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HAMBURGER-NOETHER EXPANSIONS OVER RINGS

BY

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ABSTRACT. We study Hamburger-Noether expansions over rings, obtaining some applications to equisingular deformation theory and the moduli problem of plane curve singularities, and construct a universal equation for a given equisingularity class.

Introduction. Hamburger-Noether expansions over algebraically closed fields are considered in [**1**] in the study of singularities of algebroid curves. In this paper we introduce Hamburger-Noether expansions over rings, generalizing for them most of the results exhibited in [**1**] for the field case.

First, we obtain a universal equation for an equisingularity class E of plane curve singularities, i.e., a power series $F_E(X, Y) \in B_E[[X, Y]]$, B_E a certain ring, such that for any algebraically closed field k and any algebroid curve in E over k, an equation for this curve can be obtained as the image of F_E by a homomorphism ϕ: $B_E[[X, Y]] \to k[[X, Y]]$ which is the extension to power series rings of a ring homomorphism ϕ: $B_E \to k$.

Second, we define HN-equisingular deformations of a plane algebroid curve $f_0 \in k[[X, Y]]$ over a complete local k-algebra A to be those deformations of f_0 which are equivalent to one obtained as the image of F_E (E the equisingularity class of f_0) by the extension to power series rings of a ring homomorphism $B_E \to A$. Most of the main properties of HN-equisingularity theory are immediate consequences of properties of Hamburger-Noether expansions. We remark that when the characteristic of k is zero, HN-equisingularity agrees with the usual definitions of equisingularity (see [**2–5**]).

Finally, for a given equisingularity class, we construct a total HN-equisingular family π: $X \to Y$ with section ε: $Y \to X$, of type E, in such a way that the parameter space Y is an irreducible smooth affine algebraic variety over k, and such that for any closed point $y \in Y$ the induced algebroid family $\operatorname{Spec} \hat{\mathcal{O}}_{X,\varepsilon(y)} \to \operatorname{Spec} \hat{\mathcal{O}}_{Y,y}$ is versal HN-equisingular. (The irreducibility of Y in such a construction was an open question even in characteristic zero; see [**3**].)

1. Hamburger-Noether expansions over rings. In this section A will stand for a commutative ring with unit. The multiplicative group of units of A will be denoted by A^*. The following definition of Hamburger-Noether expansion over A generalizes that given for fields in [**1**] using the notation of 3.3.4.

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DEFINITION 1.1. A Hamburger-Noether expansion over A is a set of expressions

$$
\text{(D)}\qquad
\begin{aligned}
&y = a_{01}x + \cdots + a_{0h}x^h + x^h z_1\\
&x = z_1^{h_1} z_2\\
&\cdots\\
&z_{s_1-1} = a_{s_1k_1}z_{s_1}^{k_1} + \cdots + a_{s_1d_1}z_{s_1}^{d_1} + z_{s_1}^{d_1}z_{s_1+1}\\
&\cdots\\
&z_{s_g-1} = a_{s_gk_g}z_{s_g}^{k_g} + \cdots
\end{aligned}
$$

where $s_1 < s_2 < \cdots < s_g$; $h, h_j \geqslant 1$, $1 \leqslant j < s_g$, $j \neq s_\nu$; $2 \leqslant k_\nu \leqslant d_\nu$, $1 \leqslant \nu \leqslant g$; $a_{ji} \in A$ and $a_{s_\nu k_\nu} \in A^*$. (Note that $d_\nu = h_{s_\nu}$ in the terminology of [**1**].)

REMARK 1.2. (D) provides parametric equations $x = x(t)$; $y = y(t)$, with parameter $t = z_{s_g}$ where $x(t)$, $y(t) \in k[[t]]$. These parametric equations define a ring homomorphism p: $A[[X, Y]] \to A[[t]]$, given by $p(a) = a$ if $a \in A$, $p(X) = x(t)$ and $p(Y) = y(t)$, which is continuous for the (X, Y) and (t)-topologies, respectively, on $A[[X, Y]]$ and $A[[t]]$.

THEOREM 1.3. *In the above situation*, $\ker(p)$ *is a principal ideal of* $A[[X, Y]]$ *generated by a polynomial* $f(X, Y) = Y^n + A_{n-1}(X)Y^{n-1} + \cdots + A_0(X)$, *where* $n = \operatorname{ord}_t x(t)$, $A_i(X) \in A[[X]]$ *and* $\operatorname{ord}_X A_i(X) \geqslant n - i$. *The polynomial* $f(X, Y)$ *is uniquely determined by these conditions.*

PROOF. We will use the Weierstrass division theorem (WDT) for power series over rings in the usual form (keep the proof in Zariski-Samuel [**7**, vol. 2, p. 140]): If $g(X, Y) \in A[[X, Y]]$ verifies $g(0, Y) = aY^m + a'Y^{m+1} + \cdots$ with $a \in A^*$, then for any $h(X, Y) \in A[[X, Y]]$ there exist $q(X, Y)$, $r(X, Y) \in A[[X, Y]]$, r a polynomial in Y with coefficients in $A[[X]]$ of degree at most $m - 1$, such that

$$h(X, Y) = q(X, Y)g(X, Y) + r(X, Y),$$

and, furthermore, q and r are uniquely determined. In particular, for $h(X, Y) = Y^m$, one has the Weierstrass preparation theorem (WPT): if g is as above, there exist a unique unit $U(X, Y) \in A[[X, Y]]$ and a unique monic polynomial $P(X, Y) \in A[[X]][Y]$ of degree m such that $g(X, Y) = U(X, Y)P(X, Y)$.

