GROUP ACTIONS ON ASPHERICAL $A_k(N)$-MANIFOLDS

BY

HSU-TUNG KU AND MEI-CHIN KU

Abstract. By an aspherical $A_k(N)$-manifold, we mean a compact connected manifold $M$ together with a map $f$ from $M$ into an aspherical complex $N$ such that $f^*: H^q(N; Q) \rightarrow H^q(M; Q)$ is nontrivial. In this paper we shall show that if $S^1$ acts effectively and smoothly on a smooth aspherical $A_k(N)$-manifold, $k > 1$, $N$ a closed oriented Riemannian $k$-manifold, with strictly negative curvature, and the $K$-degree $K(f) \neq 0$, then the fixed point set $F$ is not empty, and at least one component of $F = \bigcup F_j$ is an aspherical $A_k(N)$-manifold. Moreover, $\text{Sign}(f) = \sum \text{Sign}(f| F_j)$. We also study the degree of symmetry and semisimple degree of symmetry of aspherical $A_k(N)$-manifolds.

1. Introduction. Suppose $M^m$ is a compact connected topological or differentiable $m$-manifold. Following [15], $M$ is called an $A_k$-manifold, where $k$ is a nonnegative integer, if there exists $w_i \in H^i(M, Q)$, $1 \leq i \leq k$, such that $w_1 \cup \cdots \cup w_k \neq 0$. Without loss of generality, in fact, we can assume that $w_i$'s belong to the free part of $H^i(M, Z)$. Let $M$ be a compact connected differentiable manifold. The degree of symmetry $N(M)$ (resp. semisimple degree of symmetry $N^*(M)$) of $M$ is defined as the supremum of the dimensions of all compact (resp. compact semisimple) Lie groups which can act smoothly and effectively on $M$. If $M$ is a compact connected topological manifold, the degree of symmetry $N_T(M)$ and semisimple degree of symmetry $N_T^*(M)$ can be similarly defined by assuming the actions to be topological.

A space is called aspherical if its universal covering space is contractible. Burghelea (cf. [21]) has proposed to compute or estimate $N(M)$ for a connected closed differentiable $m$-manifold $M$ if there exists a degree one map $f: M^m \rightarrow N^m$, where $N$ is a closed aspherical manifold. Considerable information has been obtained in relation to this problem. If $N = T^m$, the $m$-torus., then $M$ is called hypertoral [20]. This was studied by Schultz [20,21] and Gromov and Lawson [6]. One result of Conner and Raymond in [4] corresponds to the case $M = N$ and $f$ is the identity map. If $M$ is hyperaspherical, i.e., degree of $f$ is nonzero, then Donnelly and Schultz [5] have shown that $N^*(M) = 0$. Schoen and Yau [19] have investigated the case when $N$ is a closed Riemannian manifold of nonpositive curvature which is aspherical because its universal covering is diffeomorphic to a Euclidean space. In [5],
Donnelly and Schultz have proved the following topological modified version of the Schoen-Yau result:

1.1 Theorem [5]. Let $M^m$ be a closed oriented $m$-manifold. Suppose there exists a map $f$ from $M^m$ to a closed oriented Riemannian manifold $N$ of strictly negative curvature such that $f_*: H_k(M; Q) \to H_k(N; Q)$ is nontrivial for some $k > 0$. Then

$$N_T(M) \leq \begin{cases} N_T(S^{m-k}) = \langle m - k + 1 \rangle_{SO} & \text{if } k > 1, \\ N_T(S^m) + 1 & \text{if } k = 1, \end{cases}$$

where $\langle s \rangle_{SO}$ denotes $\dim SO(s)$.

A theorem similar to the Schoen and Yau result [19] was also obtained by Browder and Hsiang [23]. Their paper also proved a “higher $\hat{A}$-genus” theorem which is analogous to our Proposition 3.10.

Let $M^m$ be a compact connected topological or differentiable $m$-manifold. We shall say that $M$ is an $A_k(N)$-manifold (resp. aspherical $A_k(N)$-manifold) if $N$ is a closed connected oriented manifold (resp. an aspherical complex), $k$ a nonnegative integer, and there exists a continuous map $f: M \to N$ such that $f_*: H^k(N; Q) \to H^k(M; Q)$ is nontrivial. It follows from the definition that any connected closed manifold is an $A_0(A)$-manifold (resp. aspherical $A_0(A)$-manifold) for any manifold (resp. aspherical complex) $N$. The following results show that both the $A_k(N)$-manifold and the aspherical $A_k(N)$-manifold may be viewed as generalized $A_k$-manifolds.

