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AN INEQUALITY WITH APPLICATIONS IN POTENTIAL THEORY

BY

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ABSTRACT. An analytic inequality (announced previously) is proved and a certain monotonicity condition is shown to be essential for its validity, contrary to an earlier conjecture. Then, a generalization of the inequality, which takes into account the extent of nonmonotonicity, is established.

1. Introduction. The purpose of this paper is to prove an inequality (see (2.1) below) which arises in connection with some potential theoretic problems. The basic result is Theorem 1 of §2, in which the fundamental inequality is established under a certain monotonicity hypothesis. In §3 we give an example which indicates that the monotonicity hypopthesis cannot be completely dropped. In §4 we prove a more general result, Theorem 2, which indicates how the key coefficient in the fundamental inequality will change if we weaken the monotonicity hypothesis. Finally, §5 gives a reinterpretation of our results in terms of information theory.

We now briefly describe the potential theoretic considerations underlying this work. Let $SH^{\langle k\rangle}$ denote the class of functions $u(z)$ subharmonic in the open unit disk D and satisfying $u(z) \leq A_u k(|z|) + B_u$ where $k(r)$, $0 < r < 1$, is a nonnegative function with $k(r) \to \infty$ as $r \to 1$. Let $A^{\langle k\rangle}$ be the corresponding class of analytic functions $f(z)$ such that $\log|f(z)| \leq A_f k(|z|) + B_f$. In [1], a characterization of zero sets for $A^{\langle k\rangle}$ was given in the particular case $k(r) = |\ln(1-r)|$, and this result yields almost immediately a characterization of the Riesz measures (generalized Laplacians) for $SH^{\langle k\rangle}$. In attempting to extend these results to a wider class of functions, it is natural to consider the particular case $k(r) = (1-r)^{-\alpha}$ where α is fixed, $0 < \alpha < 1$. Inequality (2.1) is instrumental in settling this case. A brief announcement of these ideas was given in [2]; [3] is an expanded version with more detail.

The precise role of (2.1) in these problems is too complicated to go into here. It can, however, be pointed out that unconditional validity of the inequality (as conjectured in [3]) would have led to a complete description of Riesz measures (and zero sets) for $k(r) = (1-r)^{-\alpha}$. Although the example of §3 shows that this conjecture was too optimisitic, the results of §§2, 4 can still be used to obtain partial characterizations. This will be treated in detail elsewhere.

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We have recently received a copy of a manuscript by Hinkkanen and Vaughan [**4**], in which they prove versions of the results in §§2–4. Their proofs are based on different ideas and the bounds which they obtain for the key constant (what we call the "admissible constant" in §4) differ greatly from those obtained in this paper.

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2. The monotone case. Throughout this paper, F will denote a set of points $x_0, x_1, \ldots, x_n$ $(n \geqslant 1)$ on the real line with $x_0 < x_1 < \cdots < x_n$. We will say that F satisfies the *monotonicity hypothesis* provided that the numbers $\tau_1, \tau_2, \ldots, \tau_n$ are either nondecreasing or else nonincreasing, where $\tau_i = x_i - x_{i-1}$.

This section is devoted to proving a result whose motivation and initial formulation was given in [**3**, Theorem 3].

THEOREM 1. *Suppose F satisfies the monotonicity hypothesis and that $m_0, \ldots, m_n$ are nonnegative real numbers. Suppose also that α is a constant satisfying* $0 < \alpha < 1$. *Then*

$$\int_{x_0}^{x_n} \left(\sum_{i=1}^{n} \frac{m_i}{(x - x_i)^2} \right)^{\alpha/1+\alpha} dx \leqslant \frac{20}{1-\alpha} M^{\alpha/1+\alpha} E^{1/1+\alpha} \tag{2.1}$$

where $M = \sum m_i$, and $E = \sum \tau_i^{1-\alpha}$.

We begin with a few preliminary remarks.

(1) Our statement of inequality (2.1) is more precise than that announced in [**3**], since a specific value, namely, $20/(1-\alpha)$, is given for the constant whose existence was there asserted.

(2) By symmetry, it suffices to consider the monotone increasing case. Thus, *we shall assume that* $\tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_n$.

(3) Next we observe that it suffices to prove (2.1) in the special case $M = E$. The general case with arbitrary M and E reduces to this special case if each m_i is replaced by $m_i E/M$.

(4) *We shall assume, as a matter of convenience, that* $\tau_n = 1$. The case of arbitrary τ_n reduces to this case if x_i is replaced by kx_i where $k = \tau_n^{-1}$.

