ON THE DISTRIBUTION OF THE PRINCIPAL SERIES
IN $L^2(\Gamma \backslash G)$

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ABSTRACT. Let $G$ be a semisimple Lie group of split rank one with finite center. If $\Gamma \subset G$ is a discrete cocompact subgroup, then $L^2(\Gamma \backslash G) = \sum_{\omega \in E(G)} \eta_\Gamma(\omega) \cdot \omega$. For fixed $\sigma \in \widehat{M}$, let $P(\sigma)$ denote the classes of irreducible unitary principal series $\pi_{\sigma,\nu}(\nu \in \mathcal{X})$. Let, for $s > 0$, $\psi_\sigma(s) = \sum_{\omega \in P(\sigma)} \eta_\Gamma(\omega) \cdot e^{i\lambda_\omega}$, where $\lambda_\omega$ is the eigenvalue of $\Omega$ (the Casimir element of $G$) on the class $\omega$. In this paper, we determine the singular part of the asymptotic expansion of $\psi_\sigma(s)$ as $s \to 0^+$ if $\Gamma$ is torsion free, and the first term of the expansion for arbitrary $\Gamma$. As a consequence, if $N_\sigma(r) = \sum_{\omega \in P(\sigma)} \eta_\Gamma(\omega)$ and $G$ is without connected compact normal subgroups, then

$$N_\sigma(r) \sim C_\sigma \cdot |Z(G) \cap \Gamma| \cdot \text{vol}(\Gamma \backslash G) \cdot \dim(\sigma) \cdot r^c \quad (c = \frac{1}{2} \dim G/K),$$

as $r \to +\infty$. In the course of the proof, we determine the image and kernel of the restriction homomorphism $^r* : R(K) \to R(M)$ between representation rings.

Introduction. Let $G$ be a connected, real semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $G = K.A.N.$ (respectively $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$) be an Iwasawa decomposition of $G$ (respectively $\mathfrak{g}$) and let $M$ be the centralizer of $A$ in $K$. We assume throughout this paper that $G$ has finite center and split rank one. We do not assume that $G$ is linear. Let $\hat{\mathcal{E}}(G)$ denote the set of equivalence classes of irreducible unitary representations of $G$. If $\sigma \in \hat{\mathcal{E}}(M)$, $\nu \in \mathcal{X}$, let $\pi_{\sigma,\nu}$ be the principal series representation of $G$, parametrized as in [DW, §3]. In this parametrization $\pi_{\sigma,\nu}$ is unitary if $\nu \in i\mathcal{X}$. If $\omega \in \hat{\mathcal{E}}(G)$ let $\lambda_\omega$ and $\theta_\omega$ denote, respectively, the eigenvalue of the Casimir element of $G$ on the class $\omega$ and the distributional character of $\omega$. We will abbreviate by writing $\lambda_{\sigma,\nu} = \lambda_{\sigma,\nu}$. If $\omega \in \hat{\mathcal{E}}_2(G)$, the discrete series of $G$, let $d(\omega)$ denote the formal degree of $\omega$.

For fixed $\sigma \in \hat{\mathcal{E}}(M)$ set

$$P(\sigma) = \{\pi_{\sigma,\nu} : \nu \in \mathcal{X} \text{ and } \pi_{\sigma,\nu} \text{ is irreducible}\}.$$
Let $\Gamma$ be a discrete, cocompact subgroup of $G$. The right regular representation $\pi_{\Gamma}$ of $G$ in $\mathfrak{S}(\mathfrak{g}^\ast)$ decomposes $\pi_{\Gamma} = \sum_{\omega \in \mathfrak{S}(G)} n_{\Gamma}(\omega) \cdot \omega$ and $n_{\Gamma}(\omega) < \infty$, for any $\omega \in \mathfrak{S}(G)$. If $\tau \in \mathcal{E}(K)$,

$$\phi_{\tau}(s) = \sum_{\omega \in \mathfrak{S}(G)} n_{\Gamma}(\omega) \cdot [\tau : \omega] \cdot e^{s\lambda_{\omega}}$$

defines a $C^\infty$ function on $\mathbb{R}^+$, the series converging uniformly on compacta with all derivatives $[W]$. Hence, if $\sigma \in \mathfrak{S}(M)$ is fixed, the series $\psi_{\sigma}(s) = \sum_{\omega \in \mathcal{P}(\sigma)} n_{\Gamma}(\omega) \cdot e^{s\lambda_{\omega}}$ defines a $C^\infty$ function for $s > 0$. The purpose of this paper is to study the asymptotic behavior of $\psi_{\sigma}(s)$, as $s \to 0^+$. By using the technique in [MI] we determine the singular part of the asymptotic expansion of $\psi_{\sigma}(s)$, as $s \to 0^+$, when $\Gamma$ is torsion free.

**Theorem 1.** Let $\Gamma \subset G$ be a discrete, cocompact, torsion-free subgroup. Then

$$\psi_{\sigma}(s) = \psi_{\sigma}(0) + \sum_{i=0}^{c+d-1} b_{2(i-d)+1}(\sigma) \cdot \Gamma(i + 1 - d) \cdot (4s)^{-i-1+d}$$

where $\psi_{\sigma}(s)$ extends to $\tilde{\psi}_{\sigma}(s)$, a $C^\infty$ function on $\mathbb{R}$, such that

(i) if $\text{rank } G = \text{rank } K$

$$\tilde{\psi}_{\sigma}(0) = \psi_{\sigma}(0) + \sum_{\omega \in \mathfrak{S}(\sigma)} s(\omega) \cdot n_{\Gamma}(\omega)$$

(ii) if $\text{rank } G > \text{rank } K$

$$\tilde{\psi}_{\sigma}(0) = \sum_{\omega \in \mathcal{C}(\sigma)} s(\omega) \cdot n_{\Gamma}(\omega).$$

Here $c = \frac{1}{2}\dim(G/K)$, $d = c - [c]$, $a \in \mathbb{R}^+$, and $\lambda_{\sigma}$ is the eigenvalue of the Casimir element of $M$ on the class $\sigma$. Furthermore, $b_{2(i-d)+1}(\sigma)$ denotes, for $i = 1, \ldots, c + d - 1$, the $i$th coefficient of the polynomial part of the Plancherel density associated to $\sigma$, $B_{2j}$ is the $j$th Bernoulli number, and $\varepsilon = 1$ or $-1$ depending on $\sigma$. Finally, if $\omega \in R(\sigma) \cup C(\sigma) \cup \mathcal{E}_2(G)$, then

$$s(\omega) = [\eta : \omega] = \dim \text{Hom}_K(\eta, \omega) \in \mathbb{Z}$$

where $\eta = \eta_{\sigma}$ is a virtual representation of $K$ (in particular, $s(\omega)$ depends on $\sigma$ but not on $\Gamma$).

Let $R(M)$ and $R(K)$ denote the representation rings of $M$ and $K$. We make use of the following.