1.2 Theorem. Let $M$ be a compact connected manifold. Then $M$ is an $A_k(T^k)$-manifold if and only if $M$ is an $A_k$-manifold.

Proof. Obviously, an $A_k(T^k)$-manifold is an $A_k$-manifold. To prove sufficiency, let $M$ be an $A_k$-manifold, and $w_i \in H^i(M; Z)$, $1 \leq i \leq k$, be such that $w_1 \cup \cdots \cup w_k \neq 0$. Since $H^i(M; Z) \cong [M; K(Z, i)] = [M; S^1]$, for each $w_i$, there corresponds a map $f_i: M \to S^1$ with $f_*^i(S^1) = w_i$; where $\{S^1\}$ denotes the fundamental cohomology class of $S^1$. Set $f = \prod_{i=1}^{k} f_i: M \to \prod_{i=1}^{k} S^1 = T^k$. Then $f_*^i(T^k) = \prod_{i=1}^{k} w_i \neq 0$. That is, $M$ is an $A_k(T^k)$-manifold.

In this paper we shall investigate the transformation groups on aspherical $A_k(N)$-manifolds. We shall introduce the notion of the Euler characteristic $\chi(f)$ and the $K$-degree $K(f)$ of a smooth map $f: M \to N$. If $\chi(f) \neq 0$ or $K(f) \neq 0$, then $M$ is an $A_k(N)$-manifold. Moreover, we shall show that if $S^1$ acts effectively and smoothly on a smooth $A_k(N)$-manifold $M$, $k > 1$, with fixed point set $F$, and $N$ a closed oriented Riemannian manifold with strictly negative curvature, then $\operatorname{Sign}(f) = \Sigma_j \operatorname{Sign}(f|_{F_j})$, where $F = \bigcup_j F_j$. Moreover, if $K(f) \neq 0$ for some $K$, then $F$ is not empty and at least one component of $F$ is also an aspherical $A_k(N)$-manifold. We will also generalize some results in [15] from $A_k$-manifolds to aspherical $A_k(N)$-manifolds. In particular, we shall obtain several generalizations of Theorem 1.1. For instance, if $M$ is an aspherical $A_k(N)$-manifold, $x \in H^a(M; Q)$, and $y = f^*(\tilde{y}) \in H^k(M; Q)$ such that $xy \neq 0$ and $\tilde{m} = m - k \geq 19$, then

$$N_T(M) \leq k + \langle \tilde{m} - \alpha + 1 \rangle_{SO} + \langle \alpha + 1 \rangle_{SO}$$
or
\[ N_T(M) \leq k + \dim SU(m/2 + 1). \]

If, in addition we assume that \( N \) is a closed oriented Riemannian manifold with strictly negative curvature, \( k > 1 \), and \( x, y \) and \( m \) are as above, then we have the following generalization of Theorem 1.1:
\[ N_T(M) \leq \langle m - \alpha + 1 \rangle_{SO} + \langle \alpha + 1 \rangle_{SO} \]
or
\[ N_T(M) = \dim SU(m/2 + 1), \quad M \cong CP^{m/2} \times W^{k}. \]

These bounds on \( N_T(M) \) are of course much sharper than \( \langle m + 1 \rangle_{SO} \), especially if \( \alpha \) can be chosen near \( [(m + 1)/2] \). In §4, we shall define a numerical invariant \( N(M; H) \) which is very useful to estimate the degree of symmetry of complex manifolds. We shall apply this invariant to show that if \( M \) is a complex aspherical \( A_k(N) \)-manifold and \( m = 2n + k \), then \( N(M) \leq k + \langle n + 1 \rangle_{SU} \), where \( \langle s \rangle_{SU} = \dim SU(s) \).

2. Existence of induced maps. The following result is a topological analogue of the fundamental theorem of homomorphisms for groups.

2.1 Theorem. Let \( M, N \) and \( W \) be CW complexes, \( f: M \to N \) and \( g: M \to W \) be continuous. Suppose \( g^*: H_\ast(M) \to H_\ast(W) \) is onto for \( 1 \leq i \leq \phi(N) = d \), where \( \phi(N) = \max\{j: \pi_j(N) \neq 0\} < \infty \). Then there exists a map \( h: W \to N \) such that \( hg \) is homotopic to \( f \) if and only if \( \text{Ker} g^* \subseteq \text{Ker} f^* \) for \( 1 \leq i \leq d \).