Thus, to summarize, our goal is to prove

$$\int_{x_0}^{x_n} \left(\sum \frac{m_i}{(x - x_i)^2} \right)^{\alpha/1+\alpha} dx \leqslant \frac{20}{1-\alpha} E, \tag{2.2}$$

under the hypotheses that $M = E$ and $\tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_n = 1$.

Rather than attacking (2.2) head-on, we adopt an indirect approach which is based on the easily established observation that, *if A is any positive number and if* $0 < \alpha < 1$, *then*

(2.3)

$$A^{\alpha/1+\alpha} = C \min\{Ay + y^{-\alpha} \mid 0 < y < \infty\} \quad \textit{where } C = (\alpha^{1/1+\alpha} + \alpha^{-\alpha/1+\alpha})^{-1}.$$

From this it follows that if $f(x)$ and $y(x)$ are nonnegative functions defined for $a \leqslant x \leqslant b$, then

$$\int_a^b f(x)^{\alpha/1+\alpha}\,dx \leqslant C\int_a^b \left(f(x)y(x) + y(x)^{-\alpha}\right)dx \tag{2.4}$$

provided the integrals on both sides exist, and where the constant C has the same value as in (2.3).

The proof of (2.2) can now be sketched out. We shall construct a nonnegative function $y(x)$ on $x_0 \leqslant x \leqslant x_n$ which satisfies the following inequalities:

$$\int_{x_0}^{x_n} y(x)^{-\alpha}\,dx \leqslant \frac{1}{1-\alpha/\lambda}E, \tag{2.5}$$

$$\int_{x_0}^{x_n} \frac{y(x)}{(x-x_j)^2}\,dx \leqslant \frac{2+2\lambda+2^{1/\lambda}}{1-\lambda} \quad \text{for } j = 1,\ldots,n. \tag{2.6;j}$$

In these formulas, λ is an auxiliary constant lying between α and 1. It will be used in the construction of y and will then be eliminated.

Multiplying (2.6; j) by m_j, summing, and using the assumption that $M = E$, we obtain

$$\int_{x_0}^{x_n} \sum \frac{m_i y(x)}{(x-x_i)^2}\,dx \leqslant \left(\frac{2+2\lambda+2^{1/\lambda}}{1-\lambda}\right)E. \tag{2.7}$$

Now, if we take $\lambda = (\alpha+1)/2$, our desired inequality, (2.2), follows from (2.4), (2.5), (2.7), and the observation that the constant C in (2.4) cannot exceed the value 1.

For $i = 1,\ldots,n$, let A_i denote the interval $[x_{i-1}, x_i]$ and let $K(x)$ be the left continuous step function whose value on the interior of A_i is τ_i^λ and let $\omega(x)$ be the *greatest convex minorant* of $K(x)$; thus, $\omega(x)$ is the largest convex function with the property that $\omega(x) \leqslant K(x)$ for $x_0 \leqslant x \leqslant x_n$. Now let $\Omega(x) = \omega(x)^{1/\lambda}$ for $x_0 \leqslant x \leqslant x_n$. A key property of $\Omega(x)$ may now be stated.

PROPOSITION 2.8.

$$\int_{x_0}^{x_n} \Omega(x)^{-\alpha}\,dx \leqslant \frac{1}{1-\beta}E \quad \textit{where } \beta = \alpha/\lambda.$$

The proof depends on a quite general result.

LEMMA 2.9. *Let $H(x)$ be any positive increasing piecewise left continuous function defined on an interval $[a, b]$, let $h(x)$ be the greatest convex minorant of $H(x)$, and let β be arbitrary, $0 < \beta < 1$. Then*

$$\int_a^b h(x)^{-\beta}\,dx \leqslant \frac{1}{1-\beta}\int_a^b H(x)^{-\beta}\,dx.$$

PROOF. Both integrals may be split into two parts corresponding to the set where $h(x) = H(x)$ and the set where $h(x) < H(x)$. On the first of these sets, the inequality is trivial since $1/1-\beta > 1$. The second set is the countable disjoint union of open intervals. Over the closure of each of these, the graph of h will be a line segment with endpoints on the graph of H. Thus, without loss of generality it suffices to treat the case where $h = mx + c$, $h(a) = H(a)$ and $h(b) = H(b)$. Since H is an increasing function with maximum value $mb + c$ on $[a, b]$, our inequality will be established if we can show that

$$\int_a^b (mx + c)^{-\beta}\, dx / (b - a)(mb + c)^{-\beta} \leqslant 1/1 - \beta.$$

After a bit of algebra, we find that the left-hand side can be written as

$$\frac{1}{1-\beta}\left\{1 + \frac{(ma + c)(mb + c)^{-\beta} - (ma + c)^{1-\beta}}{m(b - a)(mb + c)^{-\beta}}\right\}$$

and, since the numerator of the fraction in brackets is negative, this proves our inequality and establishes the lemma.

