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WEIGHTED ITERATES AND VARIANTS OF THE HARDY-LITTLEWOOD MAXIMAL OPERATOR

BY

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ABSTRACT. In a recent paper, M. A. Leckband and C. J. Neugebauer obtained a rearrangement inequality for a generalized maximal operator with respect to two measures. For an application they studied norm bounds for the iterated Hardy-Littlewood maximal operator with respect to two measures. In this paper this theory is further developed and other applications of the rearrangement inequality are obtained.

1. Let μ, ν be two measures on $\mathbf{R}^n$, and let there be associated with each cube $Q \subset \mathbf{R}^n$ a function ϕ_Q supported in Q. We consider the maximal operator

$$Mf(x) = \sup \int f\phi_Q \, d\nu$$

where the sup is extended over all Q with center x. If g_λ^* is the nonincreasing rearrangement of g with respect to the measure λ, i.e., $g_\lambda^*(t) = \inf\{y \colon \lambda\{|g| > y\} \leq t\}$, and if $\Phi(t) = \sup_Q\{\mu(Q)\phi_{Q,\nu}^*(\mu(Q)t)\}$, then we have proved, in **[6]**, the following theorem.

THEOREM 1. $(Mf)_\mu^*(\xi) \leq A\int_0^\infty \Phi(t) f_\nu^*(t\xi)\, dt$.

From this rearrangement inequality it is easy to get general norm inequalities. In particular, Minkowski's integral inequality gives

$$\| Mf \|_{p,\mu} \leq A\left(\int_0^\infty \frac{\Phi(t)}{t^{1/p}}\, dt\right) \| f \|_{p,\nu},$$

and thus, if $\Phi \in L(p', 1)$ (see **[2]**) we get a weighted norm inequality. At this point Muckenhoupt's A_p-condition enters; thus, if $(u, v) \in A_p$, i.e., $\int_Q u(\int_Q v^{1-p'})^{p-1} \leq C|Q|^p$ [7], and if $d\mu = u\,dx$, $d\nu = v\,dx$, $\phi_Q(x) = (1/|Q|)(\chi_Q(x)/v(x))$, then the above $Mf(x) = \sup(1/|Q|)\int_Q f(t)\, dt$, the familiar Hardy-Littlewood maximal function. We have proved in **[6]** that if $(u, v) \in A_p$, $1 < p < \infty$, then $\Phi_{u,v} \in L(p', \infty)$, and $u = v$ is in A_p, $1 < p < \infty$, if and only if $\Phi_u \in L(p', 1)$, where

$$\Phi_{u,v}(t) = \sup_Q \left\{ \frac{u(Q)}{|Q|} \left(\frac{\chi_Q}{v}\right)_\nu^* (\mu(Q)t) \right\}$$

and $\Phi_u = \Phi_{u,u}$.

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The jth iterated Hardy-Littlewood maximal function $M_j f$ turns out to be crucial in the extrapolation problem, i.e., when does $\|Mf\|_{p,u} \leqslant A_p \|f\|_{p,v}$ imply the existence of $\varepsilon > 0$ so that $\|Mf\|_{p-\varepsilon,u} \leqslant B\|f\|_{p-\varepsilon,v}$? We have shown in **[6]** that extrapolation is possible if $\|M_j\| = \mathcal{O}(A^j)$ as $j \to \infty$, where $\|M_j\|$ is the norm of M_j as an operator from L^p_v to L^p_u.

Using a dense set of functions, namely those which are nowhere constant, we obtain (Lemma 2)

$$M_{j+1}f(x) \leqslant B_n^j \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)| \frac{\log^j(|Q|/\rho_Q(y))}{j!}\,dy,$$

where

$$\rho_Q(y) = \inf\{t: y \in \{x \in Q: |f(x)| \geqslant (f\chi_Q)^*(t)\}\}.$$

From this we obtain (Theorem 2) that if $(u, v) \in A_p$, then for each $q > p$ there is a constant $0 < A_q < \infty$ such that $\|M_j f\|_{q,u} \leqslant A_q^j \|f\|_{q,v}$. This then implies that extrapolation of $\|Mf\|_{p,u} \leqslant B\|f\|_{p,v}$ is possible if and only if $\|M_j\| = \mathcal{O}(A^j)$ as $j \to \infty$ (Theorem 3).

At a fixed p, iteration may not be possible, i.e., if $\|M_j f\|_{p,u} \leqslant A\|f\|_{p,v}$ for some $j > 0$, then M_{j+1} may not be bounded on L^p_v. We will also study conditions under which iterations in this case are possible by estimating the associated $\Phi(t)$ (Theorem 4).

