EMBEDDING $L^1$ IN $L^1/H^1$

BY

J. BOURGAIN

ABSTRACT. It is proved that $L^1$ is isomorphic to a subspace of $L^1/H^1$. More precisely, there exists a diffuse $\sigma$-algebra $\mathcal{E}$ on the circle such that the corresponding expectation $E: H^\infty \to L^\infty(C)$ is onto. The method consists in studying certain martingales on the product $\mathbb{P}^N$.

1. Introduction. Let us start by fixing some terminology. As usual, $\mathbb{T}$ will denote the circle equipped with its Haar measure $m$, $H^1_0$ is the subspace of those $f \in L^1(\mathbb{T})$ for which $\hat{f}(n) = 0$ for $n \leq 0$ and $q: L^1 \to L^1/H^1_0$ is the quotient map.

We are interested in the question whether or not there exists a linear embedding of the Banach space $L^1$ in the space $L^1/H^1_0$. We briefly indicate some motivation for this problem. First, it was (and still remains) an open question if the three-space-property holds for $L^1$-embedding, i.e. suppose $X$ a Banach space, $Y$ a subspace of $X$. Is it true that whenever $L^1$ embeds in $X$, it also has to embed in either $Y$ or $X/Y$?

The problem is also unsolved in the particular case $X = L^1$ and $Y$ isomorphic to a dual space. It is not hard to show that an embedding of $L^1$ in $X/Y$ is then equivalent to the existence of a subspace $S$ of $X$, $S$ isomorphic to $L^1$ so that the quotient map $X \to X/Y$ is an isomorphism when restricted to $S$.

In the special situation $X = L^1(\mathbb{T})$ and $Y = H^1_0$, the answer was unknown for some time. There was hope that this may provide a counterexample in view of the following result, due to W. B. Johnson (see [9]).

PROPOSITION 1. No complemented subspace of $L^1/H^1_0$ is isomorphic to $L^1$.

This is a consequence of the fact that any operator $T: L^1/H^1 \to L^1$ maps weakly compact sets onto norm compact sets. Let us sketch the argument.

Consider the identity map $I: L^\infty/H^\infty \to L^1/H^1$. Then $(TI)^*: L^\infty \to H^\infty \to H^1$ is integral and therefore nuclear (since $H^1$ satisfies the Radon-Nikodym property). Consequently, also $TI$ is nuclear. Given now a weakly null sequence $(x_n)_{n=1,2,\ldots}$ in $L^1/H^1$, it follows from the lifting property (see [9] for instance) that $x_n = q(f_n)$ where $(f_n; n = 1,2,\ldots)$ is a relatively weakly compact set in $L^1(\mathbb{T})$. Therefore, for each $\varepsilon > 0$, a truncation argument provides a bounded sequence $(g_n)$ in $L^\infty$ such that $\|f_n - g_n\|_1 < \varepsilon$ for each $n$. Thus

$$\|Tx_n - TIg_n\| \leq \|T\| \|x_n - Ig_n\| < \varepsilon \|T\|.$$
Because $TI$ is nuclear, the set $\{TI(\tilde{g}_n); \ n = 1, 2, \ldots\}$ is compact for each $\varepsilon > 0$. So we conclude that $\{Tx_n\}$ is compact, as announced.

Using Proposition 1, the following is proved in [2].

**Proposition 2.** There is no almost isometric embedding of the complex $L^1$ space in $L^1/H^1$.

Thus $d(S, L^1) > \gamma > 1$ for each subspace $S$ of $L^1/H^0_0$, where $d$ is the Banach-Mazur distance (see [8, 9] for definitions). This observation allows us to define a natural distortion of $L^1$, by taking

$$\|f\|_1 = \|f\|_1 + \|q(f)\|_1, \quad f \in L^1(\mathbb{I}).$$

Say that an operator $T : X \to Y$ is a semiembedding provided $T$ is one-one and maps the closed unit ball of $X$ on a norm-closed subset of $Y$. It can be shown that a semiembedding $T : L^1 \to L^1$ has to fix an $L^1$-copy (i.e. is an isomorphism when restricted to a subspace $S$ of $L^1$, $S$ isomorphic to $L^1$). On the other hand, (see [3]):

**Proposition 3.** The restriction of the quotient map $q : L^1 \to L^1/H^0$ to the subspace $L^1_R$ of real functions in $L^1(\mathbb{I})$ is a semiembedding.

No example is known of a semiembedding of $L^1$ in a Banach space $X$ not containing $L^1$.

Our purpose is to prove the existence of a natural embedding of $L^1$ in $L^1/H^0$. There exists a diffuse $\sigma$-algebra $\mathcal{O}$ on $\mathbb{I}$ so that the restriction of $q$ to the complex $L^1(\mathcal{O})$-space is an isomorphism. More precisely:

**Theorem.** There exists an increasing sequence $(n_k)$ of positive integers, such that if $\mathcal{O}$ is the $\sigma$-algebra on $\mathbb{I}$ generated by the functions $\sigma_k(\theta) = \text{sign } \cos n_k \theta$, then the restriction of $q$ to $L^1(\mathcal{O})$ is an isomorphism. Consequently, for this $\sigma$-algebra $\mathcal{O}$, the expectation operator $E : H^\infty \to L^\infty(\mathcal{O})$ is onto.