To prove the proposition, we apply the lemma to the case where $H(x) = K(x)^{\lambda}$ and $\beta = \alpha/\lambda$. We obtain

$$\int_{x_0}^{x_n} \Omega(x)^{-\alpha}\, dx = \int_{x_0}^{x_n} \omega(x)^{-\beta}\, dx \leqslant \frac{1}{1-\beta}\int_{x_0}^{x_n} K(x)^{-\beta}\, dx$$
$$= \frac{1}{1-\beta}\sum \int_{A_i} \tau_i^{-\alpha}\, dx = \frac{1}{1-\beta}\left(\sum \tau_i^{1-\alpha}\right) = \frac{1}{1-\beta}E,$$

which is the assertion of Proposition 2.8.

Although the function Ω satisfies one of our desired inequalities, (2.5), it certainly will not satisfy any of (2.6; j). We must perform a bit of surgery on Ω to obtain a function y which satisfies all of these inequalities.

On each interval A_i we erect two construction lines running from the points $(x_{i-1}, 0)$ and $(x_i, 0)$ up to $((x_{i-1} + x_i)/2, \tau_i)$. These lines have slope 2, -2, respectively. Since $\Omega(x) \leqslant \tau_i$ on A_i, and is a convex function, these lines will cut the graph of Ω at two points whose x-coordinates are denoted a_i and b_i, respectively. Thus $\Omega(a_i) = 2(a_i - x_{i-1})$ and $\Omega(b_i) = 2(x_i - b_i)$.

We define $y(x)$ on A_i as follows:

$$y(x) = \begin{cases} K_i(x - x_{i-1})^{1/\lambda}, & x_{i-1} \leqslant x \leqslant a_i, \\ \Omega(x), & a_i \leqslant x \leqslant b_i, \\ L_i(x_i - x)^{1/\lambda}, & b_i \leqslant x \leqslant x_i, \end{cases}$$

where K_i, L_i are chosen to make y continuous.

We shall now show that y satisfies inequality (2.5). We start by writing

$$\int_{x_0}^{x_n} y(x)^{-\alpha}\, dx = \sum \int_{A_k} y(x)^{-\alpha}\, dx$$

and we split the integral over A_k into three parts corresponding to $[x_{k-1}, a_k]$, $[a_k, b_k]$, $[b_k, x_k]$.

To treat the first of these integrals, we note that

$$\begin{aligned}\int_{x_{k-1}}^{a_k} y(x)^{-\alpha}\,dx &= \frac{K_k^{-\alpha}}{1-\alpha/\lambda}(a_k - x_{k-1})^{1-\alpha/\lambda}\\ &= \frac{\Omega(a_k)^{-\alpha}}{1-\alpha/\lambda}(a_k - x_{k-1}) = \frac{\Omega(a_k)^{1-\alpha}}{1-\alpha/\lambda}\,\frac{(a_k - x_{k-1})}{\Omega(a_k)}\\ &= \frac{1}{2}\,\frac{1}{1-\alpha/\lambda}\Omega(a_k)^{1-\alpha} \leqslant \frac{1}{2}\,\frac{1}{1-\alpha/\lambda}\tau_k^{1-\alpha}.\end{aligned}$$

The same estimate applies to the integral over $[b_k, x_k]$. Thus if we sum up all these contributions, we obtain

$$2\left(\sum \frac{1}{2}\,\frac{1}{1-\alpha/\lambda}\tau_k^{1-\alpha}\right) = \frac{1}{1-\alpha/\lambda}E.$$

Using Proposition 2.8, we estimate the rest of our integral as follows:

$$\sum \int_{a_k}^{b_k} y(x)^{-\alpha}\,dx = \sum \int_{a_k}^{b_k} \Omega(x)^{-\alpha}\,dx < \int_{x_0}^{x_n} \Omega(x)^{-\alpha}\,dx \leqslant \frac{1}{1-\alpha/\lambda}E.$$

This establishes (2.5).

We turn now to the estimates (2.6; j) and consider the case $1 < j < n$ (the cases $j = 1$ and n are easier). We split the integral

$$\int_{x_0}^{x_n} \frac{y(x)}{(x - x_j)^2}\,dx$$

into four parts corresponding to the intervals

$$[x_0, b_j],\ [b_j, x_j],\ [x_j, a_{j+1}] \quad \text{and} \quad [a_{j+1}, x_n].$$

The two inner integrals can be computed exactly, the value being $2\lambda/(1-\lambda)$ in each case.

Consider now the integral $\int_{a_{j+1}}^{x_n} y(x)dx/(x - x_j)^2$. We assert that its value cannot exceed $2^{1/\lambda}/(1-\lambda)$. This follows from a general result which we may state as follows.