**Proposition 1.** Let $i^*: R(K) \to R(M)$ be the restriction homomorphism. Then $\text{Im}(i^*) = R(M)^W$, where $W = W(\mathbb{S}, \mathbb{A})$. If $\text{rank } G = \text{rank } K$, then $R(M)^W = R(M)$ and $i^*$ is surjective.
Let $\Gamma \subseteq G$ be an arbitrary discrete cocompact subgroup. We assume for simplicity, that $G$ has no nontrivial compact, connected, normal subgroups (see the remark below). Theorem 1.1 in [W], together with Proposition 1, imply

**Corollary 1.** If $Z(\Gamma) = Z(G) \cap \Gamma$ and $\sigma \in \hat{\mathcal{E}}(M)$ satisfies $\sigma|_{Z(\Gamma)} = 1$, then

$$\lim_{s \to 0^+} s^c \cdot \psi_\sigma(s) = \frac{\dim(\sigma) \cdot |Z(\Gamma)| \cdot \text{vol}(\Gamma \backslash G)}{(4\pi)^c},$$

(if $\sigma|_{Z(\Gamma)} \neq 1$ then $n_\Gamma(\pi_{\sigma, r}) = 0$, for all $v$, hence $\psi_\sigma(s) = 0$).

Let $N_\sigma(r) = \sum_{\omega \in \rho(\sigma), \omega \neq r} n_\Gamma(\omega) (r > 0)$. Corollary 1 and the Tauberian theorem for the Laplace transform imply

$$\lim_{r \to \infty} r^{-c} \cdot N_\sigma(r) = \frac{\dim(\sigma) \cdot |Z(\Gamma)| \cdot \text{vol}(\Gamma \backslash G)}{\Gamma(c + 1) \cdot (4\pi)^c}.$$  

**Remark.** When $G$ has compact normal subgroups, (1) still holds with $\dim V_{\nabla(\Gamma \cap N)} \cdot |\Gamma \cap N|$ substituting $\dim(\sigma) \cdot |Z(G) \cap \Gamma|$, where $N = \cap_{\chi \in E} x Kx^{-1}$. This follows from [W, 1.1], with a correction factor as in [BH, §6], and Proposition 1. Indeed, if $\sigma = i^\#(\eta)$, $\eta = \sum m_j \tau_j \in R(K)$, one can show that $\sum m_j \dim V_{\tau_j} \cdot \text{vol}(N \cap \Gamma) = \dim V_{\nabla(\Gamma \cap N)}$.

The asymptotic formula (1) for the spherical principal series (i.e. $\sigma = 1$) in $L^2(\Gamma \backslash G)$ was proposed by Gelfand ([G, p. 77], see also [GGP, pp. 82, 94]). It was proved by Gangolli for complex $G$, by Eaton for $G$ of split rank one, and by Duistermaat-Kolk-Varadarajan, for general $G$ ([Ga, DKV], see also [GW]). With the aid of Proposition 1, Theorem 1.1 in [W] implies the Gelfand type formula (1) for any $\sigma \in \hat{\mathcal{E}}(M)$, when $G$ is as above.

The outline of the paper is as follows. In §1, we prove Theorem 1.1 (assuming Proposition 1). The proof of Proposition 1 is given in §2. Finally, we show (Lemma 2.6) that if rank $G = \text{rank } K$, then $J = \ker i^* \neq 0$ and determine $J$ explicitly. We recall that if $\Gamma$ is torsion-free, each $\eta \in \ker i^*$ yields a finite alternating sum formula in the $n_\Gamma(\omega)$'s [M3, Theorem 1.2].

1. We first normalize Haar measures conveniently. If $\lambda$ denotes the long positive restricted root of $(\mathfrak{g}, \mathfrak{h})$ let $H \in \mathfrak{h}$ be so that $\lambda(H) = 2$. Let $a = B(H, H)$, $B$ the Killing form of $\mathfrak{g}$. Let $d\bar{x}, d\bar{k}$ denote respectively the invariant Riemannian measures on $G$ and $K$ induced by the inner product on $\mathfrak{g}$, $(X, Y) = a^{-1} \cdot B(X, Y)$. We will use on $G$ and $K$ the measures $dx = \text{vol}(K)^{-1} \cdot d\bar{x}$, $dk = \text{vol}(K)^{-1} \cdot d\bar{k}$. As usual, let $dx$ on $\Gamma \backslash G$ be so that

$$\int_{\Gamma \backslash G} \left( \sum_{\gamma} f(\gamma x) \right) dx = \int_G f(x) dx, \quad \text{for } f \in C_c(G).$$

For fixed $\sigma \in \hat{\mathcal{E}}(M)$, the Plancherel density associated to $\sigma$ can be written $\mu_\sigma(x\lambda) = q_\lambda(x) \cdot \phi_\sigma(x)$, where $q_\lambda(x)$ is a polynomial of degree $2c - 1$ and $\phi_\sigma(x) = 1, \tanh \pi x$ or $\coth \pi x$, depending on $\sigma$ [O]. Moreover, $\phi_\sigma = 1$ if and only if rank $G > \text{rank } K$. Let $d = c - [c]$, that is, $d = 0$ if rank $G = \text{rank } K$ and $d = \frac{1}{2}$, otherwise. If the Haar
measure on \( G \) is normalized as above, then
\[
q_\sigma(x) = \sum_{i=0}^{c+d-1} b_{2(i-d)+1}(\sigma) \cdot x^{2(i-d)+1}
\]
and \( b_{2c+1}(\sigma) = \text{dim}(\sigma)/(\Gamma(c) \cdot \pi^c) \) [M2, §3].

For fixed \( \tau \) set, if \( x \in G \) and \( s > 0 \),
\[
g_{\tau,s}(x) = \int_{\mathcal{E}(G)} \text{dim}(\tau)^{-1} \cdot \phi_{\tau,\omega}(x^{-1}) \cdot e^{s\lambda_\omega} \, d\mu(\omega)
\]
where \( \phi_{\tau,\omega} \) is the \( \tau \)-spherical trace function associated to \( \omega \) and \( \mu(\omega) \) is the Plancherel measure on \( \mathcal{E}(G) \). DeGeorge and Wallach (unpublished) have proved a general result which implies that \( g_{\tau,s} \in \mathcal{C}^p(G) \) (the \( p \)-Schwartz space of \( G \)) for any \( p > 0 \) (\( G \) can be of arbitrary split rank). Using this fact, one shows [M3, 1.1] that \( \theta_\omega(g_{\tau,s}) = [\tau: \omega] e^{s\lambda_\omega} \) for any \( \omega \in \mathcal{E}(G) \), where \([\tau: \omega] = \text{dim} \text{Hom}_K(\tau, \omega)\).