Now we will prove the theorem. The uniqueness of f follows from the uniqueness in WPT. To construct f, we will first consider the case in which A is a domain and, therefore, a subring of an algebraically closed field K. The expansion D defines an algebroid curve over K (see [**1**]) and a Weierstrass polynomial $f(X, Y) \in K[[X]][Y]$ of degree $n = \operatorname{ord}_t x(t)$ defining this curve. Moreover, using induction on the length $M = h + h_1 + \cdots + h_{s_g-1}$ $(h_{s_\nu} = d_\nu)$ of D, one can prove that f actually has its coefficients in A. In fact, for $M = 0$, it is obvious because D takes the form $y = a_{01}x + a_{02}x^2 + \cdots$ and so $f(X, Y) = Y - a_{01}X - a_{02}X^2 - \cdots$. In the inductive step, making the quadratic transformation $Y = a_{01}X + XY'$, we obtain $f(X, Y) = X^n f'(X, Y')$, where f' is an equation for the quadratic transform of f. If $h > 1$, by the induction hypothesis, f' has its coefficients in A and hence $f(X, Y) \in A[[X, Y]]$.

If $h = 1$, the quadratic transform of f is defined by a monic polynomial $f_1(X, Y') \in K[[Y']][X]$ of degree $n_1 = \mathrm{ord}_t z_1(t)$ which, by the induction hypothesis, has its coefficients in A. As f' and f_1 define the same curve, we have $f_1(X, Y') = U(X, Y')f'(X, Y')$ for a certain unit $U(X, Y') \in K[[X, Y']]$. Moreover, taking into account the analysis of the Newton diagram of f made in [**1**, 3.4.5], one has $f_1(0, Y') = bY'^n +$ higher degree terms, where $b = \pm a_{s_1 k_1}$ or $b = \pm a_{s_1 k_1}^{-1}$. In any case $b \in A^*$, so by WPT f' has its coefficients in A and, hence, $f(X, Y) \in A[[X, Y]]$.

Now suppose that A is not a domain. Take a set of indeterminates $\{A_{ji}\}$ where (j, i) ranges over the set of indices verifying $j = s_\nu$ and $k_\nu \leqslant i \leqslant d_\nu$ for some $\nu = 0, 1, \ldots, g$ ($s_0 = 0$, $k_0 = 0$, $d_g = \infty$), and consider the ring $B = \mathbf{Z}[\{A_{ji}\}]_H$, $H = A_{s_1 k_1} \cdots A_{s_g k_g}$. Next, since B is a domain, the Hamburger-Noether expansion over B whose coefficients are A_{ji} (and parameters the same h_j, s_ν, k_ν and d_ν) determines a polynomial $F(X, Y) \in B[[X]][Y]$ as above. On the other hand, there is a ring homomorphism ϕ: $B \to A$ such that $\phi(A_{ji}) = a_{ji}$. Thus if ϕ_*: $B[[X, Y]] \to A[[X, Y]]$ denotes the extension of ϕ to power series rings, f can be taken to be $\phi_*(F)$.

Summing up, for any A and D we have constructed a monic polynomial

$$f(X, Y) = Y^n + A_{n-1}(X)Y^{n-1} + \cdots + A_0(X) \in A[[X, Y]]$$

with $n = \mathrm{ord}_t x(t)$, $\mathrm{ord}_X A_i(X) \geqslant n - i$ and $f \in \ker(p)$. To complete the proof of the theorem, we need only check $(f) = \ker(p)$. Take $g \in \ker(p)$. By WDT we have $g = qf + r$ where $q(X, Y)$, $r(X, Y) \in A[[X, Y]]$, $r = \sum_{i=0}^{n-1} C_i(X)Y$. We claim $r = 0$. In fact, using induction on M the claim is evident for $M = 0$. In the inductive step (assuming $a_{01} = 0$ without loss of generality), the transformation $Y = XY'$ leads to

$$g(X, XY') = q(X, XY')X^n f'(X, Y') + r'(X, Y'),$$

where $r'(X, Y') = \sum_{i=0}^{n-1} X^i C_i(X)Y'^i$. If $h > 1$ the induction hypothesis applies directly, and one concludes $r = 0$. If $h = 1$, since $r'(x(t), y'(t)) = 0$, by the induction hypothesis one has

$$r'(X, Y') = q_1(X, Y')f_1(X, Y') = q_1(X, Y')U(X, Y')f'(X, Y').$$

Looking at the degrees of r' and f' as polynomials in Y' and using the uniqueness in WDT, one has $r' \equiv 0$ and, hence, $r \equiv 0$.

If f is as in the above theorem, f will be said to be the polynomial associated with D. Since the induced homomorphism $\bar{p}$: $\mathcal{O} = A[[X, Y]]/(f) \to A[[t]]$ is injective, $\mathcal{O}$ can be identified to the subring $A[[x(t), y(t)]]$ of $A[[t]]$. This suggests the study of the semigroup of values, conductor, etc. in the same way as for branches over fields.

Thus, following [**1**], we can associate to D the set of integers $\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_g$ defined by $\bar{\beta}_0 = \mathrm{ord}(x)$, $\bar{\beta}_1 = h\,\mathrm{ord}(x) + \mathrm{ord}(z_1)$ and, for $\nu \geqslant 1$,

$$\bar{\beta}_{\nu+1} = N_\nu \bar{\beta}_\nu + (d_\nu - k_\nu)\mathrm{ord}(z_{s_\nu}) + \mathrm{ord}(z_{s_\nu+1}),$$

where $N_\nu = \mathrm{ord}(z_{s_\nu}) \cdot \mathrm{ord}(z_{s_{\nu-1}})^{-1}$. Note that, in fact, the set $\{\bar{\beta}_0, \ldots, \bar{\beta}_g\}$ depends only on the parameters h_j, s_ν, k_ν, and d_ν of D. For each ν, $1 \leqslant \nu \leqslant g$, let $f_\nu(X, Y)$ be the polynomial associated with the Hamburger-Noether expansion

$$\begin{aligned}&y = a_{01}x + \cdots + a_{0h}x^h + x^h z_1\\&x = z_1^{h_1} z_2\\&\cdots\\&z_{s_{\nu-1}-1} = a_{s_{\nu-1}k_{\nu-1}} z_{s_{\nu-1}}^{k_{\nu-1}} + \cdots + a_{s_{\nu-1}d_{\nu-1}} z_{s_{\nu-1}}^{d_{\nu-1}}.\end{aligned}$$

Finally, define the conductor of $A[[t]]$ in $\mathcal{O}$ to be the set

$$\mathcal{C} = \{z \in A[[t]] \mid zA[[t]] \subseteq \mathcal{O}\}.$$

PROPOSITION 1.4. *Keeping the above notation, one has:*

(1) $f_\nu(x(t), y(t)) = c_\nu t^{\beta_\nu}$ + *higher degree terms, with* $c_\nu \in A^*$.