Lemma 2.10. *Suppose $p > 1$ and suppose m and c are positive constants such that the parabola $y = m(x + c)^p$, $(x > -c)$, intersects the line $y = 2x$ at two points, $0 < h < k$. Let a be the x-coordinate of the point where the parabola intersects the line $y = 2(x + c)$. Then*

$$\int_a^b \frac{m(x + c)^p}{x^2}\,dx \leqslant \frac{p2^p}{p-1}.$$

Proof. Since $p > 1$, a convexity argument shows that $(x + c)^p \leqslant 2^{p-1}(x^p + c^p)$. Thus

$$\int_h^a \frac{m(x + c)^p}{x^2}\,dx \leqslant m2^{p-1}\int_h^a \frac{x^p + c^p}{x^2}\,dx \leqslant m2^{p-1}\left[\frac{a^{p-1}}{p-1} + \frac{c^p}{h}\right].$$

We note that $m(a+c)^p = 2(a+c)$ so that $(a+c)^{p-1} = 2/m$. Also, $c^p < (h+c)^p = 2h/m$. Thus, the expression previously obtained cannot exceed $m2^{p-1}[2/m(p-1) + 2h/mh] = p2^p/(p-1)$, which proves the lemma.

Now to estimate

$$\int_{a_{j+1}}^{x_n} \frac{y(x)}{(x-x_j)^2}\,dx,$$

we note that for $a_{j+1} \leqslant x \leqslant x_n$, the piecewise linear function $\omega(x)$ is dominated by the linear function $Ax+B$ where $Aa_{j+1}+B = \omega(a_{j+1})$ and where $Ax_n + B = \omega(x_n) = 1$. Hence our integral is dominated by

$$\int_{a_{j+1}}^{x_n} \frac{(Ax+B)^p}{(x-x_j)^2}\,dx, \quad \text{where } p > 1.$$

If we replace $x - x_j$ by x, and use the fact that $\tau_n = 1$, we obtain an integral of the form considered in Lemma 2.10 (except that $x_n - x_j < a$, which only improves the inequality). Thus, by the lemma, we see that the value of our integral is less than $p2^p/(p-1) = 2^{1/\lambda}/(1-\lambda)$.

It remains to deal with the first of our integrals, corresponding to the interval $[x_0, b_j]$. We have

$$\int_{x_0}^{b_j} \frac{y(x)}{(x-x_j)^2}\,dx \leqslant \Omega(b_j)\int_{x_0}^{b_j}\frac{dx}{(x-x_j)^2} = \frac{\Omega(b_j)}{(x_j-b_j)}\,\frac{(b_j-x_0)}{(x_j-x_0)} < 2.$$

In summary then, when $1 < j < n$, we have

$$\int_{x_0}^{x_n} \frac{y(x)}{(x-x_j)^2}\,dx \leqslant 2 + \frac{4\lambda + 2^{1/\lambda}}{1-\lambda} = \frac{2+2\lambda+2^{1/\lambda}}{1-\lambda},$$

which proves (2.6; j) and completes the proof of Theorem 1.

3. An example. Let us introduce a somewhat more compact notation for the quantities considered in the fundamental inequality (2.1). Given a set $F = \{x_0 < x_1 < \cdots < x_n\}$ of points and a collection $\mathfrak{M} = \{m_0, m_1, \ldots, m_n\}$ of nonnegative real numbers, we denote by $S(x; F, \mathfrak{M})$, or simply by $S(x)$ if F and $\mathfrak{M}$ are understood, the function $\Sigma\, m_i/(x-x_i)^2$. The integral

$$\int_{x_0}^{x_n} S(x; F, \mathfrak{M})^{\alpha/1+\alpha}\,dx$$

will be denoted $I(F, \mathfrak{M})$, or simply I. The quantities $M(\mathfrak{M})$ and $E(F)$, or simply M and E, have the same meaning as before; $M = \Sigma\, m_i$ and $E = \Sigma\, \tau_i^{1-\alpha}$.

In this notation, Theorem 1 states that if F satisfies the monotonicity hypothesis, then $I \leqslant 20/(1-\alpha)M^{\alpha/1+\alpha}E^{1/1+\alpha}$. It is tempting to conjecture, as in **[3]**, that given α, $0 < \alpha < 1$, there is a constant C, which depends only on α, such that for any distribution, inequality (2.1) will hold: $I \leqslant CM^{\alpha/1+\alpha}E^{1/1+\alpha}$.