The inequality in Theorem 1 readily lends itself to studying restricted weak type. In particular, $\Phi \in L(p', \infty)$ shows that Mf is restricted weak type (p, p), i.e., $\|Mf\|_{p,\infty,\mu} \leqslant A\|f\|_{p,1,\nu}$. This observation is used to give a simple proof of the weak type behavior of a generalization of a maximal operator recently studied by Stein **[9]** (Theorem 6). For the usual Hardy-Littlewood maximal operator we will see (Theorem 7) that $\|Mf\|_{p,\infty,u} \leqslant C\|f\|_{p,1,v}$, $1 < p < \infty$, if and only if $\Phi \in L(p', \infty)$. We believe that this characterization is easier to use than the one found by Kerman **[5]**. The paper concludes with some variants of Theorem 1.

2. We will establish an inequality similar to Theorem 1 for the jth iterated Hardy-Littlewood maximal operator $M_j f$. It will be convenient to define a "telescoping" maximal operator $\overline{M}_j f$ as follows. First, define

$$\overline{M}_{1Q}f(x) = \sup_{x\in Q_1\subset Q} \frac{1}{|Q_1|}\int_{Q_1} |f|, \quad \overline{M}_{jQ}f(x) = \sup_{x\in Q_j\subset Q} \frac{1}{|Q_j|}\int_{Q_j} \overline{M}_{j-1,Q_j}f,$$

i.e.,

$$\overline{M}_{jQ}f(x) = \sup_{x\in Q_1\subset Q} \frac{1}{|Q_1|}\int_{Q_1} \sup_{x_2\in Q_2\subset Q_1} \frac{1}{|Q_2|}\int_{Q_2} \cdots \frac{1}{|Q_{j-1}|}\int_{Q_{j-1}} \sup_{x_j\in Q_j\subset Q_{j-1}} \frac{1}{|Q_j|}\int_{Q_j} |f(t)|\,dt\,dx_j \cdots dx_2.$$

Let $\overline{M}_1 f = Mf$, and for $j \geqslant 2$, define

$$\overline{M}_j f(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q \overline{M}_{j-1,Q}f(t)\,dt.$$

LEMMA 1. *Let $f \in L^p(\mathbf{R}^n)$ for some $p > 1$. Then there is $C_n > 0$ such that for a.e. x,*

$$M_j f(x) \leqslant C_n^j \overline{M}_j f(x), \qquad j = 1, 2, \ldots.$$

PROOF. The condition $p > 1$ assures that for a.e. x, $M_j f(x) < \infty, j = 1, 2, \ldots,$ and we will show that the lemma holds for all such x's. Assume the inequality is true for $j - 1$ and $M_j f(x) < \infty$. Then

$$M_j f(x) = M(M_{j-1} f)(x) \leqslant C_n^{j-1} M(\overline{M}_{j-1} f)(x) < C_n^{j-1}\left(\frac{1}{|Q|}\int_Q \overline{M}_{j-1} f(u)\,du + \varepsilon\right)$$

for some Q containing x. We now let

$$S = \left\{u \in Q \colon \tfrac{1}{2} M(\overline{M}_{j-1} f)(x) \leqslant \overline{M}_{j-1} f(u) < \infty\right\},$$

and we note that

$$M(\overline{M}_{j-1} f)(x) \leqslant \frac{2}{|Q|}\int_S \overline{M}_{j-1} f(u)\,du + 2\varepsilon.$$

For each $u \in S$ choose a cube Q_u with $u \in Q_u$ and

$$\overline{M}_{j-1} f(u) \leqslant \frac{1}{|Q_u|}\int_{Q_u} \overline{M}_{j-2,Q_u} f(t)\,dt + \varepsilon.$$

Since $\sup_{u \in S}\{|Q_u|\} < \infty$, select $u_0 \in S$ for which $l_{u_0} \geqslant \frac{1}{2}\sup_{u \in S} l_u$, where l_u is the sidelength of Q_u.

There are now two cases: $3Q \supset Q_{u_0}$ and $3Q_{u_0} \supset Q$. Then $5Q \supset Q_u$ and $5Q_{u_0} \supset Q_u$ for $u \in S$, respectively. Thus, in the first case,

$$M(\overline{M}_{j-1} f)(x) \leqslant \frac{2 \cdot 5^n}{|5Q|}\int_S \sup_{t \in Q_1 \subset 5Q} \frac{1}{|Q_1|}\int_{Q_1} \overline{M}_{j-2,Q_1} f(y)\,dy\,dt + 2\varepsilon$$

and, hence,

$$M_j f(x) \leqslant \frac{2 \cdot 5^n C_n^{j-1}}{|5Q|}\int_{5Q} \overline{M}_{j-1,5Q} f + 2\varepsilon \leqslant C_n^j \overline{M}_j f(x) + 2\varepsilon.$$