(2) *The set* $\{f_1(x, y)^{i_1} \cdots f_g(x, y)^{i_g}\}$, *where* $0 \leq i_\nu < N_\nu$, *is a basis of the free* $A[[x]]$*-module* $\mathcal{O}$.

(3) *The semigroup of values* $S = \{\operatorname{ord}_t g(x(t), y(t)) \mid g(X, Y) \in A[[X, Y]]\}$ *of* $\mathcal{O}$ *is minimally generated by* $\bar{\beta}_0, \ldots, \bar{\beta}_g$.

(4) $i \in S$ *if and only if there exists* $z_i \in \mathcal{O}$ *such that* $z_i = t^i$ + *higher terms.*

(5) $\mathcal{C} = (t^c)A[[t]]$, *where* c *is the "conductor" of* S, *i.e., the minimum integer* c *in* S *with the property* $j \in \mathbf{Z}, j \geq c \Rightarrow j \in S$.

PROOF. The power series $x(t), z_1(t), \ldots, z_{s_g}(t)$ have units as their leading coefficients. Since f and f_ν are associated with Hamburger-Noether expansions with coefficients in common in their $s_{\nu-1}$ first rows, taking into account the proof of 1.3, we have

$$f_\nu(x(t), y(t)) = M(t)U(t)$$

where $M(t)$ is a monomial in $x(t), z_1(t), \ldots, z_{s_{\nu-1}+1}(t)$ and $U(t)$ is a unit in $A[[t]]$. (1) is now trivial from the formulae for $\bar{\beta}_\nu$ in [**1**, 4.2]. (2) works as in the case $A = k$ (3) follows from (2), and (4) follows from (1) and (3). Finally (5) is also obtained in the usual way: $\mathcal{C} \subseteq (t^c)A[[t]]$ is trivial and, conversely, if $z \in (t^c)A[[t]]$, then using (4) and the fact that $\mathcal{O}$ is complete for the (t)-topology induced from $A[[t]]$, one has $zw \in \mathcal{O}$ for any $w \in A[[t]]$, so $z \in \mathcal{C}$.

2. Sets of Hamburger-Noether expansions. We will say that a set of Hamburger-Noether expansions $D^1, \ldots, D^m$ over A is regular if they are pairwise different and if for two indices q, r one has the following property: If D^q and D^r have exactly the s first rows in common and if $a^q_{si} \neq a^r_{si}$ for some i, then for the least integer with this property one has $a^q_{si} - a^r_{si} \in A^*$.

We will consider only regular sets of Hamburger-Noether expansions. Suppose that such a set is given. For each q take a set of indeterminates $\{A^q_{ji}\}$ over $\mathbf{Z}$ as in the proof of 1.3. For $q \neq r$, if s is as above, let e be the greatest integer such that $a^q_{si} = a^r_{si}$ for all $i < e$. Identify the variables A^q_{ji} and A^r_{ji} for those couples (j, i) such that either $j < s$ or $j = s$ and $i < e$, and set $B_{qr} = A^q_{se} - A^r_{se}$ if both A^q_{se} and A^r_{se} are defined, and $B_{qr} = 1$ otherwise. Denote by T the whole set of indeterminates A^q_{ji} considered after identifications. Set $\mathbf{B} = \mathbf{Z}[T]_H$, where $H = \prod_q H_q \cdot \prod_{q<r} B_{q,r}$ and H_q is, as in §1, the product of the elements $A^q_{s_\nu k_\nu}$ corresponding to the qth expansion.

Let $\mathbf{D}^d$ be the Hamburger-Noether expansion over $\mathbf{B}$ whose coefficients are A_{ji}^q. By construction the set $\mathbf{D}^1,\ldots,\mathbf{D}^m$ is regular, and from it one can obtain $D^1,\ldots,D^m$ by taking images of the coefficients by the ring homomorphism $\Phi\colon \mathbf{B} \to A$ given by $\Phi(A_{ji}^q) = a_{ji}^q$.

Now denote by f_q the polynomial associated with D^q, and by $x_q = x_q(t_q)$, $y_q = y_q(t_q)$ the corresponding parametric equations. For $q \neq r$ we claim that

$$(*) \qquad f_q(x_r(t_r), y_r(t_r)) = bt_r^{m_{qr}} + \text{higher terms}$$

where $b \in A^*$. In fact, replacing D^q, D^r by $\mathbf{D}^q$, $\mathbf{D}^r$, we can assume A is a domain. For any $p \in \operatorname{Spec}(A)$ consider the algebroid branches C_p^q, C_p^r over $\overline{K(p)}$ (= algebraic closure of the residual field $k(p)$) defined by the Hamburger-Noether expansions obtained from D^q, D^r by reducing coefficients modulo p. From the regularity condition and the computation of the intersection multiplicity in terms of Hamburger-Noether expansions [**1**, 2.3.3] the number (C_p^q, C_p^r) does not depend on p. Since, on the other hand, (C_p^q, C_p^r) is the order of the power series obtained from $f_q(x_r(t_r), y_r(t_r))$ by reducing coefficients modulo p, the claim follows immediately.

