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ON FIRST COUNTABLE, COUNTABLY COMPACT SPACES. I: (ω_1, ω_1^*)-GAPS

BY

PETER J. NYIKOS[1] AND JERRY E. VAUGHAN

ABSTRACT. This paper is concerned with the (ω_1, ω_1^*)-gaps of F. Hausdorff and the topological spaces defined from them by Eric van Douwen. We construct special gaps in order that the associated gap spaces will have interesting topological properties. For example, the gap spaces we construct show that in certain models of set theory, there exist countably compact, first countable, separable, nonnormal T_2-spaces.

1. Introduction. For short, we say that a countably compact, first countable, separable, nonnormal T_2-space is an *R-space*. It is known that there exist (in ZFC) countably compact, first countable, nonnormal T_2-spaces [**N**, $\mathbf{V_1}$], but these examples are not separable (hence not R-spaces) in a strong way: They are ω-bounded (i.e., every countable subset has compact closure). It is also known that under certain set-theoretic assumptions (e.g., the continuum hypothesis (CH), and Martin's Axiom plus the negation of the continuum hypothesis (MA $+\neg$ CH)) there exist R-spaces. One of the weakest such assumptions now known to us is the cardinal equality "$b = c$", which is defined below (see Corollary 1.6).

The (ω_1, ω_1^*)-gaps of F. Hausdorff come up naturally in the search for R-spaces because the topological space defined by van Douwen [$\mathbf{vD_1}$] from an (ω_1, ω_1^*)-gap is very close to being an R-space. In fact, for any gap, the associated gap space has all the properties of an R-space except possibly one: countable compactness. These spaces are, however, countably paracompact (for the definitions of gaps and gap spaces see §2). Further, these spaces exist within ZFC by virtue of

1.1. THEOREM (HAUSDORFF [**H**]). *There exists an* (ω_1, ω_1^*)*-gap.*

In this paper we study several special kinds of (ω_1, ω_1^*)-gaps (called tight gaps and big gaps, see §§3, 4) and the topological properties of the associated gap spaces and related spaces. Our results show that in many models of set theory R-spaces can be constructed from (ω_1, ω_1^*)-gaps. For example, if $c = \aleph_1$ or $c = \aleph_2$, we can construct such R-spaces (Corollary 1.7). Our results, however, do not completely answer the question, "does there exist an R-space?"

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Let c denote the cardinality of the continuum. We now define the cardinals b and p which are closely related to certain (ω_1, ω_1^*)-gaps (these and other well-known cardinals in set theory are discussed in [**vD$_2$** and **V$_2$**] and in [**He**] under different terminology). Let ω denote the set of natural numbers, $|X|$ the cardinal number of a set X, and $[\omega]^\omega$ the set of all infinite subsets of ω. A family $\mathcal{F} \subset [\omega]^\omega$ is said to have the *strong finite intersection property* (s.f.i.p.) provided for every finite $\mathcal{F}' \subset \mathcal{F}$, $|\bigcap \mathcal{F}'| = \omega$. A family $\mathcal{F}$ is *unbounded below* provided there does not exist a set $A \in [\omega]^\omega$ such that $|A - F| < \omega$ for all $F \in \mathcal{F}$. We define $p = \min\{|\mathcal{F}| : \mathcal{F} \subset [\omega]^\omega$, $\mathcal{F}$ has s.f.i.p., and $\mathcal{F}$ is unbounded below$\}$. The cardinal b is defined as follows. Let ${}^\omega\omega$ denote the set of all functions from ω into ω. For $f, g \in {}^\omega\omega$ we say $f \leqslant^* g$ provided there exists $N \in \omega$ such that for all $n \geqslant N$, $f(n) \leqslant g(n)$. A set $H \subset {}^\omega\omega$ is called *unbounded above* provided there does not exist $g \in {}^\omega\omega$ such that $f \leqslant^* g$ for all $f \in H$. We define $b = \min\{|H| : H \subset {}^\omega\omega$ and H is unbounded above$\}$.

We use the following classical inequality proved by F. Rothberger [**R**], $\omega_1 \leqslant p \leqslant b \leqslant c$.