The argument presented here is rather delicate. In order to give the reader an idea how it is organised, we briefly outline the proof. We have to introduce the $\sigma$-algebra $\mathcal{O}$ such that the inequality

$$(\ast) \quad \|h - E_{\mathcal{O}}[h]\|_1 \geq \delta\|h\|_1$$

holds for each $h \in H^1_0$. But choosing the sequence $(n_k)$ sufficiently lacunary, it is enough to verify $(\ast)$ for functions $h$ with spectrum contained in a set of the form

$$E = \{\Sigma v_k n_k; \ |v_k| \leq a_k \text{ for each } k\}$$

where $(a_k)$ is a sequence of positive integers and $(n_k), (a_k)$ satisfy the transference property. Thus the $n_k$-frequencies can be replaced by independent variables. The space $H^1_0 \cap L^1_E$ identifies with a subspace of the space $\mathcal{K} \subset L^1(\mathbb{I}^N)$ of those functions $h = \Sigma h_k$ on $\mathbb{I}^N$ such that each increment $h_k = h_k(x_1, \ldots, x_k)$ is an $H^1_0$-function in $x_k$. The required inequality now becomes

$$(\ast\ast) \quad \|h - E_{\mathcal{O}}[h]\|_1 \geq \delta\|h\|_1$$

for $h \in \mathcal{K}$, where $\mathcal{O}$ is a natural diadic product $\sigma$-algebra on $\mathbb{I}^N$ (generated by the functions $\sigma_k(x) = \text{sign } \cos x_k$).
This reduction of the problem is worked out in §4. Its purpose is to approach the problem with martingale techniques. The martingale prerequisites are given in §2. To obtain (**) we first prove \( L^1 \)-estimations for certain square functions related to \( h \) (see Lemma 4). These are derived using a “step-by-step” method (explained at the beginning of §5) and an examination of what happens at each increment. More precisely, we have to consider at this point functions of the form \( a + h - b \sigma \), where \( a, b \) are scalars, \( h \in H_0^1 \) and \( \sigma = \text{sign } \cos \).

Minorations of the \( L^{-1} \)-norm of such expressions are given in Propositions 8 and 9 below. It is only at this place that some complex function theory will be involved.

2. Martingale preliminaries. Let \( (\mathcal{F}_k)_{k=0,1,2,\ldots} \) be an increasing sequence of \( \sigma \)-algebras on a probability space \( (\Omega, \mathcal{F}, P) \) assuming \( \mathcal{F} = \bigvee_{k=1}^{\infty} \mathcal{F}_k \). Denote by \( E_k \) the expectation with respect to \( \mathcal{F}_k \). For \( f \in L^1(\mathcal{F}) \) let

\[
\begin{align*}
  f* &= \sup_k |E_k[f]| \quad \text{and} \quad S(f) = \left[ |E_0[f]|^2 + \sum_{k=1}^{\infty} |E_k[f] - E_{k-1}[f]|^2 \right]^{1/2},
\end{align*}
\]

We will use the notation \( C \) to indicate a numerical constant. Let us recall the following result, due to D. Davis (see [7]).

**Proposition 4.** \( C^{-1} \| S(f) \|_1 \leq \| f* \|_1 \leq C \| S(f) \|_1 \).

The next inequality is probably known, but we include its proof here for the sake of completeness.

**Proposition 5.** Let \( (v_k) \) be an adapted sequence of functions; thus \( v_k \) is \( \mathcal{F}_k \)-measurable for each \( k \). Then

\[
\left\| \sum |E_{k-1}[v_k]|^2 \right\|^{1/2}_1 \leq C \left\| \sum |v_k|^2 \right\|^{1/2}_1.
\]

**Proof.** It is no restriction to assume the \( \mathcal{F}_k \) finite algebras. Moreover, since one may always tensor the \( v_k \) against a Rademacher sequence, we can assume \( E_{k-1}[v_k] = 0 \) and thus \( (v_k) \) is an adapted martingale difference sequence. Since, then

\[
\left\| \sum |v_k|^2 \right\|^{1/2}_1 = \left\| \sum v_k \right\|_{H^1(\mathcal{F}_k)},
\]

it follows from the atomic decomposition property for \( H^1 \)-functions (see for instance [7, Chapter I]) and convexity, that we may take for \( \sum v_k \) a function of the form (for some positive integer \( j \))

\[
a = \frac{1}{|A|} (\varphi - E_{j-1}[\varphi])
\]

where \( A \) is an \( \mathcal{F}_j \)-atom, \( \text{supp } \varphi \subset A \) and \( \| \varphi \|_{\infty} \leq 1 \). In this case

\[
v_k = E_k[a] - E_{k-1}[a] = 0 \quad \text{for } k < j,
\]

\[
= \frac{1}{|A|} (E_k[\varphi] - E_{k-1}[\varphi]) \quad \text{for } k \geq j.
\]
Also, $E_k[\varphi]$ is supported by $A$ for $k \geq j$ and hence $v_k$ for $k > j$. Thus the left side in Proposition 5 is dominated by

$$\|v_j\|_1 + \left\| \left( \sum_{k > j} E_{k-1} \left[ |v_k|^2 \right] \right)^{1/2} \right\|_1 \leq 2 + \int_A \left( \sum_{k > j} E_{k-1} \left[ |v_k|^2 \right] \right)^{1/2} \quad \text{(by Cauchy-Schwarz)}$$

$$\leq 2 + |A|^{1/2} \left( \int \sum_{k > j} |v_k|^2 \right)^{1/2}$$

$$\leq 2 + |A|^{1/2} \|a\|_2 \leq 3,$$

proving the result.

**Proposition 6.** For $f \in H^1(\mathbb{T}_N)$, one has an inequality

$$\left( \sum \|E_k - E_{k-1}\|_2 \right)^{1/2} \leq C\|f\|_1^{1/2}\|f\|_H^{1/2}.$$

To prove this, we will first deal with the special case of the Rademacher projection on the Cantor group (in fact, only this will be used later on).

**Proposition 7.** If $D = \{1, -1\}^N$ is the Cantor group and $f \in H^1(D)$, then

$$\left( \sum \hat{f}(k)^2 \right)^{1/2} \leq C\|f\|_1^{1/2}\|f\|_H^{1/2}$$

where $\hat{f}(k) = \hat{f}(\epsilon)\epsilon_k$.

