_k \left| \frac{1}{2}\hat{h}_k(1) - \frac{2}{\pi} b_k \right|^2 \right]^{1/2} \right\|_1 \leq C\|h - f\|_1^{1/8} \|h\|_1^{7/8}.$$

Since $b_k = (b_k)_e$, a convexity argument allows us to replace, in a previous inequality, $\hat{h}_k(1)$ by $(\hat{h}_k(1))_e$. Combining with Lemma 7, we conclude

$$C^{-1}\|f\|_1 \leq \left\| \left( \sum |b_k|^2 \right)^{1/2} \right\|_1 \leq C\|h - f\|_1^{1/16} \|h\|_1^{15/16}, \qquad \|f\|_1 \leq C\|h - f\|_1,$$

and thus Proposition 10.

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