In the second case we get for $z \in 5Q_{u_0}$,

$$\varepsilon + \sup_{z \in Q_1 \subset 5Q_{u_0}} \frac{5^n}{|Q_1|}\int_{Q_1} \overline{M}_{j-2,Q_1} f \geqslant \varepsilon + \frac{5^n}{|5Q_{u_0}|}\int_{Q_{u_0}} \overline{M}_{j-2,Q_{u_0}} f \geqslant \frac{1}{2} M(\overline{M}_{j-1} f)(x),$$

and thus,

$$M(\overline{M}_{j-1} f)(x) \leqslant 2\varepsilon + \frac{2 \cdot 5^n}{|5Q_{u_0}|}\int_{5Q_{u_0}} \sup_{z \in Q_1 \subset 5Q_{u_0}} \frac{1}{|Q_1|}\int_{Q_1} \overline{M}_{j-2,Q_1} f$$
$$= 2\varepsilon + \frac{C_n}{|5Q_{u_0}|}\int_{5Q_{u_0}} \overline{M}_{j-1,5Q_{u_0}} f \leqslant 2\varepsilon + C_n \overline{M}_j f(x).$$

This completes the proof.

For $f \colon \mathbf{R}^n \to [0, \infty]$, let f^* be the rearrangement of f relative to Lebesgue measure on $\mathbf{R}^n$. We will also assume that f is nowhere constant, i.e., $|\{x \colon f(x) = a\}| = 0$,

$a > 0$. For a cube $Q \subset \mathbf{R}^n$, let $E_t = \{x \in Q\colon f(x) \geqslant (f \cdot \chi_Q)^*(t)\}$. Then $|E_t| = t$ and $t^{-1}\int_0^t (f\chi_Q)^* = t^{-1}\int_{E_t} f\chi_Q$.

LEMMA 2. *Let $f\colon \mathbf{R}^n \to [0, \infty]$ be in $L^p(\mathbf{R}^n)$ for some $p > 1$ and nowhere constant. Then there exists $B_n > 0$ such that for a.e. x,*

$$M_{j+1}f(x) \leqslant B_n^j \sup_{x \in Q} \frac{1}{|Q|}\int_Q f(y)\frac{\log^j(|Q|/\rho_Q(y))}{j!}\,dy,$$

where $\rho_Q(y) = \inf\{t\colon y \in E_t\}$.

PROOF. By Lemma 1, for a.e. x, $M_{j+1}f(x) \leqslant C_n^{j+1}\overline{M}_{j+1}f(x)$. By the n-dimensional version of Lemma 1 of [**6**] we see that

$$\overline{M}_{j+1}f(x) \leqslant A_n^j \sup_{x \in Q} \frac{1}{|Q|}\int_0^{|Q|}\frac{1}{t_1}\int_0^{t_1}\cdots\frac{1}{t_j}\int_0^{t_j}(f \cdot \chi_Q)^*(\tau)\,d\tau\,dt_j\cdots dt_1.$$

For the proof of the lemma we assume, for simplicity $j = 2$. Then

$$\frac{1}{|Q|}\int_0^{|Q|}\frac{1}{t}\int_0^t\frac{1}{s}\int_0^s(f \cdot \chi_Q)^*(u)\,du\,ds\,dt$$

$$= \frac{1}{|Q|}\int_0^{|Q|}\frac{1}{t}\int_0^t\frac{1}{|E_s|}\int_{E_s}f(x)\,dx\,ds\,\,dt = \frac{1}{|Q|}\int_0^{|Q|}\frac{1}{t}\int_{E_t}f(x)\int_{\rho_Q(x)}^t\frac{1}{s}\,ds\,dx\,dt$$

$$= \frac{1}{|Q|}\int_0^{|Q|}\frac{1}{t}\int_{E_t}f(x)\log\left(\frac{t}{\rho_Q(x)}\right)dx\,dt = \frac{1}{|Q|}\int_Q f(x)\frac{1}{2}\log^2\left(\frac{|Q|}{\rho_Q(x)}\right)dx.$$

Now let (u, v) be a pair of weights with $u \geqslant 0$ in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $0 < v < \infty$, a.e. Set $d\mu = u\,dx$, $d\nu = v\,dx$.

LEMMA 3. *With the same hypothesis as in Lemma 2,*

$$(M_{j+1}f)^*_\mu(\xi) \leqslant AB_n^j\int_0^\infty \Phi_j(t)f_\nu^*(t\xi)\,dt,$$

where

$$\Phi_j(t) = \Phi_{j,f}(t) = \sup_Q\left\{\frac{\mu(Q)}{|Q|}\left(\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)}\chi_Q(x)\right)^*_\nu(\mu(Q)t)\right\}.$$

PROOF. This is Theorem 1 with

$$\phi_Q(x) = \frac{1}{|Q|}\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)}\chi_Q(x)$$

coupled with Lemma 2.