**Proof.** We will use the theorem of [6] on the BMO-distance of a BMO-function to $L^\infty$ (in the diadic setting). The result asserts, in particular, that for $\varphi \in \text{BMO}(D)$,

$$\text{dist}_{\text{BMO}}(\varphi, L^\infty) = 0 \land \epsilon > 0 \land \exists \alpha + \beta \quad \text{such that}$$

$$\|\alpha\|_{\text{BMO}} \leq C_1 \epsilon \quad \text{and} \quad \|\beta\|_\infty \leq C_2 \max(\epsilon, \lambda_0(\epsilon))$$

where $\lambda_0 = \lambda(\epsilon)$ has to satisfy

$$\sup_I \frac{1}{|I|} \left| \{ x \in I ; |\varphi(x) - \varphi_I| > \lambda \} \right| \leq e^{-\lambda/\epsilon}$$

whenever $\lambda > \lambda(\epsilon) \quad (\varphi_I = |I|^{-1}\int_I \varphi)$.

Now take $\varphi = \sum a_k \epsilon_k$ with $\sum |a_k|^2 = 1$. It follows from the distribution property of Rademacher that for each diadic interval $I$,

$$\left| \{ \alpha \in I ; |\varphi(x) - \varphi_I| > \lambda \} \right| \leq C e^{-c\lambda^2}|I|,$$

for numerical constants $c > 0, C < \infty$. Hence $\text{dist}_{\text{BMO}}(\varphi, L^\infty) = 0$ and $\lambda(\epsilon) \sim 1/\epsilon$.

Decomposing $\varphi = \alpha + \beta$ as above, we get

$$|\langle f, \varphi \rangle| \leq |\langle f, \alpha \rangle| + |\langle f, \beta \rangle| \leq C_1\|f\|_H + C_2 \frac{1}{\epsilon} \|f\|_1.$$

Taking supremum over $\varphi$ and choosing $\epsilon = \|f\|_1^{1/2}\|f\|_{H^1}^{1/2}$, the inequality follows.
PROOF OF PROPOSITION 6. Assume \( f \) real and estimate
\[
\left( \sum_{k=1}^{K} \|(E_k - E_{k-1})[f]\|_1^2 \right)^{1/2}.
\]
Define for each \( k \),
\[
\sigma_k = \text{sign } \Delta f_k \quad \text{and} \quad b_k = \frac{1}{2}(\sigma_k - E_{k-1}[\sigma_k]).
\]
Then
\[
\|f\|_1 \geq \int \int |f| \prod_{k=1}^{K} (1 + \epsilon_k b_k) \, d\epsilon \, d\omega \geq \frac{1}{2} \int \sum_{k=1}^{K} \epsilon_k \Phi_k(\epsilon) \, d\epsilon
\]
where
\[
\Phi_k(\epsilon) = \int \prod_{j=1}^{k-1} (1 + \epsilon_j b_j) |\Delta f_k| \, d\omega.
\]
Application of Proposition 7 to the function \( \sum \epsilon_k \Phi_k(\epsilon) \) then gives
\[
\left( \sum_{k=1}^{K} \|\Delta f_k\|_1^2 \right)^{1/2} \leq C \|f\|_1^{-1/2} \left[ \int \left( \sum |\phi_k(\epsilon)|^2 \right)^{1/2} \, d\epsilon \right]^{1/2}
\]
\[
\leq C \|f\|_1^{1/2} \left[ \int \int S(f) \prod (1 + \epsilon_j b_j) \, d\omega \, d\epsilon \right]^{1/2}
\]
\[
= C \|f\|_1^{1/2} \|f\|_H^1
\]
as announced.

REMARK. The author is grateful to P. W. Jones for outlining a more explicit procedure to obtain the decomposition used in the proof of Proposition 7.

3. Some inequalities involving \( H_0^1 \)-functions. The purpose of this section is to prove the following results.

PROPOSITION 8. For \( a \in \mathbb{C} \) and \( h \in H_0^1 \), one has
\[
\|a + h\|_1 \geq \left( |a| + \delta|h| \right)^{1/2}
\]
where \( \delta > 0 \) is a fixed constant.

PROPOSITION 9. There exists \( \delta > 0 \) such that for \( a \in \mathbb{C} \), \( b \in \mathbb{C} \) and \( h \in H_0^1 \),
\[
\|a + h - b\sigma\|_1 \geq \left( |a| + \delta \left[ \text{Re}(\langle h, \sigma \rangle (\langle h, \sigma \rangle - b)) \right] \right)^{1/2},
\]
\[
\|a + h - \left( \langle h, \sigma \rangle \sigma \right)\|_1 \geq \left( |a| + \delta^2 |h| - \langle h, \sigma \rangle \sigma \right)^{1/2},
\]
where \( \sigma = \text{sign cos} \) and \( h_\epsilon(\theta) = \sum_{n=1}^{\infty} \tilde{h}(n) \cos n\theta \).
It is clear that it suffices to prove Propositions 8 and 9, with $a = 1$.

**Proof of Proposition 8.** Factoring $1 + h$ gives $1 + h = (1 + g_1)(1 + g_2)$ where $g_1, g_2 \in H_0^2$ and

$$
\|1 + h\|_1 = \left(1 + \|g_1\|^2_2\right)^{1/2} \left(1 + \|g_2\|^2_2\right)^{1/2}.
$$

Since $|h| \leq |g_1| + |g_2| + |g_1||g_2|$ the result follows from the majorations

$$
\left\|\left(1 + |g_1|^2\right)^{1/2}\right\|_1 \leq \left\|\left(1 + |g_1|^2\right)^{1/2}\right\|_2 = \left(1 + \|g_2\|^2_2\right)^{1/2} \leq 1 + h, \quad (i = 1, 2)
$$

and

$$
\left\|\left(1 + |g_1|^2|g_2|^2\right)^{1/2}\right\|_1 \leq 1 + \|g_1g_2\|_1 \leq 1 + \|g_1\|_2\|g_2\|_2 \leq 1 + h.
$$

Also to obtain Proposition 9, we will use the $L^2$-theory. Our argument here is, however, more complicated. This is the only point where explicit constructions of $H^{\infty}$-functions appear.