However, in this section, we describe an iterative procedure for constructing a sequence of distributions with the property that, as one passes from one distribution to the next, I grows at a faster rate than does the product $M^{\alpha/1+\alpha}E^{1/1+\alpha}$. Thus, no such constant C can exist.

We shall describe one step in the construction. It is, of course, understood that in successive steps new values for certain parameters will need to be chosen; this will be reemphasized when necessary.

We start with any initial distribution of points, $x_0 < x_1 < \cdots < x_n$, and masses, $m_0, \ldots, m_n$, and we let I_0, M_0, and E_0, denote the associated initial values of I, M, and E. We now fix a positive number ε which is at least small enough so that the $\varepsilon/2$-neighborhoods of the x_i are disjoint, and which will also be required to satisfy one additional smallness criterion whose precise statement is best deferred until its relevance becomes clear.

Near each x_i, we now carry out a construction which changes the distribution of mass. We select a positive integer, N, and a positive number, τ, which are required to satisfy $(N+1)\tau^{1-\alpha} = m_i$ and $N\tau < \varepsilon/2$. For convenience, we also require that the same value of N be used at each construction site $x_0, \ldots, x_n$.

Now, inside the $\varepsilon/2$-neighborhood of x_i, replace x_i by $N+1$ points (denoted for the purposes of the next computation by $y_0, \ldots, y_N$), spaced a distance τ apart from each other, and place equal masses $m = \tau^{1-\alpha}$ at each of these points.

We observe in passing that the integral associated with a single such distribution will satisfy $I \geqslant Nm_i/(N+1)$. This can be seen as follows:

$$\int_{y_0}^{y_N} \left(\sum \frac{m}{(y-y_i)^2} \right)^{\alpha/1+\alpha} dy = m^{\alpha/1+\alpha} \sum_{j=1}^{N} \int_{y_{j-1}}^{y_j} \left(\sum \frac{1}{(y-y_i)^2} \right)^{\alpha/1+\alpha} dy$$

$$> m^{\alpha/1+\alpha} \sum_{j=1}^{N} \int_{y_{j-1}}^{y_j} |y-y_j|^{-2\alpha/(1+\alpha)}\, dy$$

$$= \left(\frac{1+\alpha}{1-\alpha} \right) N\tau^{\alpha(1-\alpha)/(1+\alpha)} \tau^{(1-\alpha)/(1+\alpha)} > N\tau^{1-\alpha} = \frac{N}{N+1} m_i.$$

Having carried out this construction at each x_i, we obtain our new distribution of points and masses, and we now need to compare the new values of I, M, and E, with the initial values I_0, M_0 and E_0.

Denoting the new value of I by I_1, we may write: $I_1 = K + K_0 + \cdots + K_n$ where K_i is the contribution to I_1 of integrating over the interval spanned by those points which were added near x_i, and where K is the contribution obtained from the union of the complementary intervals. If ε has been chosen sufficiently small, then we have $K \geqslant I_0 - \delta_1$ where, at the kth step in the construction, we take $\delta_k = 1/2^k$. Using the estimate made earlier, we can write $K_i \geqslant Nm_i/(N+1)$ and since the same value of N was chosen at each x_i, we obtain the following lower bound on the growth of I:

$$I_1 \geqslant I_0 - \delta_1 + \frac{N}{N+1} M_0 \geqslant I_0 - \delta_1 + \frac{1}{2} M_0.$$

On the other hand, the total mass of our system is unchanged, $M_1 = M_0$, and, in computing the new value of E, we see that the construction near each x_i contributes $N\tau^{1-\alpha} = Nm_i/(N+1)$, so that

$$E_1 \leqslant E_0 + \frac{N}{N+1} M_0 < E_0 + M_0.$$

If we now iterate this process we obtain, after k steps, the following estimates:

$$I_k \geqslant I_0 - \frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2^k} + \frac{kM_0}{2} \geqslant I_0 - 1 + \frac{kM_0}{2}, \qquad E_k < E_0 + kM_0,$$

and $M_k = M_0$.

Hence putting

$$C_k = \frac{I_k}{M_k^{\alpha/1+\alpha} E_k^{1/1+\alpha}},$$

we have

$$C_k \geqslant \frac{I_0 - 1 + kM_0/2}{M_0^{\alpha/1+\alpha}(E_0 + kM_0)^{1/1+\alpha}}.$$

Thus, $C_k \to \infty$ as $k \to \infty$.

4. The problem of arbitrary distributions. We fix α, $0 < \alpha < 1$, and a finite set F. A number C will be called an admissible constant for F provided that, in the notation introduced in §3, $I(F, \mathfrak{M}) \leqslant CM(\mathfrak{M})^{\alpha/1+\alpha}E(F)^{1/1+\alpha}$, for any distribution of nonnegative masses $\mathfrak{M}$. Thus Theorem 1 asserts that $20/(1-\alpha)$ is an admissible constant for any set F which satisfies the monotonicity condition $\tau_1 < \tau_2 < \cdots < \tau_n$, and the example of §3 shows that (for α fixed) there is no constant which is admissible for all sets F. In this section we give a formula which shows how the admissible constant grows with the complexity of the set F.