THEOREM 2. *Let (u, v) be a pair of weights as above and assume that $\|Mf\|_{q,u} \leqslant B_q\|f\|_{q,v}$, $1 \leqslant p < q$. Then for each $q > p$ there is a constant $0 < A_q < \infty$ such that $\|M_jf\|_{q,u} \leqslant A_q^j\|f\|_{q,v}$.*

PROOF. We may assume that f is nowhere constant. We will estimate $\Phi_j(t)$ of Lemma 3 and show that $\Phi_j \in L(q', \infty)$, $q > p$. Fix $q > p$, and let $p < p_0 < q$. Since

$\| Mf \|_{p_0,u} \leqslant B_{p_0} \| f \|_{p_0,v}$, the pair $(u, v) \in A_{p_0}$, i.e.,

$$\mu(Q)\left(\int_Q v^{1-p_0'}\right)^{p_0-1} \leqslant C|Q|^{p_0} \qquad [7].$$

We next note that

$$\left(\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)}\chi_Q(x)\right)^*_\nu(t) \leqslant \frac{1}{t^{1/q'}}\left(\int_Q \frac{\log^{jq'}(|Q|/\rho Q(x))}{j!^{q'}} v^{1-q'}\,dx\right)^{1/q'}$$

$$\leqslant \frac{1}{j!t^{1/q'}}\left(\int_Q \log^{jq'r'}\left(\frac{|Q|}{\rho_Q(x)}\right)dx\right)^{1/r'q'}\left(\int_Q v^{1-p_0'}\right)^{1/q(p_0'-1)}$$

where $r = (p_0' - 1)/(q' - 1) > 1$. Since the rearrangement of $\log|Q|/\rho_Q(x)$ is $\log|Q|/t$, and $(1/a)\int_0^a \log^r(a/t)\,dt = \Gamma(r+1)$, $r > -1$, the above equals

$$\frac{\Gamma(jq'r'+1)^{1/r'q'}}{t^{1/q'}j!}|Q|^{1/r'q'}\left(\int_Q v^{1-p_0'}\right)^{1/q(p_0'-1)}.$$

Let $c_j = \Gamma(jq'r'+1)^{1/r'q'}/j!$ and observe that from Stirling's formula one gets $C_j^{1/j} \leqslant C_*$, where C_* is a constant depending on the product $q'r'$ only. All this gives

$$\frac{\mu(Q)}{|Q|}\left(\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)}\chi_Q(x)\right)^*_\nu(\mu(Q)t)$$

$$\leqslant \frac{C_*^j}{t^{1/q'}}\cdot\frac{\mu(Q)^{1/q}}{|Q|^{1-1/r'q'}}\left(\int_Q v^{1-p_0'}\right)^{1/q(p_0'-1)}.$$

We finally observe that $1 - 1/r'q' = p_0/q$, $(p_0 - 1)(p_0' - 1) = 1$, and thus $\Phi_j(t) \leqslant CC_*^j/t^{1/q'}$.

We now complete the proof and fix $s > p$. Then from Lemma 3 we get

$$\| M_{j+1} f \|_{s,u} \leqslant AB_n^j\left(\int_0^\infty \frac{\Phi_j(t)}{t^{1/s}}\,dt\right)\| f \|_{s,v}.$$

Now choose $p < q_1 < s$ and $s < q_2 < \infty$ and observe that

$$\int_0^\infty \frac{\Phi_j(t)}{t^{1/s}}\,dt = \int_0^1 + \int_1^\infty \leqslant \int_0^1 \frac{\alpha^j}{t^{1/q_1'+1/s}}\,dt + \int_1^\infty \frac{\beta^j}{t^{1/q_2'+1/s}}\,dt \leqslant \gamma^j,$$

and the proof of Theorem 2 is complete.

3. In **[6]** we have shown that if $\| M_j f \|_{p,u} \leqslant A_j \| f \|_{p,v}$ for some $1 < p, j = 1, 2, \ldots$, and if

$$\Phi(t) = \sup_Q\left\{\frac{\mu(Q)}{|Q|}\left(\frac{\chi_Q}{v}\right)^*_\nu(\mu(Q)t)\right\},$$

then

$$\Phi(2^{-N}) \leqslant C\frac{A_{j+1}}{B^j}\left(\frac{j!}{N^j}\right)2^{N/p'}.$$

From this and $A_j = \mathcal{C}(A^j)$, one gets that $\Phi \in L((p-\varepsilon)', 1)$ for some $\varepsilon > 0$, and thus $\|Mf\|_{p-\varepsilon,u} \leqslant A\|f\|_{p-\varepsilon,v}$. All this was done in the context $n = 1$, which we shall also assume for the next theorem.

THEOREM 3. *Let $(u, v) \in A_p$ for some $p > 1$. Then there is $\varepsilon > 0$ with $(u, v) \in A_{p-\varepsilon}$ if and only if* $\sup_{\|f\|_{p,v}=1}\|M_j f\|_{p,u} = \mathcal{C}(A^j)$.