Clearly under CH, $\omega_1 = p = b = c$. It is also known that there exist models of set theory in which these cardinals are different (see the discussion in [**vD$_2$**, **He** and **V$_2$**]).

We defer making further definitions until §2, and now state our main results.

Since (ω_1, ω_1^*)-gap spaces are countably paracompact, it is natural to ask if any of them are countably compact. Our first result answers that.

1.2. THEOREM. *The following are equivalent.*

(i) *There exists a countably compact* (ω_1, ω_1^*)*-gap space*,

(ii) *There exists a tight* (ω_1, ω_1^*)*-gap*,

(iii) $p = \omega_1$.

Under MA $+\neg$CH, $p = c > \omega_1$ [**B**]. Thus MA $+\neg$CH implies that there do not exist any countably compact (ω_1, ω_1^*)-gap spaces.

1.3. THEOREM (ZFC). *There exists an* (ω_1, ω_1^*)*-gap which is not tight, hence there exists an* (ω_1, ω_1^*)*-gap space which is not countably compact.*

There is another way that gap spaces can be used to get R-spaces. Note that if an (ω_1, ω_1^*)-gap space X can be densely embedded into a countably compact, first countable T_2-space Y, then Y is an R-space (see §5).

1.4. THEOREM. *If $b = c$, then every first countable, locally compact, T_2-space of cardinality $< c$ can be embedded into a countably compact, first countable T_2-space.*

Since every (ω_1, ω_1^*)-gap space is a first countable, locally compact, zero-dimensional T_2-space and has cardinality ω_1, we have

1.5. COROLLARY. *If $b = c$ and $c > \omega_1$, then every* (ω_1, ω_1^*)*-gap space can be embedded into a countably compact, first countable, T_2-space.*

Combining 1.2 and 1.4 with Rothberger's inequality ($p \leqslant b$) we have

1.6. COROLLARY. *If $b = c$, then there exists an R-space.*

W. Weiss has independently proved this result as an improvement of the similar result (using MA $+\neg$ CH) of M. Dahroug. Their proofs do not use (ω_1, ω_1^*)-gaps. We also have

1.7. COROLLARY. *If $c = \omega_1$ or $c = \omega_2$, then there exists an R-space.*

The next result shows that the embedding of gap spaces as given in 1.5. cannot be carried out in ZFC for every (ω_1, ω_1^*)-gap.

1.8. THEOREM. (i) *There exists a big (ω_1, ω_1^*)-gap if and only if $b = \omega_1$.*

(ii) *If there exists a big (ω_1, ω_1^*)-gap, then the associated gap space cannot be embedded into any countably compact, first countable T_2-space.*

Open questions. (1) Does there exist (in ZFC) an (ω_1, ω_1^*)-gap such that the associated gap space can be embedded into a countably compact, first countable T_2-space? If the answer to (1) is yes then the main problem is solved:

(2) Does there exist an R-space?

In §2, we give the basic definitions of gaps and gap spaces, and consider tight gaps and big gaps in §§3 and 4 respectively. We prove Theorem 1.4 in §5.

2. Definitions. All spaces considered in this paper are completely regular T_2-spaces. The topological terms are standard (e.g., see [**E** or **W**]) but we review them for completeness. A space X is called *countably compact* if every sequence in X has a cluster point (or equivalently if every infinite subset has an accumulation point). A space X is *first countable* if every point has a countable local base; *separable* if X has a countable dense set; *locally compact* if every point has a compact neighborhood; and *zero-dimensional* if every point has a local base of clopen sets.

The set-theoretic concepts: Recall that $[\omega]^\omega$ denotes the set of all infinite subsets of ω. For $A, B \in [\omega]^\omega$ we write $A \subset^* B$ provided $|A - B| < \omega$, and $A < B$ provided $A \subset^* B$ and $|B - A| = \omega$. A set $\Gamma = \{(A_\alpha, B_\alpha);\ a < \omega_1\} \subset [\omega]^\omega \times [\omega]^\omega$ is called an (ω_1, ω_1^*)-gap provided:

(2.0) $B_0 < \omega$;

(2.1) for all $\alpha < \omega_1$, $A_\alpha < A_{\alpha+1} < B_{\alpha+1} < B_\alpha$;

(2.2) there does not exist $H \in [\omega]^\omega$ such that $A_\alpha < H < B_\alpha$ for all $\alpha < \omega_1$.