Conversely, if A is a reduced ring and $D^1,\ldots,D^m$ is a set of Hamburger-Noether expansions over A such that $(*)$ holds for any two indices $q \neq r$, then one can prove that the set is regular. In fact, assume otherwise that there are indices q, r, s, e with $a_{ji}^q = a_{ji}^r$ if either $j < s$ or $j = s$, $i < e$ and such that $c = a_{se}^q - a_{se}^r$ is simultaneously nonzero and a nonunit. Since $c \notin A^*$, there exists $p_1 \in \operatorname{Spec} A$ with $c \in p_1$, and, on the other hand, since $c \neq 0$ and A is reduced, there exists $p_2 \in \operatorname{Spec} A$ such that $c \notin p_2$. Using [**1**, 2.3.3] again, it follows that $(C_{p_2}^q, C_{p_2}^r) < (C_{p_1}^q, C_{p_1}^r)$. This is a contradiction, since the condition $b \in A^*$ in $(*)$ implies that the number (C_p^q, C_p^r) (= order of the reduction modulo p of $f_q(x_r(t_r), y_r(t_r))$) does not depend on $p \in \operatorname{Spec} A$. Thus the set is regular.

For a regular set the intersection multiplicity (D^q, D^r) can be defined to be the number m_{qr} in $(*)$. The equality $(D^q, D^r) = (D^r, D^q)$ follows from $(*)$ by reducing modulo a maximal ideal and using the corresponding result for the case where A is a field. Moreover, the Kernel of the parametrization homomorphism

$$p\colon A[[X, Y]] \to \prod_q A[[t_q]] = A[[\underline{t}]]$$

defined by $D^1,\ldots,D^m$ is the principal ideal (f), where $f = f_1 \cdots f_m$. In fact, if $g \in \operatorname{Ker}(p)$ then by 1.3 one has $g = g_1 \cdot f_1$. Since $g(x_2(t_2), y_2(t_2)) = 0$, it follows from $(*)$ that $g_1(x_2(t_2), y_2(t_2)) = 0$ and, hence, $g_1 = g_2 \cdot f_2$. Using induction one shows $g = g_m \cdot f_1 \cdots f_m \in (f)$. It follows that the induced homomorphism

$$\bar{p}\colon \mathcal{O} = A[[X, Y]]/(f) \to A[[\underline{t}]]$$

is injective and $\mathcal{O}$ can be identified to the subring $A[[x(\underline{t}), y(\underline{t})]]$ of $A[[\underline{t}]]$. As in §1, f will be called the polynomial associated with the set $D^1,\ldots,D^m$. The conductor $\mathcal{C} = \{z \in A[[\underline{t}]] \mid zA[[\underline{t}]] \subseteq \mathcal{O}\}$ is computed in the following

LEMMA 2.1. *$\mathcal{C} = (t_1^{\delta_1})A[[t_1]] \times \cdots \times (t_m^{\delta_m})A[[t_m]]$, where $\delta_q = c_q + \sum_{q \neq r}(D^q, D^r)$ and c_q is the conductor of the semigroup defined by D^q.*

PROOF. First we will prove $(0,\ldots,0,t_q^{\delta_q},0,\ldots,0)\in\mathcal{C}$ for each q. Take, for instance, $q=1$. We must see $(w_1t_1^{\delta_1},0,\ldots,0)\in\mathcal{C}$ for any $w_1\in A[[t_1]]$. From $(*)$, the equation

$$z'f_2(x_1(t_1),y_1(t_1))\cdots f_m(x_1(t_1),y_1(t_1))=w_1t_1^{\delta_1}$$

has a solution $z'\in A[[t_1]]$ which verifies $\operatorname{ord}_{t_1}z'\geqslant\delta_1-\Sigma_{r\neq1}(D^1,D^r)=c_1$, so $z'=g(x_1(t_1),y_1(t_1))$ for some power series $g(X,Y)\in A[[X,Y]]$. Now it is evident that

$$\left(t_1^{\delta_1}w_1,0,\ldots,0\right)=g(x,y)f_2(x,y)\cdots f_m(x,y)\in\mathcal{C}.$$

Thus $(t_1^{\delta_1})A[[t_1]]\times\cdots\times(t_m^{\delta_m})A[[t_m]]\subseteq\mathcal{C}$. Conversely, if $(z_1,\ldots,z_m)\in\mathcal{C}$ then $(z_1,0,\ldots,0)\in\mathcal{C}$ and, hence, $(z_1w_1,0,\ldots,0)\in\mathcal{C}$ for all $w_1\in A[[t_1]]$ (and the same for the other indices). It follows that $z_1w_1=h_{w_1}(x_1(t_1),y_1(t_1))$ and $h_{w_1}(x_r(t_r),y_r(t_r))=0$, $r\neq1$, for some power series h_{w_1}; therefore $h_{w_1}(X,Y)=g_{w_1}(X,Y)f_2(X,Y)\cdots f_m(X,Y)$. When w_1 ranges over $A[[t_1]]$, $\operatorname{ord} g_{w_1}(x_1(t_1),y_1(t_1))$ ranges over the set of integers $\operatorname{ord}(z_1)-\Sigma_{r\neq1}(D^1,D^r)+\operatorname{ord}(w_1)$, so $\operatorname{ord}(z_1)\geqslant c_1+\Sigma_{r\neq1}(D^1,D^r)=\delta_1$.

We will consider, in the sequel, an equisingularity class E of reduced plane algebroid curves. For an equisingularity class E we mean (according to the results in **[1]**) the class of all reduced plane algebroid curves over algebraically closed fields having a given number, m, of branches which can be ordered in such a way that they have given semigroups of values and given intersection multiplicities. In fact, by extension, we can consider in E all the regular sets of Hamburger-Noether expansions over arbitrary rings corresponding to the given semigroups and intersection multiplicities.