PROOF. If $(u, v) \in A_{p-\varepsilon}$, the result follows from Theorem 2, and the converse was just mentioned and is Theorem 6 in [**6**].

There are examples which show that the norm inequality $\|Mf\|_{p,u} \leqslant A\|f\|_{p,v}$ does not admit an iteration (e.g. Theorem 4 in [**6**]).

If we let

$$\Phi_0(t) = \sup\left\{\frac{\mu(Q)}{|Q|}\left(\frac{\chi_Q}{v}\right)^*_\nu(\mu(Q)t)\right\},$$

the next theorem gives an estimate of $\Phi_j(t)$ in terms of $\Phi_0(t)$ that may allow an iteration up to a certain index.

THEOREM 4. *Let $f\colon \mathbf{R}^n \to [0, \infty]$ be in $L^p(\mathbf{R}^n)$ for some $p > 1$ and nowhere constant. Then*

$$\Phi_j(2^{-N}) \leqslant C\left[\frac{N^j}{j!}\Phi_0\left(\frac{2^{-N}}{2}\right) + N^j\right], \qquad N = 1, 2, \ldots.$$

PROOF. For a fixed N consider

$$L_{Q,j} = \frac{\mu(Q)}{|Q|}\left(\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)}\chi_Q(x)\right)^*_\nu(\mu(Q)2^{-N}).$$

Let $Q_N = \{x \in Q\colon \log(|Q|/\rho_Q(x)) > N/\log_2 e\}$ or $Q_N = \{x \in Q\colon \rho_Q(x) < |Q|2^{-N}\}$. We note that $|Q_N| = |Q|2^{-N}$ and thus

$$\begin{aligned} L_{Q,j} \leqslant C&\frac{\mu(Q)}{|Q|}\left(\frac{N^j}{j!v(x)}\cdot\chi_{Q\setminus Q_N}(x)\right)^*_\nu\left(\mu(Q)\frac{2^{-N}}{2}\right) \\ &+ \frac{\mu(Q)}{|Q|}\left(\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)}\chi_{Q_N}(x)\right)^*_\nu\left(\mu(Q)\frac{2^{-N}}{2}\right). \end{aligned}$$

The first expression on the right is at most $(N^j/j!)\Phi_0(2^{-N-1})$. The second term is zero if $\nu(Q_N) \leqslant \mu(Q)2^{-N-1}$. Hence we assume that $\nu(Q_N) > \mu(Q)2^{-N-1}$ and construct a set $S_N \subset Q_N$ such that if we let

$$\alpha_N = \left(\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)}\chi_{Q_N}\right)^*_\nu(\mu(Q)2^{-N-1}),$$

then

(i)

$$\tfrac{1}{2}\mu(Q)2^{-N-1} \leqslant \nu(S_N) \leqslant \mu(Q)2^{-N-1},$$

(ii)

$$\frac{\log^j(|Q|/\rho_Q(x))}{j!v(x)} \geqslant \alpha_N \quad \text{for } x \in S_N.$$

From this we get

$$\alpha_N \nu(S_N) \leqslant \int_{Q_N} \frac{\log^j(|Q|/\rho_Q(x))}{j!}\,dx \leqslant \int_0^{|Q_N|} \frac{\log^j(|Q|/t)}{j!}\,dt$$

$$= |Q_N| \sum_{k=0}^{j} \frac{\log^k(|Q|/|Q_N|)}{k!} \leqslant C|Q_N|\log^j\left(\frac{|Q|}{|Q_N|}\right).$$

Thus

$$\alpha_N \leqslant C\frac{|Q_N|\log^j(|Q|/|Q_N|)}{\nu(S_N)} \quad \text{and} \quad \frac{\mu(Q)}{|Q|}\alpha_N \leqslant CN^j,$$

and the proof is complete.

COROLLARY. *Let* $(u, v) \in A_p$ *for some* $1 < p < \infty$. *If* $\Phi_0(2^{-N}) \leqslant (C/N^k)2^{N/p'}$, $N = 1, 2, \ldots,$ *then* $\|M_j f\|_{p,u} \leqslant A_j \|f\|_{p,v}$, $j = 1, 2, \ldots, k-1$, *and*

$$\mu\{x: M_k f(x) > y\} \leqslant (C/y^p)\|f\|_{p,v}^p.$$

PROOF. From Lemma 3 and Minkowski's integral inequality we get

$$\|M_{k-1}\|_{p,u} \leqslant C\left(\int_0^\infty \frac{\Phi_{k-2}(t)}{t^{1/p}}\,dt\right)\|f\|_{p,v}.$$

We write $\int_0^\infty = \int_0^1 + \int_1^\infty$, and to estimate $\int_0^1$ we use Theorem 4 and note that

$$\Phi_{k-2}(2^{-N}) \leqslant C\left(\frac{2^{N/p'}}{N^2} + N^{k-2}\right).$$

From this, $\sum \Phi_{k-2}(2^{-N})/2^{N/p'} < \infty$ and $\Phi_{k-2} \in L(p', 1)$ on $[0, 1]$.