Let \( \Gamma \subset G \) be a discrete, cocompact, torsion-free subgroup. Fix \( \sigma \in \mathcal{E}(M) \). We assume first that rank \( G = \text{rank} \, K \). Then, by Proposition 1, there exists \( \eta = \sum m_j \tau_j, \quad m_j \in \mathbb{Z}, \tau_j \in \mathcal{E}(K) \) such that \( i^*(\eta) = \sigma \). Set \( g_{\eta,s} = \sum m_j g_{\tau_j,s} \). Since \( g_{\eta,s} \in \mathcal{C}^p(G) \) for any \( p > 0 \), and \( g_{\eta,s} \) is \( K \)-finite, the operator \( \pi_\Gamma(g_{\eta,s}) \) on \( L^2(\Gamma \backslash G) \) is trace-class [M1, §2], and
\[
\text{tr} \pi_\Gamma(g_{\tau,s}) = \sum_{\omega \in \mathcal{E}(G)} n_\Gamma(\omega) \cdot [\eta: \omega] \cdot e^{s\lambda_\omega},
\]
where \([\eta: \omega] = \sum m_j \cdot [\tau_j: \omega]\).

If \( \omega \in \mathcal{E}(G) \), by Langlands' classification, either \( \omega \in \mathcal{E}_2(G) \) or \( \omega \in P(\xi) \cup R(\xi) \cup C(\xi) \), for some \( \xi \in \mathcal{E}(M) \). If \( \omega \in P(\xi) \), then
\[
[\eta: \omega] = [i^*(\eta): \xi] = \begin{cases} 0, & \xi \neq \sigma, \\ 1, & \xi = \sigma. \end{cases}
\]

Hence \( \text{tr} \pi_\Gamma(g_{\tau,s}) = \psi_\sigma(s) + h_\sigma(s) \), where
\[
h_\sigma(s) = \sum_{\omega \in \mathcal{E}(G) \cup R(\sigma) \cup C(\sigma)} n_\Gamma(\omega) \cdot [\eta: \omega] \cdot e^{s\lambda_\omega}.
\]
Note that the sets \( \{ \omega \in \mathcal{E}_2(G) \mid [\eta: \omega] \neq 0 \} \), \( \{ \omega \in C(\sigma) \mid n_\Gamma(\omega) \neq 0 \} \) are finite [DW, p. 489]. Hence \( h_\sigma(s) \) is analytic.

On the other hand ([M1, 5.1], essentially)
\[
\text{tr} \pi_\Gamma(g_{\eta,s}) \sim \text{vol}(\Gamma \mid G) \cdot g_{\eta,s}(1), \quad \text{as } s \to 0^+
\]
(that is, \( \text{tr} \pi_\Gamma(g_{\eta,s}) = \text{vol}(\Gamma \mid G) \cdot g_{\eta,s}(1) = o(s^n) \) for all \( n \in \mathbb{N} \), as \( s \to 0^+ \)).

Set \( g_{\eta,s}^0 = \sum m_j \cdot g_{\tau_j,s}^0 \), where
\[
g_{\tau_j,s}^0 = \sum_{\omega \in \mathcal{E}_2(G)} d(\omega) \cdot [\eta: \omega] \cdot e^{s\lambda_\omega} \quad \text{(a finite sum)}.
\]

By choice of \( \eta \), if \( h_{\eta,s} = g_{\eta,s} - g_{\eta,s}^0 \), then
\[
h_{\eta,s}(1) = \int_{-\infty}^{+\infty} e^{s\lambda_\eta \circ \lambda} \cdot \mu_\sigma(x \lambda) \, dx,
\]
where \( \lambda_{\sigma, \pi \lambda} = -(4x^2 + |p|^2 + a \lambda_\sigma) \) [M1, p. 17]. Here if \( X_1, \ldots, X_r \) is a basis of \( \mathfrak{M} \) such that \( (X_i, X_j) = \delta_{ij} \) and \( \Delta_{\mathfrak{M}} = -\Sigma X_i^2 \), \( \lambda_\sigma \) is so that \( \sigma(\Delta_{\mathfrak{M}}) = \lambda_\sigma \cdot I \). On the other hand \( \mu_\sigma(x \lambda) = q_\sigma(x) \cdot \phi_\sigma(x) \), \( q_\sigma(x) = \sum_{i} b_{2i+1}(\sigma) \cdot x^{2i+1} \) and \( \phi_\sigma(x) = \tanh \pi x \) or \( \coth \pi x \). We may write (if \( x \neq 0 \)) \( 1 - \phi_\sigma(x) = 2/(1 + e^{2\pi x}) \), where \( \epsilon = 1 \) if \( \phi_\sigma(x) = \tanh \pi x \) (respectively \( \epsilon = -1 \), if \( \phi_\sigma(x) = \coth \pi x \)). Hence

\[
h_{\eta, \delta}(1) = e^{-s(|q|^2 + a \lambda_\sigma)} \cdot \left[ 2 \int_0^{+\infty} e^{-4sx^2} \cdot q_\sigma(x) \, dx - 4 \int_0^{+\infty} \frac{e^{-4sx^2} \cdot q_\sigma(x)}{1 + \epsilon e^{2\pi x}} \, dx \right] 
\]

\[
= e^{-s(|q|^2 + a \lambda_\sigma)} \cdot \sum_{i} b_{2i+1}(\sigma) \cdot i! (4s)^{-i-1} 
\]

\[
- \sum_{i} b_{2i+1}(\sigma) \int_0^{+\infty} \frac{4 \cdot e^{-4sx^2} \cdot x^{2i+1}}{1 + \epsilon e^{2\pi x}} \, dx 
\].

Furthermore [WW, pp. 266–268]

\[
\int_0^{+\infty} \frac{4x^{2i+1}}{1 + e^{2\pi x}} \, dx = \frac{2^{2(i+1)-1}}{i + 1} \cdot B_{2(i+1)}, 
\]

\[
\int_0^{+\infty} \frac{4x^{2i+1}}{1 - e^{2\pi x}} \, dx = - \frac{B_{2(i+1)}}{i + 1}, 
\]

\( B_{2m} \) the \( m \)th Bernoulli number. Hence

\[
\lim_{s \rightarrow 0^+} \sum_{i} b_{2i+1}(\sigma) \int_0^{+\infty} \frac{4 \cdot e^{-4sx^2} \cdot x^{2i+1}}{1 + \epsilon e^{2\pi x}} \, dx 
\]

\[
= \sum_{i} b_{2i+1}(\sigma) \left[ \frac{(\epsilon + 1) \cdot 2^{2i+1}-1}{i + 1} \right] B_{2(i+1)} 
\]

(in fact, the full asymptotic expansion

\[
\int_0^{+\infty} \frac{e^{-4sx^2} \cdot x^{2i+1}}{1 + \epsilon e^{2\pi x}} \, dx \sim \sum_{j=0}^{\infty} a_j \cdot s^j 
\]

can be written down explicitly).