PROPOSITION 2.2. (a) *Assume the coefficients a_{ji} in a Hamburger-Noether expansion lexicographically ordered, i.e., $a_{ji}\leqslant a_{j'i'}$ iff either $j<j'$ or $j=j'$ and $i\leqslant i'$ (coefficients of type $a_{s_\nu i}$ with $i<k_\nu$ are also considered). Then for the equisingularity class E there exist integers $N_1,\ldots,N_m$ depending only on E with the following property: Let A be a ring and $f,f'\in A[[X]][Y]$ two polynomials associated respectively with two regular sets $D^1,\ldots,D^m$, $D'^1,\ldots,D'^m$ of Hamburger-Noether expansions in E over A such that D^q and D'^q agree to their N_qth coefficient. Then $\mathcal{O}=A[[X,Y]]/(f)$ and $\mathcal{O}'=A[[X,Y]]/(f')$ are isomorphic A-algebras.*

(b) *There exists an integer $M>0$ depending only on E with the following property: Let A be a ring and $f,g\in A[[X,Y]]$ such that* $\operatorname{ord}(f-g)\geqslant M$ *and f is a generator of the Kernel of the parametrization homomorphism defined by a regular set of Hamburger-Noether expansions over A in E. Then g is also a generator of the Kernel of the parametrization homomorphism defined by a regular set of Hamburger-Noether expansions over A in E, and $A[[X,Y]]/(f)\simeq A[[X,Y]]/(g)$ as A-algebras.*

PROOF. Since E determines the number of coefficients in the rows of the expansions of regular sets in E, we can take N_q large enough in such a way that, for any such regular set and for the lexicographic order on the coefficients of the qth expansion, at least δ_q coefficients in the bottom row are included among the first N_q

ones. Now, for f, f' as in (a), identify $\mathcal{O}$ (resp. $\mathcal{O}'$) with the subring $A[[\underline{x}, \underline{y}]]$ (resp. $A[[\underline{x}', \underline{y}']]$) of $A[[\underline{t}]]$, where $\underline{x} = (x_1(t_1), \ldots, x_m(t_m))$, $\underline{y} = (y_1(t_1), \ldots, y_m(t_m))$ (resp. $\underline{x}' = (x_1'(t_1), \ldots, x_m'(t_m))$, $\underline{y}' = (y_1'(t_1), \ldots, y_m'(t_m)))$ are the parametric equations defined by $D^1, \ldots, D^m$ (resp. $D'^1, \ldots, D'^m$). By the choice of N_q one has

$$\operatorname{ord}\big(x_q(t_q) - x_q'(t_q)\big) \geqslant \delta_q \quad \text{and} \quad \operatorname{ord}\big(y_q(t_q) - y_q'(t_q)\big) \geqslant \delta_q,$$

so by 2.1, $\underline{x}' - \underline{x}, \underline{y}' - \underline{y} \in A[[\underline{x}, \underline{y}]] \cap A[[\underline{x}', \underline{y}']]$, hence $\mathcal{O} \simeq \mathcal{O}'$.

(b) Since the polynomial h associated with a regular set of Hamburger-Noether expansions is not a zero divisor ($h(0, Y) = Y^n$), any generator of the parametrization homomorphism is of type $u \cdot h$ where u is a unit in $A[[X, Y]]$. Thus, we can assume that f in (b) is itself the polynomial associated with a regular set $D^1, \ldots, D^m$ of Hamburger-Noether expansions. Moreover, A can be assumed to be a domain. In fact, the general case can be reduced to this by using an onto ring homomorphism ψ: $B' \to A$, B' a domain, and a regular set of Hamburger-Noether expansions over B' which is transformed by ψ in $D^1, \ldots, D^m$ (B' can be realized as a polynomial ring over the ring $\mathbf{B}$ of the beginning of this section with as many indeterminates as elements in A).

Take an algebraically closed field K containing A and consider f as defining a reduced (embedded) algebroid curve C over K. According to [**1**, 2.2.9 to 2.2.12], the Hamburger-Noether expansion D^q determines an infinite sequence of formal quadratic transformations (see [**1**, 1.5.13]). Each of these formal quadratic transformations is one of type $X = X'$, $Y = \alpha X' + X'Y'$, where from the hypothesis on D^q one has $\alpha \in A$. For any power series $h(X, Y) \in K[[X, Y]]$ define the strict transform of h to be the power series $h'(X', Y') = X'^{-s}h(X', \alpha X' + X'Y')$, where $s = \operatorname{ord}(h)$. Note that one has the following trivial facts:

(1) If $h \in A[[X, Y]]$ then $h' \in A[[X', Y']]$.

(2) If $\operatorname{ord}(h_1 - h_2) \geqslant H > s = \operatorname{ord}(h_1)$, then $\operatorname{ord}(h_1' - h_2') \geqslant H - s$.

For a sequence of a finite number of formal quadratic transformations the strict transform is defined as the successive strict transform.