For the integral $\int_1^\infty$ we use the proof in Theorem 2, where it was shown that $\Phi_{k-2} \in L(q', \infty)$, $q > p$. Thus

$$\int_1^\infty \frac{\Phi_{k-2}(t)}{t^{1/p}}\,dt \leqslant C\int_1^\infty \frac{dt}{t^{1/q'+1/p}} < \infty.$$

The weak type estimate follows again from Lemma 3 by noting that by Hölder's inequality,

$$(M_k f)_\mu^*(\xi) \leqslant C\|\Phi_{k-1}\|_{p'} \cdot \|f\|_{p,\nu} \cdot 1/\xi^{1/p}.$$

Finally,

$$\|\Phi_{k-1}\|_{p'}^{p'} = \int_0^\infty \Phi_{k-1}^{p'}\,dt = \int_0^1 + \int_1^\infty.$$

Now

$$\int_0^1 \Phi_{k-1}^{p'}\,dt \leqslant C\sum \Phi_{k-1}^{p'}(2^{-N})2^{-N},$$

and, from Theorem 4, $\Phi_{k-1}^{p'}(2^{-N}) \leqslant C(2^N/N^{p'} + N^{p'(k-1)})$. Hence $\int_0^1 < \infty$. As before, $\int_1^\infty \leqslant C\int_1^\infty dt/t^{p'/q'} < \infty$, since $q > p$.

REMARK. The above corollary can be viewed as the converse of Theorem 5 in **[6]**.

4. In this section we will show how Theorem 5 of **[6]** can be used to obtain extrapolation results from the magnitude of $\| M_j \|$. We assume $n = 1$, the setting in which Theorem 5 of **[6]** has been proved.

From Theorem 2 we have that if $\| Mf \|_{q,u} \leqslant B_q \| f \|_{q,v}$, $q > p > 1$, then $\sup_{\| f \|_{q,v}=1} \| M_j f \|_{q,u} = \mathcal{O}(A_q^j)$.

THEOREM 5. *Let $1 < p < \infty$. There is a constant $C_p > 0$ such that the following holds. If for some $r_0 > p$, $A_{r_0} \leqslant C_p/(r_0 - p)$, then $\| M_j f \|_{p,u} \leqslant B_j \| f \|_{p,v}, j = 1, 2, \ldots$.*

PROOF. From Theorem 5 in **[6]** we have constants $C > 0$, $B > 0$ so that for $r > p$,

$$\Phi_0(2^{-N}) \leqslant CA_r(BA_r)^j \left(\frac{j!}{N^j} \right) 2^{N(1/p-1/r)} \cdot 2^{N/p'}$$

$$\leqslant CA_r \left(\frac{BA_r \cdot j}{eN} \right)^j j^{1/2} 2^{N(1/p-1/r)} \cdot 2^{N/p'}$$

by Stirling's formula. Hence, if $\alpha_r = e/2BA_r, j = [\alpha_r N]$, we get

$$\Phi_0(2^{-N}) \leqslant CA_r(\tfrac{1}{2})^{\alpha_r N} (\alpha_r N)^{1/2} 2^{N(1/p-1/r)} \cdot 2^{N/p'}.$$

Hence, from Theorem 4,

$$\Phi_k(2^{-N}) \leqslant C\left(\frac{N^k}{k!} A_r \left(\frac{1}{2} \right)^{\alpha_r N} (\alpha_r N)^{1/2} 2^{N(1/p-1/r)} \cdot 2^{N/p'} + N^k \right).$$

Now let $C_p = ep^2/2B$, and let $r_0 > p$, for which $(r_0 - p)A_{r_0} \leqslant C_p$. Then

$$\alpha_{r_0} \geqslant \frac{r_0 - p}{p^2} > \frac{r_0 - p}{r_0 p} = \frac{1}{p} - \frac{1}{r_0}.$$

We claim now that

$$A_{r_0}\left(\alpha_{r_0} N\right)^{1/2} 2^{N(1/p-1/r_0)} \leqslant N^{-k-2} 2^{\alpha_{r_0} N}, \qquad N \geqslant N_0.$$

With log to the base 2 this is

$$\frac{1}{N} \log A_{r_0} + \frac{1}{N} \log\left(\sqrt{\alpha_{r_0} N}\, N^{2+k} \right) + \left(\frac{1}{p} - \frac{1}{r_0} \right) \leqslant \alpha_{r_0}.$$

Since $\alpha_{r_0} > 1/p - 1/r_0$, this is possible for $N \geqslant N_0$. Hence, $\sum \Phi_k(2^{-N})/2^{N/p'} < \infty$ and $\Phi_k \in L(p', 1)$ on $[0, 1]$. From Theorem 2, $\Phi_k \in L(q', \infty)$, $q > p$, from which $\Phi_k \in L(p', 1)$ on $[1, \infty)$. Minkowski's integral inequality applied to Lemma 3 now completes the proof.