Summing up

\[
\psi_\sigma(s) \sim \text{vol}(\Gamma \setminus G) \cdot (h_{\eta, \delta}(1) + g_{\eta, \delta}^0(1)) - h_\sigma(s), 
\]

\[
\psi_\sigma(s) \sim \text{vol}(\Gamma \setminus G) e^{-s(|q|^2 + 2\lambda_\sigma)} \cdot \left( \sum_{i} b_{2i+1}(\sigma) \cdot i! (4s)^{-i-1} \right) - g_\sigma(s), 
\]

where

\[
g_\sigma(s) = \text{vol}(\Gamma \setminus G) \cdot e^{-s(|q|^2 + 2\lambda_\sigma)} \cdot \left( \sum_{i} b_{2i+1}(\sigma) \int_0^{+\infty} \frac{4 \cdot e^{-4sx^2} \cdot x^{2i+1}}{1 + \epsilon e^{2\pi x}} \, dx \right) 
\]

\[
- \text{vol}(\Gamma \setminus G) \cdot g_{\eta, \delta}^0(1) + h_\sigma(s). 
\]

This concludes the proof of Theorem 1, in this case.
If rank $G > \text{rank } K$, let $W = W(\mathfrak{g}, \mathfrak{h}) = \{1, u\}$. If $\sigma \in \hat{\mathfrak{g}}(M)$ is such that $\sigma = \sigma^u$, then by Proposition 1, $\sigma = \iota^*(\eta_0)$, $\eta_0 \in R(K)$, and the above proof (with several simplifications) can be repeated. Moreover, in this case $\mu_\sigma(x\lambda) = q_\sigma(x)$, $\delta_\eta, s = 0$, hence $g_\sigma(s) = h_\sigma(s)$.

If $\sigma \neq \sigma^u$ then $\sigma + \sigma^u = \iota^*(\eta_0)$, $\eta_0 \in R(K)$. Define $g_{\eta, s}$ as before. In this case $g_{\eta, s} = h_{\eta, s}$. Arguing as above, one obtains

$$\sum_{\omega \in \mathcal{C}(G)} n_\Gamma(\omega) \cdot [\eta : \omega] \cdot e^{i\lambda_\omega} \sim \text{vol}(\Gamma \setminus G) \cdot h_{\eta, s}(1), \text{ as } s \to 0^+.$$ 

The left-hand side equals

$$2 \psi_\sigma(s) + 2 \sum_{\omega \in \mathcal{C}(\sigma)} n_\Gamma(\omega) \cdot e^{i\lambda_\omega}$$

since $\hat{\mathfrak{g}}_2(G) = R(\sigma) = \phi$, $\pi_{\alpha,v} = \pi_{\alpha,v,-v}$ ($\nu \in \mathfrak{h}^*_\mathfrak{c}$) and $[\eta : \omega] = 1$ if $\omega \in C(\sigma)$, in this case. Similarly,

$$h_{\eta, s}(1) = 2 \cdot e^{-x(\eta^2 + a\lambda_s)} \cdot \left( \sum_{0}^{c-1/2} b_{2i}(\sigma) \int_{-\infty}^{+\infty} e^{-4isx^2} \cdot x^{2i} dx \right)$$

$$= 2e^{-x(\eta^2 + a\lambda_s)} \cdot \left( \sum_{0}^{c-1/2} b_{2i}(\sigma) \cdot \Gamma\left(i + \frac{1}{2}\right) \cdot \left(4\pi\right)^{-i-1/2} \right).$$

This concludes the proof. We observe that, if $\sigma \in \text{Im}(\iota^*)$, Corollary 1 is an immediate consequence of Theorem 1.1 in [W] and Proposition 1 (with our normalization of measures $C_G = 1$, $C_G$ as in [W, 1.1]). If $\sigma \notin \text{Im}(\iota^*)$, then $\sigma + \sigma^u = \iota^*(\eta_0)$ and (essentially) the above argument yields the result.

2. This section is mainly devoted to the proof of Proposition 1. Assume first that rank $G > \text{rank } K$. Then rank $K = \text{rank } M$. Let $T_1 \subset M$ be a maximal torus. There is a commutative diagram

$$\begin{array}{ccc}
R(K) & \xrightarrow{\iota^*} & R(M) \\
j_{K*} \downarrow & & \downarrow j_{M*} \\
R(T_1)^{W_K} & & \\
\end{array}$$

where $j_{K*}$ is an isomorphism onto $R(T_1)^{W_K}$. If $M^* = N_K(A)$ (the normalizer of $A$ in $K$), there is $u \in M^* \cap N_K(T)$, $u \notin M$. Therefore, $W_K$ is generated by $W_M$ and $u(1 W_K/W_M|= 2)$. Thus $\text{Im}(j_{K*}) = R(T)^{W_K} = (R(T)^{W_M})^W$ and Proposition 1 is clear, in this case.

From now on, we assume that rank $G = \text{rank } K$. Fix $\mathfrak{r} \subset \mathfrak{h}$, a Cartan subalgebra, and let $\Delta = \Delta(\mathfrak{h}_\mathfrak{c}, \mathfrak{h}_\mathfrak{c})$. Then $\Delta = \Delta_c \cup \Delta_n$, where $\Delta_c (\Delta_n)$ is the set of compact (noncompact) roots. Fix $\Delta^+ \subset \Delta$ a system of positive roots, $\Delta^+ = \Delta^+_c \cup \Delta^+_n$. Let $\{X_\alpha\}_{\alpha \in \Delta^+_c}$ be a Weyl basis of $\mathfrak{g}_\mathfrak{c}$ adapted to the compact form $\mathfrak{g}_\mathfrak{c} = \mathfrak{g} \oplus i\mathfrak{g}$ [H, p. 421]. Then, if $\sigma$ denotes the conjugation of $\mathfrak{g}_\mathfrak{c}$ with respect to $\mathfrak{g}$, $\sigma X_\alpha = -X_{-\alpha}$ ($\alpha \in \Delta_c$) and $\sigma X_\alpha = X_{-\alpha}$ ($\alpha \in \Delta_n$). From now on, we fix $\beta \in \Delta^+_n$ and choose $\mathfrak{r} = R(X_\beta + X_{-\beta})$. The following lemma is not difficult.
2.1. Lemma.

\[ \mathfrak{m}_C = \ker \beta \oplus \sum_{\alpha \in \Delta, \alpha + \beta \notin \Delta} \mathbb{C} \cdot X_\alpha \oplus \sum_{\alpha \in \Delta, \alpha + 2\beta \in \Delta} \mathbb{C} (X_\alpha + c_\alpha X_{\alpha + 2\beta}) \]

where \( c_\alpha = -N_{\alpha, \beta}/N_{\alpha + 2\beta, \beta} \) and \( N_{\alpha, \beta} \) is such that \([X_\alpha, X_\beta] = N_{\alpha, \beta} \cdot X_{\alpha, \beta} \). Furthermore, \( \ker \beta \) is a Cartan subalgebra of \( \mathfrak{m}_C \) and

\[ \Delta_{\mathfrak{m}_C} = \Delta(\mathfrak{m}_C, \ker \beta) = \{ \alpha' = \alpha \mid_{\ker \beta} | \alpha \pm \beta \notin \Delta \} \cup \{ \alpha' = \alpha \mid_{\ker \beta} | \alpha + 2\beta \in \Delta \}. \]

The root spaces are \( \theta_\alpha = \mathbb{C} X_\alpha \), if \( \alpha \pm \beta \notin \Delta \) and \( \theta_\alpha = \mathbb{C} (X_\alpha + c_\alpha X_{\alpha + \beta}) \), if \( \alpha + 2\beta \in \Delta \).