Condition (2.0) is included only for reasons of symmetry: we want both A_0 and $\omega - B_0$ to be infinite and have limit points in the gap space.

Let Γ be an (ω_1, ω_1^*)-gap. We now define the topological space (of van Douwen) associated with Γ. (In [**vD$_1$**], van Douwen was interested in countable paracompactness, but in this paper we are concerned with countable compactness. Thus, we have to change slightly van Douwen's definition of a gap space in order to insure that A_0 and $\omega - B_0$ have limit points.) The underlying set of the space consists of ω and two disjoint copies of ω_1; say $X(\Gamma) = L_0 \cup L_1 \cup \omega$ where $L_i = \{i\} \times \omega_1$ $(i \in 2)$. The points in ω are to be isolated in the topology.

For $\alpha < \omega_1$ we define a local base at $\langle \alpha, 0\rangle$ and $\langle \alpha, 1\rangle$ as follows. First, for $\alpha = 0$ we put for every $F \in [\omega]^{<\omega}$, $V(\langle 0,0\rangle; F) = \{\langle 0,0\rangle\} \cup A_0 - F$, and $W(\langle 0,1\rangle; F) = \{\langle 0,1\rangle\} \cup (\omega - B_0) - F$. This insures that the infinite sets A_0 and $(\omega - B_0)$

have limit points. For $0 < \alpha < \omega_1$ define for each $\beta < \alpha$ and $F \in [\omega]^{<\omega}$

$$V(\langle \alpha, 0\rangle; \beta, F) = \{\langle \xi, 0\rangle : \beta < \xi \leqslant \alpha\} \cup (A_\alpha - A_\beta) - F,$$

$$W(\langle \alpha, 1\rangle; \beta, F) = \{\langle \xi, 1\rangle : \beta < \xi \leqslant \alpha\} \cup (B_\beta - B_\alpha) - F.$$

The topology on $X(\Gamma)$ is taken to be the topology generated by these local bases. It is easy to check that the space $X(\Gamma)$ is first countable, separable, locally compact, zero-dimensional, and T_2. Further, the disjoint closed sets L_0 and L_1 cannot be separated by disjoint open sets (or else (2.2) is violated); so $X(\Gamma)$ is not normal. All these details can be found in [$\mathbf{vD_1}$]. We note that since L_0 and L_1 are countably compact, they will be closed in any first countable T_2-space Y in which $X(\Gamma)$ can be embedded; so such a space Y will also fail to be normal.

3. Tight (ω_1, ω_1^*)-gaps.

3.1. Definition. Let $\Gamma = \{(A_\alpha, B_\alpha)\colon \alpha < \omega_1\}$ be an (ω_1, ω_1^*)-gap, and $E \in [\omega]^\omega$. If $E \subset {}^* B_\alpha - A_\alpha$ for all $\alpha < \omega_1$ we say that E is *beside the gap* Γ. A gap Γ is called *tight* provided there does not exist $E \in [\omega]^\omega$ such that E is beside the gap Γ.

3.2. Lemma. *An* (ω_1, ω_1^*)*-gap* Γ *is tight if and only if the space* $X(\Gamma)$ *is countably compact.*

Proof. Since both L_0 and L_1 are countably compact subsets of $X(\Gamma)$ the only countable subsets of $X(\Gamma)$ which do not obviously have limit points are sets $E \subset \omega$ such that $|E \cap A_\alpha| < \omega$ and $|E - B_\alpha| < \omega$ (equivalently, $E \subset {}^* B_\alpha - A_\alpha$) for all $\alpha < \omega_1$. Note that $\{B_\alpha - A_\alpha\colon \alpha < \omega_1\}$ is a decreasing tower in $[\omega]^\omega$ (i.e., $\alpha < \beta$ implies $B_\beta - A_\beta \subset {}^* B_\alpha - A_\alpha$). Thus $X(\Gamma)$ is countably compact if and only if this tower is unbounded below (i.e. Γ is a tight gap).