**Lemma 1.** Given a measurable subset $A$ of $\mathbb{P}$, there exist $H^{\infty}$-functions $\varphi$ and $\psi$ satisfying the following conditions:

(i) $|\varphi| + |\psi| \leq 1,$
(ii) $\text{Re } \psi$ is an even function on $\mathbb{P},$
(iii) $|\varphi - 1/8| < 1/100$ on the set $A,$
(iv) $\|\varphi\|_1 \leq C|A|,$
(v) $\|\text{Re } \psi - 1\|_1 \leq C|A|.$

**Proof.** Fix some (large) $M > 0$ and define the following $H^{\infty}$-functions:

$$
\tau(z) = -M \int_A \frac{e^{i\theta} + z}{e^{i\theta} - z} m(d\theta), \quad \varphi = \frac{1}{8} (1 - e^\tau)^2,
$$

$$
\psi(z) = \exp \left\{ \int_{\mathbb{P}} \log(1 - \alpha(\theta)) \frac{e^{i\theta} + z}{e^{i\theta} - z} m(d\theta) \right\}
$$

where $\alpha(\theta) = |\varphi(e^{i\theta})| \vee |\varphi(e^{-i\theta})|.$

Notice that this makes sense, because $e^\tau$ has boundary value $e^{-M(\mathcal{H} + i\mathcal{H}(\mathcal{H}))}$ ($\mathcal{H} = \text{Hilbert-transform}$) and therefore $\|\alpha\|_\infty \leq \frac{1}{2}.$

(i) is obvious. On $\mathbb{P}$, we have $\text{Re } \psi = (1 - \alpha) \cos \mathcal{H}(\log(1 - \alpha))$ and thus an even function. Since $|\varphi - 1/8| \leq 3/8 |e^\tau|$ and thus $|\varphi - 1/8| < e^{-M}$ on $A$ (iii) holds for $M$ large enough. Because on $\mathbb{P}$

$$
8|\varphi| \leq \mathcal{H} + M^2|\mathcal{H}(\mathcal{H})|^2,
$$

(iv) follows. Finally,

$$
|1 - \text{Re } \psi| \leq |\alpha| + \frac{1}{2} |\mathcal{H}(\log(1 - \alpha))|^2, \quad \|1 - \text{Re } \psi\|_1 \leq 4\|\varphi\|_1
$$

and hence (v).

We refer the reader to [4, Proposition 1.6] for the following Marcinkiewicz type decomposition.
Lemma 2. There is a constant $C < \infty$ such that for given $h \in H^1_0$ and $\lambda > 0$, there exists $h_\lambda \in H^\infty_0$ satisfying:

(i) $|h_\lambda| \leq C|h|$,
(ii) $\|h_\lambda\|_\infty \leq C\lambda$,
(iii) $\|h - h_\lambda\|_1 \leq C_{\|h_\lambda\|_1} |h|$.

Let $h$ be as in Proposition 9. For $\lambda > 0$, define $A_\lambda = |h| > \lambda$. Application of Lemma 1 to the set $A_\lambda$ provides $H^\infty$-functions $\varphi_\lambda, \psi_\lambda$. We are now ready to prove

Lemma 3. $\|1 + h - b\sigma\|_1 \geq 1 + c\int_{A_\lambda} |h| + c\lambda^{-2}\|\text{Im}(h_\lambda - b\sigma)\|_2^2$ if $\lambda > K$ and $|b| < \lambda/K$ ($c > 0$ and $K < \infty$ being numerical constants).

Proof. First, since $1 - b\sigma$ is even and $\text{Im} \psi_\lambda$ odd, we find

$$\|1 + h - b\sigma\|_1 \geq \|1 + h - b\sigma\|_1 + \int (1 - b\sigma)\psi_\lambda$$

$$\geq \frac{1}{9} \int_{A_\lambda} |h| - (1 + |b|) + \int (1 - b\sigma)\text{Re} \psi_\lambda$$

$$\geq \frac{1}{9} \int_{A_\lambda} |h| - \frac{1}{9} (1 + |b|)|A_\lambda| + 1 - (1 + |b|)\|1 - \text{Re} \psi_\lambda\|_1$$

$$\geq \frac{1}{9} \int_{A_\lambda} |h| - C(1 + |b|)|A_\lambda| + 1$$

for some constant $C$. Thus, choosing $K$ large enough, we get

$$(\ast) \quad \|1 + h - b\sigma\|_1 \geq 1 + \frac{1}{10} \int_{A_\lambda} |h|.$$  

Fix some small constant $\delta > 0$. Since we always have

$$\|1 + af\|_1 \leq \|1 + f\|_1 \quad \text{for } 0 \leq a \leq 1 \text{ and } f \text{ of mean } 0,$$

it follows that

$$\|1 + h - b\sigma\|_1 \geq \|1 + \delta\lambda^{-1}(h - b\sigma)\|_1 \geq \|1 + \delta\lambda^{-1}(h_\lambda - b\sigma)\|_1 - \delta\lambda^{-1}\|h - h_\lambda\|_1.$$  

Because $\delta\lambda^{-1} |h_\lambda - b\sigma| \ll 1$ the inequality

$$(1 + t)^{1/2} \geq 1 + t/3 \quad \text{for } 0 \leq t \leq 1$$

yields

$$\|1 + \delta\lambda^{-1}(h_\lambda - b\sigma)\| \geq \left[1 + \delta\lambda^{-1}\text{Re}(h_\lambda - b\sigma)\right] \left[1 + \frac{1}{2\lambda^2}\lambda^{-2}(\text{Im}(h_\lambda - b\sigma))^2\right].$$

Therefore, also

$$(\ast\ast) \quad \|1 + h - b\sigma\|_1 \geq 1 + \frac{1}{20}\delta^{-2}\int_{A_\lambda} |h|.$$  

The required minoration clearly follows combining $(\ast)$ and $(\ast\ast)$.