Proof. Suppose \( \text{Ker} g^* \subseteq \text{Ker} f^* \) for \( 1 \leq i \leq d \). Let \( \{a_n: M \to M_n\} \), \( \{b_n: W \to W_n\} \) and \( \{c_n: N \to N_n\} \) be the Postnikov systems of the complexes \( M, W \) and \( N \), respectively. By definition we have homotopy commutative diagrams
\[
\begin{array}{ccc}
M & \overset{a_{n+1}}{\longrightarrow} & M_{n+1} \\
\downarrow & & \downarrow \\
M_n & & M_n
\end{array}
\]
and fibrations \( K(\pi_{n+1}(M), n + 1) \to M_{n+1} \to M_n \), etc. There exist maps \( \{f_n: M_n \to N_n\} \) and \( \{g_n: M_n \to W_n\} \) such that
\[
\begin{array}{ccc}
M_{n+1} & \overset{f_{n+1}}{\longrightarrow} & N_{n+1} \\
\downarrow & & \downarrow \\
M_n & & M_n
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
M_{n+1} & \overset{g_{n+1}}{\longrightarrow} & W_{n+1} \\
\downarrow & & \downarrow \\
M_n & & M_n
\end{array}
\]
commute, and
\[
\begin{array}{ccc}
M & \overset{f}{\longrightarrow} & N \\
\downarrow a_n & & \downarrow c_n \\
M_n & \overset{f_n}{\longrightarrow} & N_n \\
\downarrow a_n & & \downarrow b_n \\
M & \overset{g}{\longrightarrow} & W \\
\downarrow a_n & & \downarrow b_n \\
M_n & \overset{g_n}{\longrightarrow} & W_n
\end{array}
\]
are homotopy commutative. We shall inductively construct the maps $h_n: W_n \to N_n$ so that $h_n g_n = f_n$. Assume that $h_n$ has been constructed. Consider the following diagram where the vertical maps are fibrations:

$$
\begin{array}{c}
\text{K}(\pi_{n+1}(M), n + 1) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\text{M}_{n+1} \\
\text{M}_n \\
\end{array}
\begin{array}{c}
\text{K}(\pi_{n+1}(W), n + 1) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
W_{n+1} \\
W_n \\
\end{array}
\begin{array}{c}
\text{K}(\pi_{n+1}(N), n + 1) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
N_{n+1} \\
N_n \\
\end{array}
$$

By hypotheses and the fundamental theorem of homomorphism for groups, there exists a homomorphism $h_{n+1}: \pi_{n+1}(W) \to \pi_{n+1}(N)$ such that $h_{n+1} g_{n+1} = f_{n+1}$. Since $\left[ \text{K}(\pi, n + 1), \text{K}(\pi', n + 1) \right] \cong \text{Hom}(\pi, \pi')$, there exists a map $h_{n+1}$ such that $h_{n+1} g_{n+1} = f_{n+1}$. The maps $h_{n+1}$ and $h_n$ induce a map $h_{n+1}: W_{n+1} \to N_{n+1}$ such that $f_{n+1} \simeq h_{n+1} g_{n+1}$. Since $d = \phi(N)$, $c_d$ is a weak homotopy equivalence, hence it is a homotopy equivalence. Let $\phi: N_d \to N$ be a homotopy inverse of $c_d$. Then the map $h = \phi h_d b_d$ satisfies $h g = f$.

If $N$ is a $K(\pi_1(N), 1)$-complex, i.e. aspherical complex, then $\phi(N) = 1$. The special case when $N$ is an aspherical complex was proved in [5].

2.2 Proposition. Assume that $M$ is an aspherical $A_k(N)$-manifold and $\pi_1(M)$ abelian. Let $T^k$ act effectively on $M$ with nonempty fixed point set $F(T^k, M)$. Then there exists a map $h: M/T^k \to N$ such that $f = h \pi$ and $s \leq m - k$, where $\pi: M \to M/T^k$ is the natural projection.