From this it is clear that if Γ is a tight gap, then $p = \omega_1$ (this proves Theorem 1.2 (ii) $\to$ (iii)).

Proof of Theorem 1.2. (iii) $\to$ (ii). *If* $p = \omega_1$, *then there is a tight* (ω_1, ω_1^*)*-gap.* If $p = \omega_1$, then there is a maximal decreasing tower $\{T_\alpha\colon \alpha < \omega_1\} \subset [\omega]^\omega$ (i.e. $\alpha < \beta$ implies $T_\beta < T_\alpha$ and there does not exist any $H \in [\omega]^\omega$ such that $H \subset {}^*T_\alpha$ for all $\alpha < \omega_1$). Clearly, it suffices to prove that there exists an (ω_1, ω_1^*)-gap $\Gamma = \{(A_\alpha, B_\alpha)\colon \alpha < \omega_1\}$ such that $B_\alpha - A_\alpha = {}^* T_\alpha$ (i.e., $B_\alpha - A_\alpha \subset {}^* T_\alpha \subset {}^* B_\alpha - A_\alpha$) for all $\alpha < \omega$. The maximality of the tower is not needed in the construction of the gap so we state the following more general result.

3.3. Lemma. *If* $\{T_\alpha\colon \alpha < \omega_1\}$ *is a family of infinite subsets of* ω *such that* $T_0 < \omega$ *and* $\alpha < \beta < \omega_1$ *imply* $T_\beta < T_\alpha$, *then there exists an* (ω_1, ω_1^*)*-gap* $\Gamma = \{(A_\alpha, B_\alpha)\colon \alpha < \omega_1\}$ *such that* $B_\alpha - A_\alpha = T_\alpha$ *for all* $\alpha < \omega_1$.

This lemma can be proved by making minor modifications in Hausdorff's proof that there exists an (ω_1, ω_1^*)-gap. A somewhat different proof of a more general result has been given independently by A. Błaszczyk and A. Szymański [**BS**].

3.4. Definition. Let N be a countably infinite set, and $\{(A_\alpha, B_\alpha)\colon \alpha < \omega_1\}$ a family of pairs of infinite subsets of N. Such a family is called an (ω_1, ω_1^*)*-gap in* N provided (2.0), (2.1) and (2.2) hold for this family with N substituted for ω.

It follows from Hausdorff's Theorem 1.1, that every countable, infinite set N has an (ω_1, ω_1^*)-gap in N.

PROOF OF THEOREM 1.3 (ZFC). *There exists an (ω_1, ω_1^*)-gap which is not tight, hence there exists an (ω_1, ω_1^*)-gap space which is not countably compact.* Partition ω into three infinite sets M, N, and P. Let $\{(A'_\alpha, B'_\alpha)\colon \alpha < \omega_1\}$ be any (ω_1, ω_1^*)-gap in N. Define $A_\alpha = A'_\alpha \cup M$, and $B_\alpha = B'_\alpha \cup M \cup P$ for all $\alpha < \omega_1$. Then $\{(A_\alpha, B_\alpha)\colon \alpha < \omega_1\}$ is an (ω_1, ω_1^*)-gap having the infinite set P beside the gap.

4. Big (ω_1, ω_1^*)-gaps.

4.1. DEFINITION. An (ω_1, ω_1^*)-gap $\Gamma = \{(A_\alpha, B_\alpha)\colon \alpha < \omega_1\}$ is called *big* provided there exists a family $\{E_n\colon n \in \omega\} \subset [\omega]^\omega$ of pairwise disjoint sets such that each E_n is beside the gap Γ, and for every $D \subset \omega$ such that $|D \cap E_n| = \omega$ for infinitely many $n \in \omega$, there exists $\alpha < \omega_1$ such that $|A_\alpha \cap D| = \omega$. This concept is motivated by

4.2. LEMMA. *If Γ is a big (ω_1, ω_1^*)-gap, then $X(\Gamma)$ cannot be embedded in a countably compact, first countable T_2-space.*