Now, take numbers N_q verifying the requirements in (a) and, moreover, each N_q large enough in such a way that for all q, the curve C is nonsingular at the N_qth infinitely near point of its qth branch. For fixed q, consider the sequence consisting of the N_q first quadratic transformations determined by D^q. This sequence takes the original indeterminates X, Y into new ones X^*, Y^* where the notation X^*, Y^* is chosen in such a way that the $(N_q + 1)$th transformation is one of type $X^* = X^{*\prime}$, $Y^* = \alpha X^{*\prime} + X^{*\prime}Y^{*\prime}$. Since after the sequence of N_q transformations the algebroid curve C becomes nonsingular at the N_qth infinitely near point of its qth branch, the strict transform of f by the sequence is a power series $f^*(X^*, Y^*)$ in $A[[X^*, Y^*]]$ (fact (1) above) of order 1. Moreover, since $X^* = 0$ is transversal to the curve defined by f^* (look at the form of the $(N_q + 1)$th transformation described above), one has $f^*(0, Y^*) = cY^* +$ higher order terms, with $c \in A$, $c \neq 0$. We claim that $c \in A^*$. In fact, for any $p \in \operatorname{Spec} A$ the objects D_p^q, f_p obtained by reducing coefficients modulo p, are exactly in the same situation as the original ones D^q, f. By the choice made for N_q and because of the condition $(*)$, for any p, $\operatorname{Spec}(\overline{k(p)}[[X, Y]]/(f_p))$ is also nonsingular at the N_qth infinitely near point of the branch whose Hamburger-Noether expansion is D_p^q. This means that the power series $(f_p)^* = (f^*)_p$ is of order 1 in $\overline{k(p)}[[X^*, Y^*]]$ and it has a nonzero element as coefficient of Y^*. This element is nothing but the class of c modulo p, so $c \not\equiv 0 \pmod p$ for all $p \in \operatorname{Spec} A$ and, hence, $c \in A^*$, which proves the claim.

We will derive (b) from the above claim. Take $g \in A[[X, Y]]$ such that $\operatorname{ord}(f-g) \geqslant M = 2 + \Sigma_q M_q$, where M_q is the sum of the multiplicities of C at the N_q first infinitely near points of its qth branch. (Note that M only depends on E.) By fact (2), if g^* is the strict transform of g by the sequence, one has $\operatorname{ord}(f^* - g^*) \geqslant 2$, so $g^*(0,0) = f^*(0,0) = 0$ and $g^*(0, Y^*) = cY^* +$ higher order terms. Applying WPT to g^*, one finds a unit v in $A[[X^*, Y^*]]$ and elements a_i in A such that

$$g^*(X^*, Y^*) = \left(Y^* - a_1 X^* - \cdots - a_i X^{*i} - \cdots\right) v(X^*, Y^*).$$

Consider the Hamburger-Noether expansion D'^q with the same parameters h_j, s_ν, k_ν, d_ν as D^q whose N_q first coefficients are those of D^q and the rest are the elements a_i. By the choice of N_q, the set $D'^1, \ldots, D'^m$ is regular over A in E (use [**1**, 2.3.3]), and from (a) its associated polynomial $h(X, Y)$ verifies $A[[X, Y]]/(f) \simeq A[[X, Y]]/(h)$, so in particular $\operatorname{ord}(f) = \operatorname{ord}(h)$ (denote it by n). To complete the proof of (b) we need only check that the ideals (f) and (g) are the same. First, $g \in (h)$ since g is in the Kernel of the parametrization homomorphism of D'^q for all q (this follows from the fact that g is in the Kernel of the parametrization homomorphism of the Hamburger-Noether expansion given by the only row $y^* = a_1 x^* + \cdots + a_i x^{*i} + \cdots$), so $g = u \cdot h$, $u \in A[[X, Y]]$. From 1.3 and the assumption $\operatorname{ord}(f-g) \geqslant M > n$, one has $\operatorname{ord}(g) = \operatorname{ord}(f) = \operatorname{ord}(h) = n$, $g(0, Y) = Y^n +$ higher order terms, and $h(0, Y) = Y^n$, so $u(0, Y)$ is a unit in $A[[Y]]$ and, hence, $u(X, Y)$ is a unit in $A[[X, Y]]$.

3. Universal equations of equisingularity classes. Let E be an equisingularity class of reduced plane curve singularities given by semigroups $S^1, \ldots, S^m$ and intersection multiplicities m_{qr}, $1 \leqslant q, r \leqslant m$. Take a set of Hamburger-Noether expansions $D^1, \ldots, D^m$ in E and, from it, construct the ring $\mathbf{B}$ and the expansions $\mathbf{D}^1, \ldots, \mathbf{D}^m$ as in §2. $\mathbf{B}$ and $\mathbf{D}^1, \ldots, \mathbf{D}^m$ depend only on E and not on the choice of the set $D^1, \ldots, D^m$. In fact, since $S(D^q) = S^q$, the parameters h_j, s_ν, k_ν and d_ν in D^q are determined by S^q, so the set of indeterminates $\{A_{ji}^q\}$ is also determined by S^q. On the other hand $\mathbf{D}^q$ and $\mathbf{D}^r$ verify $(\mathbf{D}^q, \mathbf{D}^r) = m_{qr}$, $S(\mathbf{D}^q) = S^q$ and $S(\mathbf{D}^r) = S^r$, so taking into account the computation of the intersection multiplicity in [**1**, 2.3.3], the pairs A_{ji}^q, A_{ji}^r of identified variables are totally determined by m_{qr} (note that the multiplicity sequences for the branches whose Hamburger-Noether expansions are $\mathbf{D}^q, \mathbf{D}^r$ are determined respectively by S^q and S^r). Thus, $\mathbf{B}$ and $\mathbf{D}^1, \ldots, \mathbf{D}^m$ depend only on the data S^q, m_{qr} and, consequently, we will use the notation B_E for $\mathbf{B}$.

Definition 3.1. The polynomial $F_E(X, Y) \in B_E[[X]][Y]$ associated with the above set $\mathbf{D}^1, \ldots, \mathbf{D}^m$ will be called the Hamburger-Noether universal equation of the class E, and the affine scheme $\operatorname{Spec}(\mathcal{O}_E)$, $\mathcal{O}_E = B_E[[X, Y]]/(F_E)$, the Hamburger-Noether scheme of E.

The following theorem is evident.