5. We will show in this section how Theorem 1 can be used to study the restricted weak type behavior of a general maximal operator. From Theorem 1 one obtains from $\Phi \in L(p', \infty)$ that

$$(Mf)_\mu^*(\xi) \leqslant B \int_0^\infty \frac{f_\nu^*(t\xi)}{t^{1/p'}} dt = \frac{B}{\xi^{1/p}} \| f \|_{p,1,\nu}.$$

This is the same as $\|Mf\|_{p,\infty,\mu} \leqslant B\|f\|_{p,1,\nu}$, or Mf is restricted weak type (p, p). With this observation it will be easy to obtain the weak type behavior of a maximal operator generalizing the one recently studied by E. M. Stein **[9]**.

THEOREM 6. *Let $\mu > 0$, $\nu \geqslant 0$ be two Borel measures. Let $1 \leqslant q \leqslant p$, and let $M_{pq}f(x) = \sup \|f\chi_Q\|_{p,q,\nu}/\|\chi_Q\|_{p,q,\mu}$, where the* sup *is extended over all cubes centered at x. Then*

$$\|M_{pq}f\|_{p,\infty,\mu} \leqslant A\|f\|_{p,q,\nu} \quad \text{or} \quad \mu\{x: M_{pq}f(x) > y\} \leqslant (C/y^p)\|f\|^p_{p,q,\nu}.$$

PROOF. Note that $\|\chi_Q\|_{p,q,\mu} = \mu(Q)^{1/p}$. It is easy to verify that

$$M_{pq}f(x) = [M_{r,1}f^q(x)]^{1/q}, \qquad r = p/q,$$

and thus we need only show that $\mu\{x: M_{r,1}f(x) > y\} \leqslant (C/y^r)\|f\|^r_{r,1,\nu}$. Note that $\|f\chi_Q\|_{r,1,\nu} \sim \int f\chi_Q\psi_Q\,d\nu$ for some ψ_Q with $\|\psi_Q\|_{r',\infty,\nu} = 1$ or $\psi^*_{Q,\nu}(t) \leqslant 1/t^{1/r'}$. Hence, if $\phi_Q = \chi_Q\psi_Q/\mu(Q)^{1/r}$, then $\mu(Q)\phi^*_{Q,\nu}(\mu(Q)t) \leqslant 1/t^{1/r'}$, from which $\Phi \in L(r', \infty)$.

For the usual Hardy-Littlewood maximal operator, $\Phi \in L(p', \infty)$ actually characterizes the restricted weak type behavior. If (u, v) is a pair of weights,

$$\phi_Q(x) = \frac{1}{|Q|}\frac{\chi_Q(x)}{v(x)}, \quad d\mu = u\,dx, \quad d\nu = v\,dx.$$

then $\sup\int f\phi_Q\,d\nu = \sup|Q|^{-1}\int_Q f\,dx = Mf(x)$. Let $\Phi(t) = \sup_Q\{\mu(Q)\phi^*_{Q,\nu}(\mu(Q)t)\}$.

THEOREM 7. *Let $1 < p < \infty$. Then $\|Mf\|_{p,\infty,\mu} \leqslant C\|f\|_{p,1,\nu}$ if and only if $\Phi \in L(p', \infty)$.*

PROOF. We need only show that the norm inequality implies $\Phi \in L(p', \infty)$. We use the technique of **[3]** and choose $f \geqslant 0$ with $\|f\|_{p,1,v} = 1$ and $\int_Q f = \int f(\chi_Q v^{-1})v \geqslant C\|\chi_Q v^{-1}\|_{p',\infty,v}$. For $x \in Q$ we have $Mf(x) \geqslant (C/|Q|)\int_Q f$. Our assumption is $\mu\{x: Mf(x) > y\} \leqslant (C/y^p)\|f\|^p_{p,1,v}$ and, hence, we get, with $y = |Q|^{-1}\int_Q f$,

$$\mu(Q) \leqslant C\|f\|^p_{p,1,v}\left(\frac{1}{|Q|}\int_Q f\right)^{-p} \leqslant \frac{2C|Q|^p}{\|\chi_Q v^{-1}\|^p_{p',\infty,v}}.$$

From this it follows that

$$\frac{\mu(Q)}{|Q|}\left(\frac{\chi_Q}{v}\right)^*_\nu(\mu(Q)t) \leqslant \frac{\mu(Q)^{1/p}}{|Q|}\frac{1}{t^{1/p'}}\left\{\sup_{\tau>0}\tau^{1/p'}\left(\frac{\chi_Q}{v}\right)^*_\nu(\tau)\right\} \leqslant C/t^{1/p'},$$

and $\Phi \in L(p', \infty)$.