Proof. It is known that $\pi_{*1}: \pi_1(M) \to \pi_1(M/T^k)$ is surjective [2]. Thus $\pi_1(M/T^k)$ is abelian because $\pi_1(M)$ is abelian. Since $F(T^k, M)$ is not empty, $\pi_*: H^1(M/T^k; Q) \to H^1(M; Q)$ is surjective [2]. Equivalently, $\pi_*: H_1(M; Q) \to H_1(M/T^k; Q)$ is injective. But $\pi_*$ is also surjective [2], hence $\pi_*$ an isomorphism. Hence it is not difficult to see that $\text{Ker} \pi_{*1}$ is a finite group. Since $\pi_1(N)$ is torsion free, $\text{Ker} \pi_{*1} \subset \text{Ker} f_{*1}$. Since $M/T^k$ has the homotopy type of a finite complex [5], it follows from Theorem 2.1 that there exists a map $h: M/T^k \to N$ such that $h \pi \simeq f$. As $M$ is an aspherical $A_k(N)$-manifold, $h_*: H^k(N; Q) \to H^k(M/T^k; Q)$ is nontrivial. Thus we have $k \leq \dim M/T^k = m - s$, or $s \leq m - k$ as desired.

2.3 Proposition. Suppose that $M$ is an aspherical $A_k(N)$-manifold and $G$ a compact semisimple Lie group acting almost effectively on $M$ with $G(x)$ as a principal orbit. Then there exists a map $h: M/G \to N$ such that $f = h \pi$ and $\dim G(x) \leq m - k$. 
Proof. Let \(i: G \to M\) be the orbit map defined by \(i(x) = x(m), m \in M\). According to [5], we have an isomorphism
\[
\pi_1(M/G) \cong \left( \pi_1(M)/i_*\pi_1(G) \right)/P,
\]
where \(P\) is a finite normal subgroup of \(\pi_1(M)/i_*\pi_1(G)\). Since \(\pi_1(N)\) is torsion free, we have \(\text{Ker } \pi_* \subset \text{Ker } f_*\). Again \(k \leqslant \dim M/G = m - \dim G(x)\), and the proof is complete.

From the proof of [5, Theorem 3.5] we can see that the following holds.

2.4 Proposition. Let \(M\) be an \(A_k(N)\)-manifold, \(k > 1\), and \(N\) a closed oriented Riemannian manifold of strictly negative curvature. Suppose \(G\) is a compact connected Lie group acting almost effectively on \(M\) with \(G/H\) as a principal orbit. Then there exists a map \(h: M/G \to N\) such that \(f = h\pi\) and \(\dim C/\pi < m - k\).

3. Euler characteristic, \(K\)-degree of a map, and fixed point set. Let \(M^m\) and \(N^k\) be closed connected oriented manifolds, and \(f: M \to N\) be a smooth map. Let \(x \in N\) be a regular value of \(f\), and \(W = f^{-1}(x)\). Define the Euler characteristic \(\chi(f)\) of the map \(f\) by \(\chi(f) = \chi(W)\mod 2\), where \(\chi(W)\) denotes the Euler characteristic of \(W\), and set \(\chi(f) = 0\) if \(W = \emptyset\). If \(m = 4r + k\), and \(K = \{K_n\}\) is a multiplicative sequence defined by Hirzebruch in [7], define the \(K\)-degree \(K(f)\) of \(f\) to be the following number:
\[
K(f) = \begin{cases} 
\langle K_n(W), [W] \rangle \in Q, & \text{if } W \neq \emptyset, \\
0 & \text{if } W = \emptyset.
\end{cases}
\]
Since the oriented cobordism class of \(W\) is independent of the choice of the regular value \(x\), \(K(f)\) is well defined. If \(r = 0\), then \(K_0 = 1\) and \(K(f)\) is simply the degree of \(f\). Define the signature of \(f\) by \(\text{Sign}(f) = L(f)\), the \(L\)-genus of \(W\). The special case \(K = \mathbb{A}\) is defined in [6]. To prove that \(\chi(f)\) is well defined, let \(W' = f^{-1}(y)\), \(y\) is another regular value of \(f\). Then there is a compact oriented manifold \(V\) with boundary \(\partial V = W \cup W'\). But \(H'(V, W; Q) \cong H_{v-1}(V, W'; Q)\) where \(v = \dim V\). Hence, if \(v\) is odd, \(\chi(V, W) = -\chi(V, W')\), and so \(\chi(W') = \chi(W)\mod 2\).

We shall denote the normal bundle of \(W\) in \(M\) by \(v\), \(p: v \to M\) the projection and \(\eta: M \to T(v)\) the natural collapsing map, where \(T(v)\) is the Thom space of \(v\).