Proof of Proposition 9. First

$$\|1 + h - b\sigma\|_1 \geq d(b\sigma, H^1) \geq \left\{ \kappa b \right\} \left\{ \int_{-\pi}^{\pi} \sigma(\theta)e^{i\theta}d\theta \right\} = \frac{2}{\pi} |b|.$$
and hence, also,
\[ \| 1 + h - b \sigma \|_1 \geq \frac{1}{2} \| 1 + h \|_1 \geq \frac{1}{2} \| h \|_1. \]

Notice that the right member of (i), (ii) is bounded by \( 1 + 2 \delta \| h \|_1 \). Since now \( \| 1 + h - b \sigma \|_1 \geq \frac{1}{2} \| h \|_1 + \frac{1}{2} | b | \), it follows that (i) (resp. (ii)) are satisfied for \( | b | \geq 6 \) (resp. \( | \langle h, \sigma \rangle | \geq 6 \)). Hence, we may assume \( | b | \leq M \) in (i), \( | \langle h, \sigma \rangle | \leq M \) in (ii) where \( M \) is some numerical constant.

Fix a constant \( \lambda > KM \) and put \( k = h(x) \) for simplicity. Using again Lemma 2(iii), the right member of (i) can be majorized by
\[
1 + 2 \delta^2 \left[ | \text{Re} \langle h, \sigma \rangle |^2 + | \text{Im} \langle h, \sigma \rangle - b |^2 \right]^{1/2}
\leq \left[ 1 + 2 \delta^2 \left( | \text{Re} \langle k, \sigma \rangle |^2 + | \text{Im} \langle k, \sigma \rangle - b |^2 \right) \right]^{1/2} + 2 \delta C \int_{A_{\lambda}} |h|.
\]

Taking Lemma 3 into account, we see that it suffices to check the inequality
\[
| \text{Re} \langle k, \sigma \rangle |^2 + | \text{Im} \langle k, \sigma \rangle - b |^2 \leq \| \text{Im} (k - b \sigma) \|_2^2
\]
which is straightforward:
\[
\| \text{Im} (k - b \sigma) \|_2^2 = \frac{1}{2} \sum_{n > 0} | \text{Im} \hat{k} (n) - 2 \text{Im} b \hat{\sigma} (n) |^2 + \frac{1}{2} \sum_{n > 0} | \text{Re} \hat{k} (n) |^2
\]
while
\[
| \text{Re} \langle k, \sigma \rangle | \leq \sum_{n > 0} | \text{Re} \hat{k} (n) | \hat{\sigma} (n) \leq \frac{1}{\sqrt{2}} \left( \sum_{n > 0} | \text{Re} \hat{k} (n) |^2 \right)^{1/2},
\]
\[
| \text{Im} \langle k, \sigma \rangle - b | \leq \sum_{n > 0} | \text{Im} \hat{k} (n) - 2 \text{Im} b \hat{\sigma} (n) | \hat{\sigma} (n) \leq \frac{1}{\sqrt{2}} \left( \sum_{n > 0} | \text{Im} \hat{k} (n) - 2 \text{Im} b \hat{\sigma} (n) |^2 \right)^{1/2}.
\]

For the right member of (ii), a similar reasoning reduces the question to the verification of
\[
\int | \text{Re} \langle k, \sigma \rangle - \langle k, \sigma \rangle \sigma |^2 \leq \| \text{Im} (k - b \sigma) \|_2^2,
\]
which the reader will easily do.

4. Reduction of the problem. In this section, we will reduce the problem of proving that certain elements of \( L^1(\Pi) \) normed by the quotient norm \( L^1/H^1 \) to the verification of an inequality for certain functions in \( L^1(\Pi^N) \), where \( \Pi^N = \Pi \times \Pi \times \cdots \) is the product group. Denote by \( E_k (k = 1, 2, \ldots) \) the expectation with respect to the \( k \) first variables \( (x_1, x_2, \ldots, x_k) \), where \( x = (x_1, x_2, \ldots) \) is the product variable.

We consider the subspace \( \mathcal{K} \) of \( L^1(\Pi^N) \) of those functions \( h \) such that for each \( k \) the difference \( E_k [h] - E_{k-1} [h] \) is an \( H^0_0 \)-function with respect to \( x_k \). Thus \( h \) is of the form
\[
h = \sum h_k \quad \text{where} \quad h_k = \sum_{n > 0} \hat{h}_k (n) e^{inx_k}
\]
and the \( \hat{h}_k (n) \) are functions of \( x_1, \ldots, x_{k-1} \).
Again let \( \sigma = \sign \cos \) and \( \sigma_k(x) = \sigma(x_k) \) for each \( k \). Let \( \mathcal{F} \) be the \( \sigma \)-algebra on \( \prod^N \) generated by the \( \sigma_k \). In the next section, we show the following

**Proposition 10.** There is a constant \( c > 0 \) s.t. \( \| h - E_{\sigma}[h] \|_1 \geq c \| h \|_1 \) for all \( h \in \mathcal{F} \).

This fact obviously implies

**Corollary 11.** \( \inf_{h \in \mathcal{F}} \| f - h \|_1 \geq c' \| f \|_1 \) for all \( f \in L^1(\mathcal{F}) \).

For \( a, n \) positive integers, \( \mathcal{F}_a \) will be the Fejér kernel

\[
F_a(\theta) = \sum_{|j| < a} \frac{a + 1 - |j|}{a + 1} e^{ij\theta}
\]

and \( F_{a,n}(\theta) = F_a(n\theta) \).