6. In this section we will present two generalizations of Theorem 1 to abstract measure spaces.

(i) Consider $(X, \mathfrak{M}, \mu, \nu)$ and a measurable map $T: X \to \mathbf{R}^n$ such that $\mu(T^{-1}(Q)) < \infty$ for every cube $Q \subset \mathbf{R}^n$. Associate with each cube Q a measurable function $\phi_{T^{-1}(Q)}: X \to [0, \infty)$, with $\operatorname{supp}\phi_{T^{-1}(Q)} \subset T^{-1}(Q)$, and define, for $f: X \to [0, \infty]$ measurable, the maximal operator

$$\mathfrak{M}f(x) = \sup\int f\phi_{T^{-1}(Q)}\,d\nu,$$

where the sup is extended over all cubes Q with center $T(x)$.

THEOREM 8.

$$(\mathfrak{M}f)_\mu^*(\xi) \leq A\int_0^\infty \Phi(t) f_\nu^*(t\xi)\,dt$$

where

$$\Phi(t) = \sup_Q \left\{\mu(T^{-1}(Q))(\phi_{T^{-1}(Q)})_\nu^*\left(\mu(T^{-1}(Q))t\right)\right\}.$$

For the proof choose for each $x \in E_\tau = \{x\colon \mathfrak{M}f(x) > \tau\}$, a cube Q_x centered at $T(x)$ for which $\int f\phi_{T^{-1}(Q_x)}\,d\nu > \tau$. The Besicovitch covering theorem gives us a countable collection $\{Q_j\}$ with $T(E_\tau) \subset \bigcup Q_j$, $\Sigma\chi_{Q_j} \leq C$. Now proceed exactly as in the proof of Theorem 1, replacing $\mu(Q_j)$ there by $\mu(T^{-1}(Q_j))$.

(ii) We will again consider an abstract measure space $(X, \mathfrak{M}, \lambda)$ and a measurable map $T\colon X \to \mathbf{R}^n$ with $\lambda(T^{-1}(Q)) < \infty$, Q cube in $\mathbf{R}^n$. Let $\nu \geq 0$ be a measure on $\mathbf{R}^n$ and associate with each cube $Q \subset \mathbf{R}^n$ a ν-measurable function ϕ_Q with supp $\phi_Q \subset Q$. For $f\colon \mathbf{R}^n \to [0, \infty]$ and $x \in X$ we define

$$M_T f(x) = \sup \int f\phi_Q\,d\nu,$$

where again the sup is extended over all cubes centered at $T(x)$. As in Theorem 8 one can establish

$$(M_T f)_\lambda^*(\xi) \leq A\int_0^\infty \Phi(t) f_\nu^*(t\xi)\,dt$$

where

$$\Phi(t) = \sup_Q \left\{\lambda(T^{-1}(Q))\phi_{Q,\nu}^*\left(\lambda(T^{-1}(Q))t\right)\right\}.$$

As an application we consider $(\mathbf{R}^n, \mu, \nu)$ as in Theorem 1 and assume that $\Phi_0(t) = \sup_Q\{\mu(Q)\phi_{Q,\nu}^*(\mu(Q)t)\}$ is in $L(p', 1)$ for some $1 < p < \infty$. Then we have $\|Mf\|_{p,\mu} \leq A\|f\|_{p,\nu}$, where $Mf(y) = \sup \int f\phi_Q\,d\nu$, Q centered at y.

THEOREM 9. *If* $\lambda(T^{-1}(Q)) \leq C\mu(Q)$, $Q \subset \mathbf{R}^n$, *then* $\Phi(t) \in L(p', 1)$, *and hence* $\|M_T f\|_{p,\lambda} \leq A_p\|f\|_{p,\nu}$.

PROOF. We simply observe that

$$\begin{aligned}\lambda(T^{-1}(Q))\phi_{Q,\nu}^*\left(\lambda(T^{-1}(Q))t\right) &\leq \frac{1}{t}\int_0^{C\mu(Q)t} \phi_{Q,\nu}^*(\tau)\,d\tau \\ &= \frac{1}{t}\mu(Q)\int_0^{Ct} \phi_{Q,\nu}^*(\mu(Q)\tau)\,d\tau \leq C\frac{1}{Ct}\int_0^{Ct}\Phi_0(\tau)\,d\tau \\ &= C\Phi_0^{**}(Ct) \qquad [2].\end{aligned}$$

REMARK. The above hypothesis is a type of Carleson measure condition [1].

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