THEOREM 3.2. *For any algebraically closed field k and for any algebroid curve in E over k, an equation for it is obtained as the image of $F_E(X, Y)$ by the extension of power series rings of a certain ring homomorphism ϕ: $B_E \to k$. Moreover, if $\mathcal{O}$ is the local ring of such a curve, one has $\mathcal{O} \simeq \mathcal{O}_E \hat{\otimes}_{B_E} k$, where k is considered as an B_E-algebra by means of ϕ and the completion is taken with respect to the ideal $(x \otimes 1, y \otimes 1)$, $x = X + (F_E)$, $y = Y + (F_E)$.*

REMARKS 3.3. (1) ϕ is determined by an embedding $\operatorname{Spec} \mathcal{O} \to \operatorname{Spec} k[[X, Y]]$ for which $X = 0$ is transversal, and by an ordering $(C_1, \ldots, C_m)$ over the branches for which $S(C_q) = S^q$ and $(C_q, C_r) = m_{qr}$. Note that, in general, such an ordering is not unique.

(2) The polynomial $f(X, Y) \in A[[X]][Y]$ associated with a regular set of Hamburger-Noether expansions over A in E is obtained in the same way from $F_E(X, Y)$ by the extension to power series rings of a ring homomorphism $B_E \to A$ and, conversely, any image of F_E in this way is coming from a certain regular set of Hamburger-Noether expansions.

(3) If we remove coefficients in $\mathbf{D}^q$ keeping only the first N_q ones (Proposition 2.2 (a)), the ring $\mathbf{B}_0$, constructed with the new variables in the same way as $\mathbf{B}_E$, is noetherian and the corresponding F_0 and $\mathcal{O}_0 = B_0[[X, Y]]/(F_0)$ have the properties of 3.2, although obviously Remark 3.3(1) does not hold.

4. Equisingular deformations. Let $f_0 \in k[[X, Y]]$ be a power series defining a reduced algebroid curve over an algebraically closed field k, and A a complete noetherian local k-algebra having k as a coefficient field. Denote by m_A the maximal ideal of A, and by res: $A \to A/m_A$ the residual map.

A deformation of f_0 over A is a power series $f(X, Y) \in A[[X, Y]]$ such that $f_0 = \operatorname{res}_*(f)$, where res_* denotes the extension of res to power series rings. Two deformations f and f' of f_0 over A are said to be equivalent (or "similar" according to [2]) if there exist power series $u, g, h \in A[[X, Y]]$ with $g - X, h - Y \in m_A[[X, Y]]$, u a unit, such that $u(X, Y)f'(g(X, Y), h(X, Y)) = f(X, Y)$.

For the rest of the section we will assume that the y-axis is transversal to $\operatorname{Spec} k[[X, Y]]/(f_0)$, i.e., $\operatorname{ord} f_0(X, Y) = \operatorname{ord} f_0(0, Y)$. This is not a loss of generality, because after a linear change of variables we can reach this situation.

DEFINITION 4.1. A deformation of f_0 over A will be said to be HN-equisingular if it is similar to one obtained as the image of the Hamburger-Noether universal equation $F_E(X, Y)$ of the equisingularity class E of f_0 by the extension to power series rings of a ring homomorphism $B_E \to A$.

REMARKS 4.2. (1) If f is HN-equisingular, then modulo similarity, f can be assumed to be the polynomial associated with a regular set of Hamburger-Noether expansions over A in the class E. It follows, in particular, that the deformation has a section (i.e., an A-homomorphism s: $A[[X, Y]]/(f) \to A$) which, according to the above assumption, is the trivial one (i.e., $s(X + (f)) = s(Y + (f)) = 0$). This assumption will be kept in the sequel.

(2) When k has characteristic zero and A is a regular ring, HN-equisingularity agrees with equisingularity in the Zariski sense (see [5]), i.e., f is HN-equisingular iff f and f_0 define equisingular curves over the algebraic closure of the quotient field of A.

When k has characteristic zero, HN-equisingularity agrees with P-equisingularity introduced by Nobile in [**2** and **3**]. The deformation f over the noetherian local k-algebra A (with trivial section) is called P-equisingular if $f = f_1 \cdots f_m$, and each f_q admits a parametrization of Puiseux type over A,

$$(P^q) \qquad x = t_q^{n_q}, \quad y = \sum_j b_j^q t_q^j, \qquad b_j^q \in A,\ q = 1, 2, \ldots, m,$$

in such a way that if $\mathrm{res}_* P^q$ denotes the parametrization over k obtained from P^q taking residues modulo m_A, then P^q and $\mathrm{res}_* P^q$ have the same characteristic exponents and $(P^q, P^r) = (\mathrm{res}_* P^q, \mathrm{res}_* P^r)$, $q \neq r$ (taking the intersection multiplicities in the natural sense; see [**3**]). The proof of the coincidence for both definitions is based on the following points: (1) A can be assumed to be regular, because the lifting property (see 4.3(b) below) holds for both definitions. (2) If $m = 1$ and f is HN-equisingular, then, as the leading coefficient of $x(t)$ is a unit, f has a Puiseux parametrization P. Since f and $\mathrm{res}_* f$ are equisingular algebroid branches, the characteristic exponents of P and $\mathrm{res}_* P$ are the same. Conversely, if f is P-equisingular, then from P, using induction and the Abhyankar inversion formula [**2**, 2.2], one constructs a Hamburger-Noether expansion for f. (3) If $m > 1$ and f is HN-equisingular, then f is P-equisingular follows from $(*)$ in §2. Conversely, if f is P-equisingular one has $(*)$ and, hence, since A is regular and therefore reduced, one has the regularity condition.

Finally, as P-equisingularity agrees with Wahl's definition of equisingularity along the trivial section [**4**], one concludes that for characteristic zero all the definitions are essentially the same. In particular, the functor

$$A \to \{\text{HN-equisingular deformations of } f \text{ over } A\}/\text{similarity}$$

over the category of local artin k-algebras is the same as that of the Wahl equisingualr deformations functor.