We consider sequences of positive integers \((n_k)\), \((a_k)\) satisfying the following conditions: (\( \mathcal{G} \) denotes again the \( \sigma \)-algebra on \( \prod \) generated by the functions \( \sigma(n_k\theta) \).)

(i) The transference property, i.e. let \( E = (\Sigma' v_k n_k; (v_k) \in F) \) where \( F \) is the subset \( \{ (v_k), |v_k| \leq a_k \} \) of the dual group of \( \prod^N \). Then the operator

\[
T: L^1_E(\Pi) \to L^1_F(\Pi^n), \quad T(f)(x) = \sum_{(v_k) \in F} \hat{f}(\sum v_k n_k) e^{i(\sum v_k x_k)}
\]

satisfies

\[
\frac{1}{2} \| f \|_1 \leq \| T(f) \|_1 \leq 2 \| f \|_1.
\]

Moreover, \( T(f) \in \mathcal{F} \) for \( f \in L^1_E \cap H_0^1 \).

(ii) Defining for each \( k \),

\[
\xi_k = \sigma \ast F_{a_k}, \quad K = \prod_k F_{a_k,n_k},
\]

\[
R(\theta, \psi) = \prod_k \left[ 1 + \xi_k(n_k\theta)\sigma(n_k\psi) \right],
\]

one has

(\( \alpha \)) \( \sum \| \xi_k - \sigma \|_1 \leq \epsilon \),

(\( \beta \)) \( \| K \|_1 = 1 \).

For \( f \in L^1(\mathcal{G}) \),

(\( \gamma \)) \( \| f - f \ast K \|_1 \leq \epsilon \| f \|_1 \),

(\( \delta \)) \( \| f - R(f) \|_1 < \epsilon \| f \|_1 \) where \( R(f)(\theta) = \int f(\psi)R(\theta, \psi)m(d\psi) \) (where \( \epsilon > 0 \) is a small constant).

The reader will easily convince himself that the realisation of these conditions is straightforward. Details on the transference property can be found in [1].

Let us now show that the sequence \((n_k)\) satisfies the Theorem. Thus, fix \( f \in L^1(\mathcal{G}) \) and \( h \in H_0^1 \). We get, by (ii),

\[
\| f - h \|_1 \geq \| f \ast K - h \ast K \|_1 \geq \| R(f) - h \ast K \|_1 \geq 2\epsilon \| f \|_1.
\]

Notice that \( R(f) \in L^1_E \). By (i),

\[
\| R(f) - h \ast K \|_1 \geq \frac{1}{2} \| T(R(f)) - h_1 \|_1
\]

where \( h_1 = T(h \ast K) \in \mathcal{F} \).
Now
\[ T(R(f))(x) = \int f(\psi) \prod \left[ 1 + \xi_k(x_k) \sigma(n_k \psi) \right] m(d\psi). \]

By (ii)(a), we see that for any \((\pm 1)\)-sequence \((\tau_k)\)
\[ \| \otimes (1 + \tau_k \xi_k) - \prod (1 + \tau_k \sigma_k) \|_1 < \varepsilon \]
implying that
\[ \| T(R(f)) - f_1 \| < 2\varepsilon\|f\| \quad \text{where} \quad f_1 = E[T(R(f))]. \]

It follows then from Corollary 11 that
\[ \| f_1 - h \|_1 \geq \frac{1}{2} \| f_1 - h \|_1 - 3\varepsilon\|f\|_1 \geq \frac{c}{2} \| f_1 \|_1 - 3\varepsilon\|f\|_1 \]
\[ \geq \frac{c}{2} \| T(R(f)) \|_1 - 4\varepsilon\|f\|_1 \geq \frac{c}{4} \| f_1 \|_1 - 5\varepsilon\|f\|_1 \geq c'\|f\|_1 \]
taking \(\varepsilon > 0\) small enough.

5. Proof of the Theorem. It remains to prove Proposition 10. So fix \(h = \sum h_k \in \mathcal{H}\) where
\[ h_k = \sum_{n>0} \hat{h}_k(n)(x_1, \ldots, x_{k-1}) e^{inx_k}. \]

We also define
\[ [h_k]_e = \sum \hat{h}_k(n) \cos n x_k, \]
\[ [h_k]_o = \sum \hat{h}_k(n) \sin n x_k, \]
\[ \langle h_k, \sigma_k \rangle = \sum \hat{h}_k(n) \sigma(n) \]
(which is thus a function of \(x_1, \ldots, x_{k-1}\)). If \(f = E[h]\), then \(f = \Sigma b_k \cdot \sigma_k\), where
\[ b_k = b_k(x_1, \ldots, x_{k-1}) = E[h_k, \sigma_k]. \]

Using E. Stein’s theorem on \(H^1\)-multipliers (see [11]), it is easily seen that
\(\| h \|_1 \sim \| S(h) \|_1 \) (\(S = \) square function with respect to the natural decomposition).

We give a direct proof of this fact, based on Proposition 8.