PROOF. Let $\Gamma = \{(A_\alpha, B_\alpha)\colon \alpha < \omega_1\}$ and let $\{E_n\colon n \in \omega\}$ be pairwise disjoint infinite subsets of ω satisfying 4.1. Suppose that $X(\Gamma)$ can be embedded in a countably compact, first countable T_2-space Y (we assume $X(\Gamma) \subset Y$). Each E_n has an accumulation point y_n in Y and further $y_n \in Y \setminus X(\Gamma)$ because E_n is beside the gap Γ. Since $X(\Gamma)$ is locally compact, $Y - X(\Gamma)$ is closed in Y. Thus, there is a point $y \in Y - X(\Gamma)$ which is a cluster point of the sequence $\{y_n\colon n \in \omega\}$. By passing to subsequences where necessary, we may assume without loss of generality that (1) $\{y_n\colon n \in \omega\}$ converges to y in Y, and (2) each E_n converges to y_n (i.e., y_n is the unique accumulation point of E_n in Y). Now let $\{U_n\colon n \in \omega\}$ be a local base for y in Y. Pick an increasing sequence $\{i_n\colon n \in \omega\}$ of natural numbers such that the sets $D_{i_n} = U_n \cap E_{i_n}$ are infinite. Put $D = \bigcup\{D_{i_n}\colon n \in \omega\}$.

By (4.1) there exists (a first) $\alpha < \omega_1$ such that $|A_\alpha \cap D| = \omega$. Thus the point $\langle \alpha, 1\rangle \in X(\Gamma)$ is a limit point of D. But this is impossible because D has all its limit points in $Y \setminus X(\Gamma)$.

PROOF OF THEOREM 1.8. *There exists a big (ω_1, ω_1^*)-gap if and only if $b = \omega_1$.* First we assume that $b = \omega_1$. Let $\{f_\alpha\colon \alpha < \omega_1\}$ be a family of strictly increasing functions from ω into ω which has no upper bound in the $<^*$ order on ${}^\omega\omega$. Let N be a copy of ω disjoint from $\omega \times \omega$. Let $\{(A'_\alpha, B'_\alpha)\colon \alpha < \omega_1\}$ be any (ω_1, ω_1^*)-gap on N. Define $B_\alpha = B'_\alpha \cup (\omega \times \omega)$ and $A_\alpha = A'_\alpha \cup \{\langle i, j\rangle \in \omega \times \omega\colon j \geq f_\alpha(i)\}$ for all $\alpha < \omega_1$. Then $\{(A_\alpha, B_\alpha)\colon \alpha < \omega_1\}$ is a big gap on the countable set $(\omega \times \omega) \cup N$; so there exists a big (ω_1, ω_1^*)-gap.

Conversely, assume that $\{(A_\alpha, B_\alpha)\colon \alpha < \omega_1\}$ is a big (ω_1, ω_1^*)-gap in ω. Let $\{E_n\colon n < \omega\}$ be a family of mutually disjoint infinite subsets of ω satisfying the property of a big gap in (4.1). Since for every $n < \omega$ and $\alpha < \omega_1$ $E_n \subset^* B_\alpha - A_\alpha$, we may define functions f_α by the rule: $f_\alpha(n)$ is the first integer such that

$$j \geq f_\alpha(n) \rightarrow (j \in E_n \rightarrow j \in B_\alpha - A_\alpha).$$

We show that $\{f_\alpha: \alpha < \omega_1\}$ has no upper bound in ${}^\omega\omega$, and thus $b = \omega_1$. Suppose that there is a function g such that $f_\alpha \leqslant {}^* g$ for all $\alpha < \omega_1$. Define $D_n = \{x \in E_n: x > g(n)\}$. Clearly D_n is an infinite subset of E_n for all $n < \omega$. Since the gap is big, there exists $\alpha < \omega_1$ such that $(A_\alpha \cap \bigcup_n D_n)$ is infinite. Further, since each D_n is beside the gap, $A_\alpha \cap D_n$ is finite for all $n < \omega$. Thus, there are infinitely many n such that $A \cap D_n \neq \varnothing$. Since $f_\alpha \leqslant {}^* g$ there exists $n < \omega$ such that for all $m \geqslant n$ we have $f_\alpha(n) \leqslant g(n)$. Now pick an $n > m$ such that there is $j \in A_\alpha \cap D_n$. We have $j > g(n) \geqslant f_\alpha(n)$; so by definition of f_α, $j \in B_\alpha - A_\alpha$. This contradicts that $j \in A_\alpha$, and that completes the proof.