It is helpful to introduce some descriptive terminology. A *block partition* of a set $F = \{x_0 < x_1 < \cdots < x_n\}$ is a subdivision of F into consecutive blocks $F_1, \ldots, F_m$ which share their common endpoints. Thus $F_1 = \{x_i \mid 0 \leqslant i \leqslant n_1\}$, $F_2 = \{x_i \mid n_1 \leqslant i \leqslant n_2\}, \ldots,$ and $F = \bigcup F_i$. The *endpoint set* is the set $G = \{x_0, x_{n_1}, x_{n_2}, \ldots, x_{n_m} = x_n\}$ consisting of the endpoints of the blocks.

LEMMA. *Suppose $F_1, F_2, \ldots, F_m$ is a block partition of F, as above, and that C_1 is admissible for each F_i, $i = 1, \ldots, m$. Suppose also that C_2 is admissible for the endpoint set G. Then $C_1 + C_2$ is admissible for F itself.*

PROOF. Let $\mathfrak{M} = \{m_i \mid i = 0, \ldots, n\}$ be a system of masses; we must show

$$I(F, \mathfrak{M}) \leqslant (C_1 + C_2)M(\mathfrak{M})^{\alpha/1+\alpha}E(F)^{1/1+\alpha}.$$

We begin by splitting the integral into m integrals corresponding to the m blocks $F_1, \ldots, F_m$. Fixing the kth such integral, split $S(x; F, \mathfrak{M})$ into two parts, one corresponding to "interior" points of F_k and a remainder. Thus we are considering, for a fixed k,the quantity

$$\int_{[F_k]} \left(\sum_{(F_k)} \frac{m_i}{(x - x_i)^2} + \sum_{F-(F_k)} \frac{m_i}{(x - x_i)^2} \right)^{\alpha/1+\alpha} dx \tag{4.1}$$

where $\int_{[F_k]}$ denotes the integral over the closed interval determined by the endpoints of F_k, $\Sigma_{(F_k)}$ means that we consider only i's for which x_i lies between the endpoints of F_k, and $\Sigma_{F-(F_k)}$ corresponds to i's for which x_i is either an endpoint or is exterior to F_k.

Since $\alpha/1+\alpha<1$, the function $u^{\alpha/1+\alpha}$ is subadditive, and the integral (4.1) is dominated by

$$(4.2)\qquad \int_{[F_k]}\left(\sum_{(F_k)}\frac{m_i}{(x-x_i)^2}\right)^{\alpha/1+\alpha}dx+\int_{[F_k]}\left(\sum_{F-(F_k)}\frac{m_i}{(x-x_i)^2}\right)^{\alpha/1+\alpha}dx.$$

By hypothesis, the first term does not exceed $C_1M_k^{\alpha/1+\alpha}E_k^{1/1+\alpha}$ where $M_k=\sum_{(F_k)}m_i$ and where $\Sigma_k=\sum_{[F_k]}\tau_i^{1-\alpha}$. Therefore, if we sum (4.2) over $k=1,\dots,m$, the total contribution from the first terms is bounded by $C_1\sum M_k^{\alpha/1+\alpha}E_k^{\alpha/1+\alpha}$, which, by Hölder's inequality, is no greater than $C_1\hat{M}^{\alpha/1+\alpha}E^{1/1+\alpha}$ where $\hat{M}=\sum_{F-G}m_i$ (i.e., the mass at each endpoint has been replaced by zero) and where $E=E(F)=\sum_{i=1}^n\tau_i^{1-\alpha}$. Since $\hat{M}<M(\mathfrak{M})=\sum_{i=1}^n m_i$, we have found that: *the sum of the first terms in* (4.2) *is no greater than* $C_1M(\mathfrak{M})^{\alpha/1+\alpha}E(F)^{1/1+\alpha}$.

We must now deal with the contribution of the second terms. To do this we need briefly to explore the effect of a certain kind of *redistribution of mass*. Suppose c is a point lying in an open interval (a,b) and that m_1, m_2, m_3 are nonnegative numbers, which we think of as masses concentrated at a, c, b, respectively. We wish to redistribute the mass m_2 equitably (and canonically) at the endpoints. We can write c uniquely as a convex combination of a and b, $c=\lambda a+\mu b$ where $0\leqslant\lambda$, μ and $\lambda+\mu=1$. Using the coefficients λ and μ, we shall define our equitable redistribution of mass to be the one which places total mass $m_1+\lambda m_2=M_1$ at the point a, total mass $m_3+\mu m_2=M_3$ at b, and mass 0 at the point c.