Fix \(1 > \varepsilon > 0\) and a positive sequence \((s_k)_{k=1, 2, \ldots} \in L^\infty(\mathbb{N})\) satifying
\(\| (\Sigma s_k^2) \|_\infty \leq \varepsilon\). Fixing a positive integer \(K\), we get, using Proposition 8,
\[ \| E_K[h] \|_1 = \| E_{K-1}[h] + h_k \|_1 \]
\[ \geq \left\| \left( \| E_{K-1}[h] \|^2 + \delta^2 |h_k|^2 \right)^{1/2} \right\|_1 \]
\[ \geq \| E_{K-1}[h](1 - s_K^2)^{1/2} \|_1 + \delta \| h_k s_k \|_1 \]
\[ \geq \| E_{K-1}[h] \|_1 + \delta \| h_k s_k \|_1 - \| E_{K-1}[h] s_k^2 \|_1. \]

Iterating,
\[ \| h \|_1 \geq \delta \sum \| h_k s_k \|_1 - \sum \| E_{K-1}[h] s_k^2 \|_1 \]
\[ \geq \delta \sum \| h_k s_k \|_1 - \varepsilon^2 \max_k \| E_k[h] \|_1. \]
Taking supremum over the sequences \((s_k)\), it follows that
\[
\|h\|_1 \geq \delta \varepsilon \|S(h)\|_1 - \varepsilon^2 \max_k |E_k[h]|_1
\]
and choosing
\[
\varepsilon^2 = \frac{\|h\|_1}{\max |E_k[h]|_1},
\]
we get
\[
\|S(h)\|_1 \leq \delta^{-1} \|h\|_1^{1/2} \max |E_k[h]|_1^{1/2}.
\]
Hence, by Proposition 4, \(\|S(h)\|_1 \leq \delta^{-2} \|h\|_1\) as required.

Before continuing, notice that since \(\mathcal{F}\)-expectation is a contraction, \(\|S(f)\|_1 \leq \|S(h)\|_1\). Since for each \(k\), \(\cdots \langle h_k, \sigma_k \rangle \leq E_{k-1}[\|h_k\|]\), application of Proposition 5 yields
\[
\left\| \left( \sum |\langle h_k, \sigma_k \rangle|^2 \right)^{1/2} \right\|_1 \leq C \|h\|_1.
\]
If we now apply the previous procedure using Proposition 9, the following inequalities are derived.

**Lemma 4.**

\begin{align*}
(1) & \quad \left\| \left\{ \sum_k \left| \frac{\text{Re} \langle h_k, \sigma_k \rangle (\langle h_k, \sigma_k \rangle - b_k)}{|\langle h_k, \sigma_k \rangle| + |b_k|} \right|^{2/2} \right\}^{1/2} \right\|_1 \leq C \|h - f\|_1^{1/2} \|h\|_1^{1/2}, \\
(2) & \quad \left\| \left\{ \sum_k |[h_k]_k - \langle h_k, \sigma_k \rangle \sigma_k|^2 \right\}^{1/2} \right\|_1 \leq C \|h - \sum \langle h_k, \sigma_k \rangle \sigma_k\|_1^{1/2} \|h\|_1^{1/2}.
\end{align*}

**Proof.** Let us show how (I) follows from Proposition 9(i). The argument for (II) is analogous. Fix \(0 < \varepsilon < 1\) and a sequence \((s_k)_{k=1,2,\ldots}\) of positive \(L^\infty\)-functions on \(\mathbb{N}\) satisfying \(\|(\sum s_k^2)^{1/2}\|_\infty \leq \varepsilon\). Fix an integer \(k\) and apply Proposition 9(i) in the variable \(x_k\). We get
\[
\|E_k[h - f]\|_1 = \|E_{k-1}[h - f] + h_k - b_k \sigma_k\|_1
\]
\[
\geq \left\| \left\{ \left\| E_{k-1}[h - f] \right\|^2 + \delta^2 \text{Re} \langle h_k, \sigma_k \rangle (\langle h_k, \sigma_k \rangle - b_k) \right\} \left\| E_{k-1}[h - f] \right\|_1^{1/2} \right\|_1^{1/2}
\]
\[
\geq \|E_{k-1}[h - f]\|_1 + \delta \left\| \text{Re} \langle h_k, \sigma_k \rangle (\langle h_k, \sigma_k \rangle - b_k) s_k \right\|_1
\]
\[
- \|E_{k-1}[h - f] s_k\|_1.
\]
Iterating and using the same considerations as in the beginning of this section it follows that the left member of (I) is dominated by
\[
\delta^{-1} \varepsilon^{-1} \|h - f\|_1 + \text{const.} \varepsilon \|S(h - f)\|_1,
\]
and hence, choosing $\epsilon$ appropriately, by the right member of (I). We first make use of (I) to show

**Lemma 5.** \(|\Sigma \langle h_k, \sigma_k \rangle - b_k^2\|/2\|_1 \leq C\|h - f\|_4\|h\|_8^{3/4}\).**

**Proof.** Write

\[
2 \Re \frac{\langle h_k, \sigma_k \rangle (\langle h_k, \sigma_k \rangle - h_k)}{|\langle h_k, \sigma_k \rangle| + |b_k|} = \xi_k - |b_k|
\]

where

\[
\xi_k = \frac{|\langle h_k, \sigma_k \rangle - b_k|^2}{|\langle h_k, \sigma_k \rangle| + |b_k|} + |\langle h_k, \sigma_k \rangle|.
\]

By the triangle inequality, the left side of (I) dominates

\[
\left\| \left( \sum |\xi_k|^2 \right)^{1/2} - \left( \sum |b_k|^2 \right)^{1/2} \right\|_1.
\]

Also, since $b_k = E[\langle h_k, \sigma_k \rangle],

\[
\left\| \left( \sum |b_k|^2 \right)^{1/2} \right\|_1 \leq \left\| \left( \sum |\langle h_k, \sigma_k \rangle|^2 \right)^{1/2} \right\|_1.
\]

Write

\[
\left[ \sum (\xi_k^2 - |\langle h_k, \sigma_k \rangle|^2) \right]^{1/2}
\]

\[
= \left[ \left( \sum \xi_k^2 \right)^{1/2} + \left( \sum |\langle h_k, \sigma_k \rangle|^2 \right)^{1/2} \right]^{1/2} \left[ \left( \sum \xi_k^2 \right)^{1/2} - \left( \sum |\langle h_k, \sigma_k \rangle|^2 \right)^{1/2} \right]^{1/2}
\]

and apply Cauchy-Schwarz. From (I) and previous observations

\[
\left\| \left[ \sum (\xi_k^2 - |\langle h_k, \sigma_k \rangle|^2) \right]^{1/2} \right\|_1 \leq C\|h\|_4\|h - f\|_4\|h\|_8^{3/4} = C\|h - f\|_4\|h\|_8^{3/4}.
\]