Let \( \Delta_{\mathfrak{m}_C}^+ \subset \Delta_{\mathfrak{m}_C} \) be the positive system induced by \( \Delta^+ \). Let also \( T_1 = \exp(\ker \beta \cap \mathfrak{h}) \), a maximal torus of \( M^0 \) (the connected component of 1 in \( M \)).

2.2. Lemma. Let \( G \) be semisimple, of split rank one, and such that \( \text{rank } G = \text{rank } K \).

Let \( W = W(\mathfrak{g}, \mathfrak{a}) = \{ 1, u \} \). Then \( \sigma = \sigma^u \), for any \( \sigma \in \mathcal{E}(M) \).

Proof. In [KS, §16] the lemma is verified for \( G = \text{Spin}(2n, 1) \), \( G = \text{SU}(n, 1) \) and \( G = \text{Sp}(n, 1) \). We give a different proof. It is well known that \( M = Z(G) \cdot M_0 \).

Moreover, \( W \) is generated by \( u = \exp(\pi i H_{\beta}/(\beta, \beta)) \). If \( \sigma \in \mathcal{E}(M) \), then \( \chi_{\sigma^*}(x) = \chi_{\sigma^*}(x) \) for any \( x \in M \), since this holds for \( x \in T_1 \) and \( x \in Z(G) \). Hence \( \sigma^* = \sigma \).

Remark. In [KS, Theorem 12.5] Knapp and Stein prove that if \( G \) is a linear group of split rank one, \( \pi_{a, \nu} \) is reducible only if \( \nu = 0 \). Moreover, \( \pi_{a, 0} \) is reducible if and only if (i) \( \sigma = \sigma^u \), (ii) \( \mu_{\sigma}(0) > 0 \). Lemma 2.2 says that if \( \text{rank } G = \text{rank } K \), (i) is automatic. If \( \text{rank } G > \text{rank } K \) it is no longer true that \( \sigma = \sigma^u \). In fact, \( \sigma = \sigma^u \) forces \( \mu_{\sigma}(0) = 0 \), hence \( \pi_{a, 0} \) is irreducible.

We next prove a lemma. Let \( K_1 \) be a Lie group with finitely many components, such that \( A_1(K_1) \) is compact. Let \( K_2 \subset K_1 \) be a closed subgroup. As usual, let \( R(K_i) \) and \( \mathcal{E}(K_i) \) denote, respectively, the representation ring and the unitary dual of \( K_i \) \((i = 1,2) \). Let \( S \) be a closed subgroup of \( Z(K_1) \) (the center of \( K_1 \)) such that \( S \subset K_2 \). Then \( R(K_2/S) \) can be identified with the subring of \( R(K_2) \) generated by those representations \( \tau \) of \( K_1 \) such that \( S \subset \ker \tau \). Let \( i^* : R(K_1/S) \to R(K_2/S) \), \( i^* : R(K_1) \to R(K_2) \) denote the restriction homomorphisms.

2.3. Lemma. \( \text{Im}(i^*_\mathcal{E}) = \text{Im}(i^*) \cap R(K_2/S) \).

Proof. Let \( \tau \in \mathcal{E}(K_1) \). If \( i^*(\tau) = \sum r_j \cdot \xi_j \) \((r_j \neq 0) \) we say that \( \xi_j \) is a \( K_2 \)-type of \( \tau \).

We note that if \( \tau \) has a \( K_2 \)-type \( \xi \) such that \( \xi \mid_S = 1 \), then \( \tau \mid_S = 1 \). Indeed, since \( S \) is central in \( K_1 \), then \( \text{Ind}^S_{\mathfrak{k}_2} \xi \mid_S = 1 \). Thus \( \tau \mid_S = 1 \), too. As a consequence, if \( \tau, \gamma \in \mathcal{E}(K_1) \) have a common \( K_2 \)-type and \( \tau \mid_S = 1 \), then \( \gamma \mid_S = 1 \).

We now prove the lemma. Let \( \eta \in R(K_1) \) be such that \( i^*(\eta) \in R(K_2/S) \). If \( \eta = \sum t_j \cdot \gamma_j \), \( i^* \eta = \sum t_j \cdot \sigma_j \), set \( \mathcal{E}_{\eta}(K_1) = \{ \tau_1, \ldots, \tau_k \} \), \( \mathcal{E}_{i^*(\eta)}(K_2) = \{ \sigma_1, \ldots, \sigma_l \} \). By assumption \( \sigma_j \mid_S = 1, j = 1, \ldots, l \).

Define inductively

\[ \mathcal{S}_1 = \{ \gamma \in \mathcal{E}_{\eta}(K_1) \mid \gamma \text{ contains a } K_2 \text{-type in } \mathcal{E}_{i^*(\eta)}(K_2) \} \],

\[ \mathcal{S}_{i+1} = \{ \gamma \in \mathcal{E}_{\eta}(K_1) \mid \gamma \text{ has a common } K_2 \text{-type with some } \tau \in \mathcal{S}_i \} \].
Then \( S_1 \subset S_2 \subset \cdots \subset S_n(K_1) \). By the above observation, if \( t \in S_j \) for some \( j \), then \( t|_{K_1} = 1 \). Thus, if \( S_n = S_j(K_1) \) for some \( n \in \mathbb{N} \), then \( \eta \in R(K_1/S) \) and the lemma is proved. Otherwise, there exists \( n \) such that \( S_n = S_{n+1} \neq S_{n+1}(K_1) \). It is then easy to see that if \( \eta' = \sum_{\tau_j \in S_j} m_j \tau_j \), then \( i^*(\eta') = 0 \). Thus \( i^*(\eta) = i^*(\eta - \eta') \) and \( \eta - \eta' \in R(K_1/S) \). We note that in general it is not true that \( \ker i^* \subset R(K_1/S) \), as the example \( K_1 = S^1, K_2 = S = \{ \pm 1 \} \) already shows.

2.4. Lemma. Let \( G \) be a simply connected Lie group of split rank one. Assume that \( \text{rank } G = \text{rank } K \) and \( \mathfrak{g} \neq \mathfrak{g}/(2, \mathbb{R}) \). Then \( M \) is simply connected.

PROOF. By applying the long exact sequence in homotopy to the fibration \( M \to K \to K/M \), one readily obtains \( \pi_0(M) = \pi_1(M) = \{ 1 \} \) (\( K/M \) is diffeomorphic to the unit sphere in \( \mathfrak{g} \) and \( \dim \mathfrak{g} \geq 4 \)).