3.1 Theorem. Let \(f: M^m \to N^k\) be a smooth map. Then
\begin{enumerate}
\item \(K(f) = \langle K_n(M) \cup f^*[N], [M] \rangle \mod 2\) if \(m = 4r + k\),
\item \(\chi(f) = (\langle p \eta \rangle^*e(W) \cup f^*[N], [M] \rangle \mod 2\), where \([M]\) denotes the fundamental class of \(M\), and \(e(W)\) the Euler class of the tangent bundle \(TW\). In particular, if \(K(f) \neq 0\), or \(\chi(f) \neq 0\), then \(M\) is an \(A_k(N)\)-manifold.
\end{enumerate}

Proof. Let \(\nu'\) denote the normal bundle of \(x\) in \(N\) and \(U \in H^k(T(\nu))\) and \(U' \in H^k(T(\nu'))\) be the Thom classes. The map \(f\) induces a bundle map \(b: \nu \to \nu'\) and hence a map \(T(b)\) such that \(T(b)^*U' = U\) (cf. [3, II 2.8]). The natural collapsing map \(\eta': N \to T(\nu')\) has degree 1, hence \(\eta'^*U' = [N]\). Since \(T(b)\eta = \eta'f\), it follows that
\[
f^*[N] = f^*\eta'^*U' = \eta^*T(b)^*U' = \eta^*U.
\]
By using the Poincaré duality we can easily show that $j_*[W] = [M] \cap \eta^*U$, where $j: W \to M$ is the inclusion. Since the normal bundle $\nu$ is trivial, hence $j^*K_r(M) = K_r(W)$. It follows that

$$K(f) = \left< K_r(W), [W] \right> = \left< j^*K_r(M), [W] \right> = \left< K_r(M), j_*[W] \right>$$

$$= \left< K_r(M), [M] \cap \eta^*U \right> = \left< K_r(M) \cup \eta^*U, [M] \right>$$

$$= \left< K_r(M) \cup f^*(N), [M] \right>.$$

This completes the proof of (a).

Let $p: T(\nu) \to W$ be projection. Since $\eta$ has degree 1, $\eta_*[M] \cap U = [W]$. It follows that

$$\langle e(W), [W] \rangle = \langle e(W), \eta_*[M] \cap U \rangle = \langle (p\eta)^*e(W) \cup f^*(N), [M] \rangle.$$

Hence $\chi(f) = \langle (p\eta)^*e(W) \cup f^*(N), [M] \rangle \mod 2$.

The main result of this section is the following:

3.2 Theorem. Suppose $M^m$ and $N^k$ are closed oriented connected manifolds, where $N$ is a Riemannian manifold with strictly negative curvature and $k > 1$. Let $f: M \to N$ be a smooth map such that $K(f) \neq 0$ for some multiplicative sequence $K$. Then for any smooth action of $S^1$ on $M$, the fixed point set $F$ is not empty, and at least one component of $F$ is an aspherical $A_k(N)$-manifold. Moreover, we can orient each component $F_j$ of $F$ so that $\operatorname{Sign}(f) = \sum_j \operatorname{Sign}(f| F_j)$.

From now on we shall always assume that $N$ is an aspherical complex and call an aspherical $A_k(N)$-manifold simply an $A_k(N)$-manifold. In view of Theorem 3.1, Theorem 3.2 is a special case of the following:

3.3 Theorem. Let $G = S^1$ act effectively and smoothly on a smooth $A_k(N)$-manifold, $k > 1$, and $N$ a closed oriented Riemannian manifold with strictly negative curvature.

(a) Suppose $K(M)$ is a polynomial in the Pontrjagin classes of $M$ with rational coefficients such that $\langle z \cup K(M), [M] \rangle \neq 0$ where $z = f^*(\bar{z}) \in H^k(M; Q)$. Then the fixed point set $F$ of $G$ is not empty, and at least one component of $F$ is also an $A_k(N)$-manifold.

(b) We can orient each component of $F = \bigcup F_j$ so that

$$\operatorname{Sign}(f) = \sum_j \operatorname{Sign}(f| F_j).$$

This theorem is an immediate consequence of the following two theorems.

3.4 Theorem. Suppose $G = S^1$ acts effectively and smoothly on a smooth $A_k(N)$-manifold $M$, and there exists a map $h: M/G \to N$ such that $f \simeq h\pi$.

(a) If $\langle z \cup K(M), [M] \rangle \neq 0$ where $z$ and $K$ are as in Theorem 3.3 and the fixed point set $F$ of $G$ is not empty. Then at least one component of $F$ is also an $A_k(N)$-manifold.

(b) We can orient each component $F_j$ of $F$ so that

$$\operatorname{Sign}(f) = \sum_j \operatorname{Sign}(f_j), \quad f_j = f| F_j.$$