Now suppose we fix (for the moment) a point, x, *not lying in* $[a,b]$. Using the fact that the function $g(w)=1/(x-w)^2$ is a convex function on each of the intervals $(-\infty,x)$ and (x,∞), it is easily checked that

$$\frac{m_1}{(x-a)^2}+\frac{m_2}{(x-c)^2}+\frac{m_3}{(x-b)^2}\leqslant\frac{M_1}{(x-a)^2}+\frac{M_3}{(x-b)^2}.$$

By repeating this process, *we may redistribute the mass from any number of interior points to the endpoints, and the corresponding inequality will still hold.*

Returning now to our problem, recall that we need to estimate the sum, over k, of the second terms in (4.2). To begin with, we assert that, for each k

$$(4.3)\qquad \int_{[F_k]}\left(\sum_{F-(F_k)}\frac{m_i}{(x-x_i)^2}\right)^{\alpha/1+\alpha}dx\leqslant\int_{[F_k]}\left(\sum\frac{M_{n_i}}{(x-x_{n_i})^2}\right)^{\alpha/1+\alpha}dx,$$

where, in the right-hand side, the sum is taken over the endpoint set and where the mass of each block has been redistributed at its endpoints. To see this, we first apply the redistribution principle just established to obtain an intermediate inequality where all the mass, except that in the block F_k, is redistributed. But then, the right-hand side only increases if we now increase the mass at the endpoints of F_k by redistribution, so that (4.3) holds and, in particular, the integrand on the right no longer depends on k.

If we now sum over k and use the hypothesis that C_2 is admissible for the endpoint set, G, we see that the contribution of these terms is no greater than

$$\int_{x_1}^{x_{n_m}} \left(\frac{M_{n_i}}{(x - x_{n_i})^2} \right)^{\alpha/1-\alpha} dx \leqslant C_2\left(\sum M_{n_i} \right)^{\alpha/1+\alpha} E(G)^{1/1+\alpha}.$$

Now the total mass has not been changed; that is,

$$\sum_{i=1}^{m} M_{n_i} = \sum_{i=1}^{n} m_i = M(\mathfrak{M}).$$

But, $E(G) = \sum_{i=1}^{m} T_i^{1-\alpha}$ where $T_i = x_{n_i} - x_{n_{i-1}}$, and we certainly have $E(G) \leqslant \sum_{i=1}^{M} \tau_i^{1-\alpha} = E(F)$. Thus, the *total contribution of the second terms in* (4.2) *is no greater than* $C_2 M(\mathfrak{M})^{\alpha/1+\alpha} E(G)^{1/1+\alpha}$. This result, combined with the estimate previously established, proves the lemma.

With each finite point set F, we can associate a positive integer $r(F)$, called the *rank of monotonicity* of F, as follows: $r(F) = 1$ if F satisfies the monotonicity condition (i.e., the τ_i are either nondecreasing or nonincreasing); and, inductively, $r(F)$ is defined to be the least integer k for which F admits a block partition $F_1, \ldots, F_m$ with endpoint set G such that $r(F_i) \leqslant k - 1$ and $r(G) = 1$.

Combining this definition, Lemma 5.1, Theorem 1 and an obvious induction argument, we obtain a satisfying generalization of Theorem 1.

THEOREM 2. *If F is any finite set, then* $20r(F)/(1 - \alpha)$ *is an admissible constant for F.*

We end this section by noting that, even if we know nothing about F except its cardinality, it is possible to say something about $r(F)$. For it is easily established by induction that if $3 \leqslant \operatorname{card} F \leqslant 2^n + 1$, then $r(f) \leqslant n$.

5. An application in potential theory. The connection between the results of this paper and the characterization of Blaschke regions for certain classes of functions was treated at some length in the announcement [3], where the notion of harmonic entropy was introduced.

In this section we introduce a closely related but more elementary notion, *α-entropy*, and we show how our results readily lead to some useful estimates of this quantity. To explain the use of the term "entropy", which comes from Information Theory, we consider the following model.

Given α, F, and $\mathfrak{M}$, as usual, we define the associated *signal function*, $S(x; F, \mathfrak{M})$, and *noise function*, $N(x, F)$, by

$$S(x; F, \mathfrak{M}) = \sum \frac{m_i}{(x - x_i)^2}, \qquad N(x; F) = d(x, F)^{-(1+\alpha)},$$

where d denotes Euclidean distance on the real line.

We interpret the quantity $M = \sum m_i$, as the *strength* of the signal.