2.5. Proof of Proposition 1. By Lemma 2.2, in order to prove Proposition 1, we must show that \( i^*: R(K) \to R(M) \) is surjective, if \( \text{rank } G = \text{rank } K \). By Lemma 2.3 (applied to \( (K_1, K_2) = (K, M) \)), it is enough to verify this under the assumption that \( K \) (hence \( G \)) be simply connected. Now, since \( G \) has split rank one, we may assume that \( G \) is simple and, on the other hand, if \( \mathfrak{g} = \mathfrak{g}/(2, \mathbb{R}) \), it is clear that \( i^* \) is surjective. We thus assume that \( G \) is simple, simply connected, \( \text{rank } G = \text{rank } K \) and \( \mathfrak{g} \neq \mathfrak{g}/(2, \mathbb{R}) \).

It will be enough to show, by Lemma 2.4, that the fundamental representations of \( \mathcal{R}_c \) are restrictions of virtual representations of \( \mathfrak{R}_c \). We give a proof by case-by-case verification. Though a direct proof would be desirable, by this method, one finds explicitly \( \eta \in R(K) \) with \( i^*(\eta) = \sigma \), for each fundamental representation \( \sigma \) of \( \mathfrak{R}_c \).

Since, by Theorem 1, the coefficients \( a_i (i > 0) \) of the asymptotic expansion of \( \psi_\sigma(s) \) involve the numbers \( \lfloor \eta : \omega \rfloor (\omega \in \mathfrak{c}(G)) \), the knowledge of \( \eta \) may be of some use.

From now on we identify, via the Killing form, the imaginary dual of \( \mathfrak{S} \) with a convenient subspace of \( \mathbb{R}^n \), so that the usual inner product of \( \mathbb{R}^n \) corresponds to a multiple of the Killing form. Let \( \{ e_1, \ldots, e_n \} \) be the canonical basis of \( \mathbb{R}^n \). We often denote by \( 1 \) the trivial representation (of any Lie algebra). We will make use of the well-known branching formulas (see [Z, pp. 128–132 or B, 10]).

(i) \( \mathfrak{g} = \mathfrak{g} \mathfrak{u}(n, 1) (n \geq 2) \).

\[
i \mathfrak{g}^* = \left\{ \sum_{i=1}^{n+1} t_i e_i | t_1 + \cdots + t_{n+1} = 0 \right\}, \quad \mathfrak{H}_c \cong \mathfrak{u}(n), \quad \mathfrak{M}_c \cong \mathfrak{u}(n-1),
\]

\[
\Delta_c^+ = \{ e_i - e_j | 2 \leq i < j \leq n+1 \}, \quad \Delta_n^+ = \{ e_i - e_j | 2 \leq i < n+1 \}, \quad \beta = e_1 - e_2.
\]

The centers of \( \mathfrak{H} \) and \( \mathfrak{M} \) correspond, respectively, to \( R(e_1 - \frac{1}{n} (e_2 + \cdots + e_{n+1})) \) and \( R(e_1 + e_2 - 2(e_3 + \cdots + e_{n+1}))/\langle n - 1 \rangle \). Any \( a \in \mathbb{R} \) defines a character \( \phi_a \) (\( \phi_a' \)) on \( \mathfrak{H} (\mathfrak{M}) \) by the rule

\[
\phi_a \left( e_1 - \frac{1}{n} \sum_{i=2}^{n+1} e_i \right) = ia \phi_a' \left( (e_1 + e_2) - \frac{2\sum_{i=3}^{n+1} e_i}{n - 1} \right) = ia \phi_a'.
\]

Hence \( \phi_a (\phi_a') \) defines a one-dimensional representation of \( \mathfrak{H}_c (\mathfrak{M}_c) \) and it is easy to verify that \( i^*(\phi_a) = \phi_a' \).
The fundamental representations are \( \lambda_i = \varepsilon_2 + \cdots + \varepsilon_i \) (for \( i = 1, \ldots, n-2 \)) and \( \lambda'_j = \varepsilon_3 + \cdots + \varepsilon_j \) (for \( j = 1, \ldots, n-1 \)), for \( \mathfrak{R}_C \) and \( \mathfrak{W}_C \). The branching formulas imply

\[
i^*(\lambda_2) = \phi_1 \oplus \phi_2 \oplus \lambda'_3, \\
i^*(\lambda_i) = \phi_{2i-3} \otimes \lambda'_i \oplus \phi_{2i-2} \otimes \lambda'_{i+1}, \quad 3 \leq i \leq n-1, \\
i^*(\lambda_n) = \phi_{2n-3} \otimes \lambda'_n \oplus \phi_{2n-2}.
\]

where \( \phi'_j = \phi_{2j} \) (\( a_j \) can be easily computed). Since \( \text{Im}(i^*) \) contains \( \phi'_a \) for any \( a \), this clearly implies that \( \lambda'_j \in \text{Im}(i^*) \) for \( 3 \leq j \leq n \).

(ii) \( \mathfrak{S} = \mathfrak{S}(2n, 1) \).

\[
i^* = \left\{ \sum_{i=1}^n t_i \varepsilon_i | t_i \in \mathbb{R} \right\}, \quad \Delta^+ = \{ \varepsilon_i | 1 \leq i \leq n, \varepsilon_i \pm \varepsilon_j | 1 \leq j \leq n \},
\]

\[
\Delta_c^+ = \{ \varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq n \}, \quad \Delta_n^+ = \{ \varepsilon_i | 1 \leq i \leq n \},
\]

\( \beta = \varepsilon_1 \), \( \Delta_{\mathfrak{N}}^+ = \{ \varepsilon_i | 2 \leq i \leq n, \varepsilon_i \pm \varepsilon_j | 2 \leq j \leq n \} \).

Fundamental weights:

\( \mathfrak{R}_C \): \( \lambda_i = \varepsilon_1 + \cdots + \varepsilon_i \) (for \( i = 1, \ldots, n-2 \)), \( \lambda_\pm = \frac{1}{2} (\varepsilon_1 + \cdots + \varepsilon_{n-1} \pm \varepsilon_n) \)

\( \mathfrak{W}_C \): \( \lambda'_i = \varepsilon_2 + \cdots + \varepsilon_i \) (for \( i = 2, \ldots, n-1 \)), \( \lambda'_+ = \frac{1}{2} (\varepsilon_2 + \cdots + \varepsilon_n) \).

By the branching formulas

\[
i^*(\lambda_i) = \lambda'_i \oplus \lambda'_{i+1}, \quad i = 1, \ldots, n-2, (\lambda'_1 = 1), i^*(\lambda_\pm) = \lambda'_+.
\]

Hence \( \lambda'_{i+1} = i^*(\lambda_i - \lambda_{i-1} + \lambda_{i-2} - \cdots \pm 1) \), \( \lambda'_+ = i^*(\lambda_\pm) \). We include the case \( \mathfrak{S} = \mathfrak{S}(2n + 1, 1) \), for completeness.