PROPOSITION 4.3. (a) *If f is HN-equisingular then* $A[[X, Y]]/(f)$ *is a flat A-algebra.*

(b) *Let ψ: $A' \to A$ be a surjective morphism of complete local noetherian k-algebras and f a HN-equisingular deformation over A. Then f has a HN-equisingular lift to A'.*

(c) *Let f be a HN-equisingular deformation of f_0 over A and $g \in A[[X, Y]]$ a power series such that* $\mathrm{ord}(f - g) \geqslant M_E$ *(E the equisingularity class of E, and M_E the integer of* 2.2(b)). *Then g is a HN-equisingular deformation of* $g_0 = \mathrm{res}_* g$.

(d) *For any f_0, there exists a versal HN-equisingular deformation with regular basis, i.e., a HN-equisingular deformation $f \in A[[X, Y]]$ where A is regular, such that for any other HN-equisingular deformation $g \in B[[X, Y]]$ of f_0, there exists a k-algebra homomorphism ϕ: $A \to B$ such that g is similar to $\phi_* f$.*

PROOF. (a), (b) and (c) are trivial. To see (d), let E be the equisingularity class of f_0, $D_0^1, \ldots, D_0^m$ the expansions corresponding to the branches of f_0, and $N_1, \ldots, N_m$ the integers of 2.2(a). In D_0^q replace the first N_q coefficients a_{ji}^q by $a_{ji}^q + z_{ji}^q$ (z_{ji}^q being indeterminates) and keep the rest fixed. Now, as in §3, for each couple of indices q, r identify those indeterminates z_{ji}^q and z_{ji}^r which must be necessarily equal in order that the intersection multiplicity be the number m_{qr} given by E. We obtain a regular set of Hamburger-Noether expansions over the regular ring $A = k[[\{z_{ji}^q\}]]$ which gives rise to a HN-equisingular deformation f of f_0 over A. By 2.2 it is evident that f is versal.

5. Total equisingular familes. Let k be an algebraically closed field. A family of plane curve singularities is a system (π, X, Y, ε) where $\pi\colon X \to Y$ is a flat morphism of algebraic schemes over k, $\varepsilon\colon Y \to X$ a section of π, and for any geometric point y of Y the fiber X_y is a reduced plane curve. The family is said to be HN-equisingular if for every closed point $y \in Y$ the induced algebroid family

$$\pi_y\colon \operatorname{Spec}\left(\hat{\mathcal{O}}_{X,\varepsilon(y)}\right) \to \operatorname{Spec}\left(\hat{\mathcal{O}}_{Y,y}\right)$$

is isomorphic to one of type

$$p\colon \operatorname{Spec}(A[[X,Y]]/(f)) \to \operatorname{Spec} A,$$

where $A = \hat{\mathcal{O}}_{Y,y}$ and f is a HN-equisingular deformation of f_0 along the trivial section s_0 of p in such a way that ε corresponds to s_0. The HN-equisingular family has type E if for all closed points $y \in Y$, $\operatorname{Spec}(\hat{\mathcal{O}}_{X_y,\varepsilon(y)})$ has type E. Finally, the HN-equisingular family of type E is total if for any algebroid curve f_0 in the class E there exists a closed point $y \in Y$ such that $k[[X,Y]]/(f_0)$ is isomorphic to $\hat{\mathcal{O}}_{X_y,\varepsilon(y)}$.

Total equisingular families of type E arise in a natural way in the study of the moduli problem of plane curve singularities, i.e., the problem of classifying all the plane curve singularities in E modulo formal isomorphism. As an application of the above sections, we will show such families can be generated from Hamburger-Noether expansions.

Let E be a fixed equisingularity class, and $N_1,\ldots,N_m$ integers as in 2.2(a). Take the ring $\mathbf{B}_0 = \mathbf{Z}[\{A_{ji}^q\}]_H$ where the indeterminates A_{ji}^q and H are chosen as in §3, except that now for each q we only consider the N_q first A_{ji}^q's. Denote by $F_0(X,Y) \in \mathbf{B}_0[[X]][Y]$ the polynomial associated with the Hamburger-Noether expansions whose coefficients are actually A_{ji}^q. Now take a polynomial $F'(X,Y) \in \mathbf{B}_0[X,Y]$ such that $\operatorname{ord}(F_0 - F') \geqslant M$ (M the integer of 2.2(b)), and set $X = \operatorname{Spec}(\mathbf{B}_0[X,Y]/(F'))$, $Y = \operatorname{Spec} B_0$ and $\pi\colon X \to Y$, $\varepsilon\colon Y \to X$ the morphisms associated, respectively, with the natural injection $\mathbf{B}_0 \to \mathbf{B}_0[X,Y]/(F')$ and the surjection $\mathbf{B}_0[X,Y]/(F') \to \mathbf{B}_0$ of ideal (X,Y). For any algebraically closed field k, the base change $\mathbf{Z} \to k$ induces a family of plane curve singularities $(\pi^k, X^k, Y^k, \varepsilon^k)$ over k.

THEOREM 5.1. *$(\pi^k, X^k, Y^k, \varepsilon^k)$ is a total HN-equisingular family of type E. Moreover, Y^k is a smooth irreducible affine algebraic variety over k and for any closed point $y \in Y^k$ the induced family π_y: $\operatorname{Spec}(\hat{\mathcal{O}}_{X^k,\varepsilon^k(y)}) \to \operatorname{Spec}(\hat{\mathcal{O}}_{Y^k,y})$ is a versal HN-equisingular deformation.*

PROOF. It is trivial from the results in the above sections.

REMARK 5.2. Nobile [3] constructed such a family in the case of characteristic zero, but he did not prove that the parameter space can be taken to be irreducible. On the other hand, note that the family of 6.1 works for characteristic $p > 0$ and, furthermore, it is independent of the field since it proceeds from a scheme theoretical family.

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