Regarding F and α as fixed, and $\mathfrak{M}$ as variable, it is natural to ask for an estimate of the minimum signal strength needed to overcome the noise. We call this quantity the *α-entropy*, or if α is fixed simply the *entropy*, of F and denote it by $\mathcal{E}(F)$; so, to be precise

$$\mathcal{E}(F) = \min\{M(\mathfrak{M}) \mid S(x; F, \mathfrak{M}) \geqslant N(x; F) \text{ for } x \in [x_0, x_n] \setminus F\}.$$

It turns out that the entropy is of the same order of magnitude as the quantity which we earlier denoted by $E(F)$.

THEOREM 3. *Let C be an admissible constant for F. Then*

$$\left(\frac{2^\alpha}{C(1-\alpha)}\right)^{(1+\alpha)/\alpha} E(F) \leqslant \mathcal{E}(F) \leqslant 2^\alpha E(F).$$

PROOF. The inequality on the right is established directly as follows. Given $F = \{x_0 < x_1 < \cdots < x_n\}$, let $m_0 = k\tau_1^{1-\alpha}$, $m_n = k\tau_n^{1-\alpha}$, and, for $0 < j < n$, let $m_j = k(\tau_i^{1-\alpha} + \tau_{i+1}^{1-\alpha})$ where $k = 2^{\alpha-1}$. We assert that for this collection $\mathfrak{M} = \{m_i\}$ of masses, we will have $S(x; F, \mathfrak{M}) \geqslant N(x; F)$. To see this, suppose $x \in [x_0, x_n] \setminus F$ and choose $x_{i_0} \in F$ so that $d(x, F) = |x - x_{i_0}|$. To illustrate how the argument goes, let us assume that $1 \leqslant i_0 \leqslant n$ and that x lies to the left of x_{i_0} so that $|x - x_{i_0}| \leqslant \frac{1}{2}\tau_{i_0}$.

In this case, we have

$$S(x; F, \mathfrak{M}) > \frac{m_{i_0}}{(x - x_{i_0})^2} \geqslant \frac{k\tau_{i_0}^{1-\alpha}}{(x - x_{i_0})^2} \geqslant \frac{k(2|x - x_{i_0}|)^{1-\alpha}}{(x - x_{i_0})^2}$$
$$= 2^{1-\alpha} k |x - x_{i_0}|^{1-\alpha} = N(x; F).$$

A similar argument works if x lies to the left of the nearest point in F. Thus, for this distribution $\mathfrak{M}$, we have $S(x; F, \mathfrak{M}) \geqslant N(x; F)$ for $x \in [x_0, x_n] \setminus F$.

We also have

$$M(\mathfrak{M}) = k\left(\tau_1^{1-\alpha} + \left(\tau_1^{1-\alpha} + \tau_2^{1-\alpha}\right) + \cdots + \left(\tau_{n-1}^{1-\alpha} + \tau_n^{1-\alpha}\right) + \tau_n^{1-\alpha}\right)$$
$$= 2kE(f) = 2^\alpha E(F),$$

which proves the right-hand inequality.

To establish the other inequality, we assume that we have a collection of masses $\mathfrak{M}$ for which $S(x; F, \mathfrak{M}) \geqslant N(x; F)$. Let us raise both sides to the power $\alpha/1 + \alpha$ and integrate from x_0 to x_n. The left side is, of course, just $I(F, \mathfrak{M})$. The right side can be computed exactly.

$$\int_{x_0}^{x_n} N(x; F)^{\alpha/1+\alpha} dx = 2 \sum_{i=1}^{n} \int_{x_{i-1}}^{(x_i + x_{i-1})/2} \frac{dx}{(x - x_{i-1})^\alpha} = \frac{2^\alpha}{1-\alpha} E(F).$$

Thus we have $2^\alpha E(F)/(1-\alpha) \leqslant I(F, \mathfrak{M}) \leqslant CM(\mathfrak{M})^{\alpha/1+\alpha} E(F)^{1/1+\alpha}$, since C was assumed to be an admissible constant for F. After a bit of algebra we find that

$$M(\mathfrak{M}) \geqslant \left(\frac{2^\alpha}{C(1-\alpha)}\right)^{(1+\alpha)/\alpha} E(F).$$

This establishes the left-hand inequality and completes the proof of the theorem.

We remark that, since one choice of C is $20r(F)/(1-\alpha)$, and since $0 < \alpha < 1$ so that $(1+\alpha)/\alpha > 2$ then the conclusion of Theorem 3 can very simply be stated: $(20r(F))^{-2/\alpha} E(F) \leqslant \mathcal{E}(F) \leqslant 2E(F)$.

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