(iii) \( \mathfrak{S} = \mathfrak{S}(2n + 1, 1) \).

\[
i^* = \left\{ \sum_{i=1}^{n+1} t_i \varepsilon_i | t_i \in \mathbb{R} \right\}, \quad \Delta^+ = \{ \varepsilon_i \pm \varepsilon_j | 1 \leq j \leq n + 1 \},
\]

\[
\Delta_c^+ = \{ \varepsilon_i \pm \varepsilon_j | 2 \leq i < j \leq n \}, \quad \Delta_n^+ = \{ \varepsilon_i \pm \varepsilon_j | 2 \leq i < j \leq n + 1 \}.
\]

Fundamental weights:

\( \mathfrak{R}_C \): \( \lambda_i = \varepsilon_2 + \cdots + \varepsilon_i \) (for \( i = 2, \ldots, n \)), \( \lambda_+ = \frac{1}{2} (\varepsilon_2 + \cdots + \varepsilon_{n+1}) \)

\( \mathfrak{W}_C \): \( \lambda'_i = \varepsilon_2 + \cdots + \varepsilon_i \) (for \( i = 2, \ldots, n + 1 \)), \( \lambda'_+ = \frac{1}{2} (\varepsilon_2 + \cdots + \varepsilon_n \pm \varepsilon_{n+1}) \).

Moreover, \( i^*(\lambda_i) = \lambda'_i \oplus \lambda'_{i-1} \) (for \( 2 \leq i \leq n \)), \( i^*(\lambda_+) = \lambda'_+ \oplus \lambda'_- \).

Hence \( \lambda'_i = i^*(\lambda_i - \lambda_{i-1} + \lambda_{i-2} - \cdots \pm 1) \in \text{Im}(i^*) \) (for \( i = 2, \ldots, n \)).

Recall [Hu, p. 188] that \( \lambda'_+ \oplus \lambda'_- = \lambda'_n \oplus \lambda'_{n-2} \oplus \cdots \).

Thus \( \lambda'_+ \oplus \lambda'_- \in \text{Im}(i^*) \). On the other hand, if \( W = \{ 1, u \} \) one knows that \( \lambda'_{2i} = \lambda'_i (i = 2, \ldots, n-1), (\lambda'_\pm)^u = \lambda'_\pm \).

Hence \( \text{R}(M)^W = \mathbb{Z}[\lambda'_2, \ldots, \lambda'_{n-1}][\lambda'_+, \lambda'_-]^W \) is a polynomial ring over \( \mathbb{Z}[\lambda'_2, \ldots, \lambda'_{n-1}] \) in the symmetric functions \( \lambda'_+ \oplus \lambda'_- \). Hence, if \( M \) is simply connected (i.e. \( G = \text{Spin}(2n + 1, 1) \)) \( \text{Im}(i^*) = \text{R}(M)^W \).
The case $G = SO(2n, 1)$ follows from Lemma 2.3.

$$i\mathfrak{E} = \mathfrak{d} \rho(n, 1)$$

$$(iv) \mathfrak{S} = \mathfrak{d} \rho(n, 1) (n \geq 2).$$

$$\mathfrak{L} \cong \mathfrak{d} \rho(1) \times \mathfrak{d} \rho(n), \quad i\mathfrak{S}^* = \left\{ \sum_{i=1}^{n+1} l_i e_i | l_i \in \mathbb{R} \right\}.$$
DISTRIBUTION OF THE PRINCIPAL SERIES IN $L^2(\Gamma \backslash G)$

(dimensions 7, 21, and 8). The branching formulas are

$$i^*(\lambda_i) = \lambda'_i + 1, \quad i^*(\lambda_2) = \lambda'_1 \oplus \lambda'_2 \oplus \lambda'_3, \quad i^*(\lambda_+) = \lambda'_1 \oplus \lambda'_+ \oplus 1$$

and

$$i^*(\lambda_3) = (\lambda'_1 + \lambda'_+) \oplus \lambda'_2 \oplus \lambda'_+.$$

Therefore, $\lambda'_+ = i^*(\lambda_1 - 1)$, $\lambda'_1 = i^*(\lambda_+ - \lambda_1)$, $\lambda'_2 = i^*(\lambda_2 - \lambda_+ + 1)$ and $i^*$ is surjective. We sketch the proof of the branching formulas.

A basis for the unipotent radical of the Borel subalgebra of $\mathfrak{m}_C$ defined by $\Delta_{\mathfrak{m}}$ is $X_{e_i + e_j}$, $2 \leq i \leq 4$, $X_{e_i - e_j}$, $2 \leq i < j \leq 4$, $X_{e_i + e_j - e_k}$, $X_{e_i + e_j + c_2}$, and $X_{e_i - e_j}$, where the constants $c_i$ are as in Lemma 2.1.

Since $\lambda_1 = \frac{1}{2} \beta + \lambda'_+$, the restriction of $\lambda_1$ contains $\lambda'_+$. Since $\dim(\lambda_1) = 9$, $\dim(\lambda'_+) = 8$, the first identity is clear.

Now $\lambda_2 = (-\frac{1}{2}) \beta + \lambda'_+$. Hence $i^*(\lambda_2)$ contains $\lambda'_+$. Since any weight of $\lambda_2$ is of the form $\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$ one checks that any vector of weight $\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)$ is $\mathfrak{m}_C$-dominant. Thus, $\lambda'_+$ restricted to $\mathfrak{m}_C$ contains $\lambda'_1$. Since $\dim(\lambda'_1) = 7$, the third identity follows. Now we study $\lambda_2 = \epsilon_1 + \epsilon_2$, restricted to $\mathfrak{m}_C$. This is the adjoint representation of $\mathfrak{g}_C$ with weights $\pm \epsilon_i$, $1 \leq i \leq 4$, $\pm \epsilon_j$, $1 \leq i < j \leq 4$, and 0, with multiplicity 4. Clearly, $i^*(\lambda_2)$ contains the $\mathfrak{m}_C$-module with highest weight $\lambda'_2$. On the other hand, it is easily checked that any vector of weight $\epsilon_1$ is $\mathfrak{m}_C$-dominant. Since $\epsilon_1 \in \ker i^* = \lambda'_+$, then $i^*(\lambda_2) = \lambda'_2 \oplus \lambda'_+ \oplus \mu$, a representation of dimension 7. Now if $v_1 \neq 0$ is of weight $\epsilon_2 + \epsilon_3$ and $v_2 \neq 0$ is of weight $\epsilon_1 - \epsilon_4$, then $X_{e_i - e_j}(v_1)$ and $X_{e_i + e_j}(v_2)$ are nonzero vectors of weight $\epsilon_1 + \epsilon_2$. Hence, we can choose $v_1$ and $v_2$ so that $c_1 X_{e_i - e_j}(v_1) + c_2 X_{e_i + e_j}(v_2) = 0$. It is easy to verify that with this choice $v_1 + v_2$ is $\mathfrak{m}_C$-dominant. Since

$$\epsilon_2 + \epsilon_3 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4) - \beta, \quad \epsilon_1 - \epsilon_4 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4) + \beta,$$

the $\mathfrak{m}_C$-submodule spanned by $v_1 + v_2$ has highest weight $\lambda'_1$. This proves the third identity, since $\dim(\lambda'_1) = 7$. We omit the proof of the last one, since from the first three already concludes that $i^*$ is surjective.