Since for each $k$,

\[
\xi_k^2 - |\langle h_k, \sigma_k \rangle|^2 = (\xi_k + |\langle h_k, \sigma_k \rangle|) \frac{|\langle h_k, \sigma_k \rangle - b_k|^2}{|\langle h_k, \sigma_k \rangle| + |b_k|} \geq C|\langle h_k, \sigma_k \rangle - b_k|^2.
\]

Lemma 5 is proved.

The left side of Lemma 5 dominates $\|f - \Sigma \langle h_k, \sigma_k \rangle \sigma_k\|_1$.

**Lemma 6.** $\|\Sigma h_k \|_1$ and $\|\Sigma [h_k]_e - b_k \sigma_k^2 \|_1 \leq C\|h - f\|_4\|h\|_8^{3/8}$.**

**Proof.** Since $\Sigma [h_k]_0 = h - \Sigma [h_k]_e$, the first inequality is a consequence of the second. Write

\[
\left\| \left[ \sum [h_k]_e - b_k \sigma_k^2 \right]^{1/2} \right\|_1 \leq \left\| \left[ \sum [h_k]_e - \langle h_k, \sigma_k \rangle \sigma_k^2 \right]^{1/2} \right\|_1
\]

\[
+ \left\| \left[ \sum \langle h_k, \sigma_k \rangle - b_k \right]^{1/2} \right\|_1.
\]
which by Lemmas 4(II) and 5 is estimated by

\[ C\|h - \sum \langle h_k, \sigma_k \rangle \sigma_k\|_1^{1/2}\|\sigma\|_1^{1/2} + C\|h - f\|_1^{1/4}\|h\|_1^{3/4} \leq C\|h - f\|_1^{1/8}\|h\|_1^{7/8}. \]

Define for \( u \in L^1(\mathbb{R}^n) \),

\[ (u)_e(x) = \int_D u(e_1x_1, e_2x_2, \ldots) \, de \]

(\(=\) the natural projection on the even part in \( x_1, x_2, \ldots \)).

**Lemma 7.** \( \|\sum_k |(h_k(1))_e|^2\|_1^{1/2} \leq C\|h - f\|_1^{1/16}\|h\|_1^{5/16}. \)

**Proof.** At this point, we will make use of Proposition 7. Fix \( x \in \mathbb{R}^n \) and remark that the sequence of functions in \( e \in D \),

\[ [h_k]_0(e_1x_1, e_2x_2, \ldots), \]

is a martingale difference sequence.

Moreover, the \( k \)th Rademacher coefficient is clearly given by

\[ \sum_{n>0} (h_k(n))_e(x) \sin nx_k \]

and Proposition 7 yields

\[ \left[ \sum_k \left( \sum_{n>0} (h_k(n))_e(x) \sin nx_k \right)^2 \right]^{1/2} \]

\[ \leq C \left[ \int \left[ \sum_k [h_k]_0(e \cdot x) \right]^2 \, de \right]^{1/2} \left[ \int \left[ \sum_k [h_k]_0(e \cdot x) \right] \, de \right]^{1/2}. \]

Integration in \( x \), application of Cauchy-Schwarz and Lemma 6, gives

\[ (+) \left\| \left[ \sum_k \left( \sum_{n>0} (h_k(n))_e \sin nx_k \right)^2 \right]^{1/2} \right\| \leq C\|h - f\|_1^{1/16}\|h\|_1^{7/16}\left[ \sum_k [h_k]_0^2 \right]^{1/2}. \]

Also

\[ \left\| \left[ \sum_k [h_k]_0^2 \right]^{1/2} \right\| \leq C\|h\|_1. \]

On the other hand, we can multiply the \( k \)th increment in the left member of (\( + \)) by \( \sin x_k \) and then take \( E_{k-1} \)-expectation. Proposition 5 shows that

\[ \left\| \left[ \sum_k [h_k]_e^2 \right]^{1/2} \right\| \leq C\|h - f\|_1^{1/16}\|h\|_1^{5/16}, \]

proving Lemma 7.

Now, rewriting

\[ \left[ \sum_k [h_k]_e - b_k \sigma_k \right]^2 = \left[ \sum_k \sum_{n>0} h_k(n) \cos nx_k - b_k \sigma_k \right]^2 \]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
multiplication of the kth increment by $\cos x_k$ and taking $E_{k-1}$-expectation yields (by Proposition 5 and Lemma 6)

$$\left\| \sum_k \left| \frac{1}{2} \hat{h}_k(1) - \frac{2}{\pi} b_k \right|^{1/2} \right\|_1 \leq C \| h - f \|_1^{1/8} \| h \|_8^{7/8}.$$  

Since $b_k = (b_k)_e$, a convexity argument allows us to replace, in a previous inequality, $\hat{h}_k(1)$ by $(\hat{h}_k(1))_e$. Combining with Lemma 7, we conclude

$$C^{-1} \| f \|_1 \leq \left\| \left( \sum \left| b_k \right|^2 \right)^{1/2} \right\|_1 \leq C \| h - f \|_1^{1/16} \| h \|_1^{15/16}, \quad \| f \|_1 \leq C \| h - f \|_1,$$

and thus Proposition 10.

REFERENCES


DEPARTMENT OF MATHEMATICS, VRIJE UNIVERSITEIT BRUSSEL, PLEINLAAN 2-F7, 1050 BRUSSELS, BELGIUM