4.3. REMARK. It is easy to see that there exists a non-big gap (in ZFC). If $p = \omega_1$, a tight gap is not big. If $p > \omega_1$, then $b > \omega_1$ (by Rothberger's inequality); so every gap is not big by Theorem 1.8.

5. Proof of Theorem 1.4. If $b = c$, then every first countable, locally compact, T_2-space of cardinality $< c$ can be embedded into a countably compact, first countable, locally compact, zero-dimensional T_2-space.

5.1. LEMMA (VAN DOUWEN [$\mathbf{vD_2}$]). *If X is a first countable regular space of cardinality $< b$, then X has property D (i.e., for every closed, discrete sequence $\{x_i: i < \omega\}$ in X, there exists a discrete family $\{U_i: i < \omega\}$ of open sets in X such that $x_j \in U_i \Leftrightarrow j = i$).*

Now let (X, T) be a topological space satisfying the hypothesis of 1.4. We construct a space Y by transfinite induction in the manner of Ostaszewski **[O]** starting with X at the first step. The underlying set of Y will be $X \cup c$ where c is the cardinal number 2^ω, and where we consider X and c as disjoint sets. Let $\{H_\alpha: \omega \leqslant \alpha < c\}$ list $[c]^\omega$ such that $H_\alpha \subset \alpha$ for all $\omega \leqslant \alpha < c$. Let $\{E_\alpha: \omega \leqslant \alpha < c\}$ list $[X]^\omega$. This requires only that $|X| \leqslant c$, but we need $|X| < c$ in order to apply 5.1, and to have X zero-dimensional. In order to catch up with the listings, put $X_n = X$ and $T_n = T$ for all $n < \omega$. Assume for all $\alpha < \gamma$, where $\omega \leqslant \alpha < \gamma < c$, we have defined topologies T_α on sets X_α such that

(1) $X_\alpha = X \cup \alpha$ (for $\omega \leqslant \alpha < \gamma$).

(2) (X_α, T_α) is a first countable, locally compact, zero-dimensional T_2-space.

(3) $\beta < \alpha < \gamma$ implies (X_β, T_β) is an open subspace of (X_α, T_α).

(4) $\omega \leqslant \alpha < \alpha + 1 < \gamma$ implies that both H_α and E_α have a limit point in $(X_{\alpha+1}, T_{\alpha+1})$.

In order to construct T_γ on $X_\gamma = X \cup \gamma$, we proceed in two cases (γ a successor or limit ordinal) as in many Ostaszewski type constructions (see [$\mathbf{vD_2}$, **O**, $\mathbf{V_1}$]). For completeness we sketch the proof. If γ is a successor ordinal, say $\gamma = \alpha + 1$, and if both H_γ and E_γ have limit points in (X_α, T_α) we let α be isolated in (X_γ, T_γ). If one or both of H_γ, E_γ is closed discrete, we use zero-dimensionality, local compactness, and Lemma 5.1 to get a discrete family of compact clopen sets $\{V_i: i \in \omega\}$ each of which meets H_γ and/or E_γ (whichever is closed discrete) in an infinite set. We let a local base at α be $\{W_n: n < \omega\}$ where $W_n = \bigcup\{V_i: i > n\} \cup \{\alpha\}$. In case γ is a limit ordinal, define a subset U of X_γ to be open in T_α if and only if $U \cap X_\alpha \in T_\alpha$ for all $\alpha < \gamma$. It is possible that for some $\alpha < c$, the space (X_α, T_α) is countably compact (even (X_0, T_0)), and we could stop right there. Our construction, however, would continue by adding isolated points for countably many steps and then pick up again adding limit points. Thus we can get the final space $Y = (X_c, T_c)$ to have underlying set $X \cup c$. The space Y is a countably compact, first countable, locally compact, zero-dimensional T_2-space containing $X = X_0$ as a subspace. We leave the details to the reader.

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Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208

Department of Mathematics, University of North Carolina at Greensboro, Greensboro, North Carolina 27412