We conclude the paper by computing $\ker(i^*: R(K) \to R(M))$ explicitly. Recall that each $\eta \in \ker(i^*)$ yields an alternating sum formula in the multiplicities $r_i(\omega)$, if $\Gamma$ is torsion-free [M3, 1.2]. We assume from now on that $G$ is a connected, semisimple Lie group of split rank one, with finite center. If $K$ is compact and $\tilde{K} \to K$, a finite covering, we identify $\delta(K)$ with $\{\tau \in \delta(\tilde{K})|\ker p \subset \ker \tau\}$ and $R(K)$ with the corresponding subring of $R(\tilde{K})$. Let $T$ be a maximal torus of $K$ and $\tilde{T} = p^{-1}(T)$.

2.6. Lemma. (i) If rank $G >$ rank $K$, then $\ker i^* = 0$.

(ii) If rank $G =$ rank $K$, let $\tilde{G} \to G$ be a finite covering so that $\delta_{\eta} = \frac{1}{2}(\Sigma_{\Delta^+} \alpha)$ is a weight of $\tilde{T} = p^{-1}(T)$. Then $\ker i^* = R(K) \cap R(\tilde{K}) \cdot \eta_1$, where $\eta_1 \in R(\tilde{K})$ is such that $\eta_1(t) = t^{r_\beta(t) - 1})$, $t \in \tilde{T}$.

Proof. As noted at the beginning of the section, if rank $G >$ rank $K$, $i^*: R(K) \to R(M)^W$ is an isomorphism.
We thus assume that rank $G = \text{rank } K$. We also assume that $\delta_n$ is a weight of $T$. The lemma is obvious once it is proved in this case.

Let $\beta \in \Delta_n^+$ and $\mathfrak{A} = \mathbb{R}(X_\beta + X_{-\beta})$, as above.

If $\eta \in \ker i^*$, then $\eta(t) = 0$ for $t \in T_\beta$, since $T_\beta \subset M$. Therefore ([A, 6.4], essentially), there is $\eta' \in R(T)$ so that

$$\eta(t) = (t^\beta - 1) \cdot \eta'(t), \quad t \in T.$$

Since $\eta' = \eta$ ($s \in W_K$), then $\eta(t) = 0$, for $t \in sT_\beta = T_{s\beta}$. If $s\beta \neq \pm \beta$, then $\dim T_\beta \cap T_{s\beta} < \dim T_\beta$. Thus, by continuity, $\eta'(t) = 0$, $t \in T_{s\beta}$. Hence, $\eta'(t) = (t^{s\beta} - 1) \cdot \eta''(t)$, for some $\eta'' \in R(T)$.

We may thus write

$$(*) \quad \eta = \prod_{\gamma \in \Psi} (t^\gamma - 1) \cdot \eta' \quad (\eta' = \eta'(\Psi) \in R(T)),$$

where $\Psi$ is any subset of $W_K \cdot \beta$ such that $\Psi \cap -\Psi = \emptyset$.

Since $\emptyset$ is of split rank one, then either $K \subset C$ acts irreducibly on $\mathfrak{P}_C$, or $\mathfrak{P}_C = \mathfrak{P}^+ + \mathfrak{P}^-$, where $\mathfrak{P}^+ = \Sigma_{\Delta_n^+} \mathfrak{P}_a, \mathfrak{P}^- = \Sigma_{\Delta_n^-} \mathfrak{P}_a$ and $\mathfrak{P}_a$ are irreducible subspaces. Furthermore, all noncompact roots have the same length [KW, 12.1]. Thus $W_K \cdot \beta = \Delta_n^+$ or $W_K \cdot \beta = \Delta_n^-$, since $W_K$ acts transitively on weights of a fixed length.

Then, if $\Psi = \Delta_n^+$ in $(*)$, we may write

$$\eta = \eta_0 \cdot \eta'' \quad \text{with } \eta_0(t) = \prod_{\gamma \in \Delta_n^+} (t^\gamma - 1), \eta'' \in R(T),$$

or

$$\eta = \eta_1 \cdot \eta', \quad \text{where } \eta_1(t) = t^{-\delta_n} \cdot \eta_0(t) \in R(T)^{W_K} \text{ and } \eta' \in R(T)^{W_K}. $$

On the other hand, $M = Z(G) \cdot M^0$ ($M^0$, the connected component of 1 in $M$) and $T_\beta = Z(G) \cdot T_\beta^0$ ($T_\beta^0 = \exp(\ker \beta \cap \emptyset$), a maximal torus of $M^0$). Hence, $M = \cup \{x \cdot T_\beta \cdot x^{-1} \mid x \in M\}$ and $\eta_1 \in \ker i^*$, since $\eta_1(t) = 0$ for $t \in T_\beta$. Thus ker $i^* = R(K) \cdot \eta_1$, as asserted.

**EXAMPLES.** (i) $G$ simply connected. Then ker $i^* = R(K) \cdot \eta_1$.

(ii) $G = SL(2, \mathbb{R})$.

Then $K = T = \left\{ k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right\}$, $\mathfrak{S}(K) = \{\tau_n \mid \tau_n(k(\theta)) = e^{i\theta} \tau_n\}, \Delta = \Delta_n = \{\pm \alpha\}, \ k(\theta)^n = e^{2i\theta} = \tau_n(k(\theta))$. Hence ker $i^* = R(K) \cdot (\tau_n - \tau_{-n})$, as in [M3, Lemma 2.1].

(iii) $\emptyset = \mathbb{S}(n, 1)$.

Then $W_K \cdot \beta = \Delta_n^+$ and $\eta_0(t) = \prod_{\Delta_n^+}(t^\gamma - 1) \in R(T)^{W_K}$. Hence ker $i^* = R(K) \cdot \eta_0$

(if $\delta_n$ is a weight of $K$, $\eta_0$ and $\eta_1$ differ by a unit in $R(T)^{W_K} \simeq R(K)$).

(iv) $G = SO(2n, 1)$.

In the notation of 2.5(ii), by Lemma 2.6,

$$\ker i^* = \{ \eta = \eta' \otimes \eta_1 \mid \eta' \in R \text{Spin}(2n), \eta \in RSO(2n) \}.$$

where $\eta_1 = \lambda_+ - \lambda_- \in R \text{Spin}(2n)$. Now

$$R \text{Spin}(2n) = \mathbb{Z}[\lambda_1, \ldots, \lambda_n][\lambda_+, \lambda_-] \subset RSO(2n)[\lambda_+, \lambda_-]$$

[Hu, Chapter 13].
DISTRIBUTION OF THE PRINCIPAL SERIES IN $L^2(\Gamma \backslash G)$

It is then easy to check that $\eta' \otimes \eta_1 \in \text{RSO}(2n)$ if and only if $\eta' = \eta^+ \otimes \lambda_+ + \eta^- \otimes \lambda_- \eta^+ \in \text{RSO}(2n)$. That is,

$$\ker i^* = \left\{ (\eta^+ \otimes \lambda_+ + \eta^- \otimes \lambda_-) \otimes (\lambda_+ - \lambda_-) | \eta^+ \in \text{RSO}(2n) \right\}.$$ 

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