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A RESTRICTION THEOREM FOR SEMISIMPLE LIE GROUPS OF RANK ONE

BY

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ABSTRACT. Let $\mathfrak{g}_{\mathbf{R}} = \mathfrak{k}_{\mathbf{R}} + \mathfrak{p}_{\mathbf{R}}$ be a Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g}_{\mathbf{R}}$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding complexification. Also let $\mathfrak{a}_{\mathbf{R}}$ be a maximal abelian subspace of $\mathfrak{p}_{\mathbf{R}}$ and let $\mathfrak{a}$ be the complex subspace of $\mathfrak{p}$ generated by $\mathfrak{a}_{\mathbf{R}}$. We assume $\dim \mathfrak{a}_{\mathbf{R}} = 1$. Now let G be the adjoint group of $\mathfrak{g}$ and let K be the analytic subgroup of G with Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$. If $S'(\mathfrak{g})$ denotes the ring of all polynomial functions on $\mathfrak{g}$ then clearly $S'(\mathfrak{g})$ is a G-module and a fortiori a K-module. In this paper, we determine the image of the subring $S'(\mathfrak{g})^K$ of K-invariants in $S'(\mathfrak{g})$ under the restriction map $f \mapsto f|_{\mathfrak{k}+\mathfrak{a}}$ ($f \in S'(\mathfrak{g})^K$).

1. Introduction. Consider a reductive Lie algebra $\mathfrak{g}_{\mathbf{R}}$ over $\mathbf{R}$, a fixed Cartan decomposition $\mathfrak{g}_{\mathbf{R}} = \mathfrak{k}_{\mathbf{R}} + \mathfrak{p}_{\mathbf{R}}$ and a maximal abelian subspace $\mathfrak{a}_{\mathbf{R}}$ of $\mathfrak{p}_{\mathbf{R}}$. Extend $\mathfrak{g}_{\mathbf{R}}$ to a Cartan subalgebra $\mathfrak{h}$ of the complexification $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g}_{\mathbf{R}}$ in the usual way. By ϕ we shall denote the set of roots of the pair $(\mathfrak{g}, \mathfrak{h})$ and by W_ϕ the corresponding Weyl group, whereas Δ will denote the set of roots of the pair $(\mathfrak{g}_{\mathbf{R}}, \mathfrak{a}_{\mathbf{R}})$ and W_Δ the corresponding Weyl group. Let G be the adjoint group of $\mathfrak{g}$ and let K be the analytic subgroup of G with Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$. If H is a group and V a finite dimensional H-module over $\mathbf{C}$, let $S'(V)$ denote the ring of all polynomial functions on V and let $S'(V)^H$ denote the subring of H invariants.

Fundamental for many questions in representation theory is the following Chevalley's Restriction Theorem:

(i) *The operation of restriction from* $\mathfrak{g}$ *to* $\mathfrak{h}$ *induces an isomorphism of* $S'(\mathfrak{g})^G$ *onto* $S'(\mathfrak{h})^{W_\phi}$;

(ii) *The operation of restriction from* $\mathfrak{p}$ *to* $\mathfrak{a}$ *induces an isomorphism of* $S'(\mathfrak{p})^K$ *onto* $S'(\mathfrak{a})^{W_\Delta}$.

Also we have a theorem of the same nature due to Helgason: If $\mathfrak{g}_{\mathbf{R}}$ is a classical semisimple Lie algebra (with real or complex structure) then the restriction from $\mathfrak{h}$ to $\mathfrak{a}$ maps $S'(\mathfrak{h})^{W_\phi}$ onto $S'(\mathfrak{a})^{W_\Delta}$. This does not hold in general for the real forms of the exceptional Lie algebras E_6, E_7, E_8, but in any event, the elements in $S'(\mathfrak{a})^{W_\Delta}$ are all obtained from rational invariants on $\mathfrak{h}$ by restriction. In fact we have the following result of Harish-Chandra and Helgason: *Let* $Q(S'(\mathfrak{h})^{W_\phi})$ *and* $Q(S'(\mathfrak{a})^{W_\Delta})$ *denote the quotient fields of* $S'(\mathfrak{h})^{W_\phi}$ *and* $S'(\mathfrak{a})^{W_\Delta}$ *respectively*; *then the restriction from* $\mathfrak{h}$ *to* $\mathfrak{a}$ *induces a mapping of* $Q(S'(\mathfrak{h})^{W_\phi})$ *onto* $Q(S'(\mathfrak{a})^{W_\Delta})$ (for all this see §2.1.5 in **[4]**).

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This paper is concerned with the determination of the image of the homomorphism of $S'(\mathfrak{g})^K$ into $S'(\mathfrak{k}+\mathfrak{a})$ induced by restriction from $\mathfrak{g}$ to $\mathfrak{k}+\mathfrak{a}$. In [3] a suitable element $b \in S'(\mathfrak{g})^K$ is defined and the following theorem of Kostant is proved: *Let $S'(\mathfrak{g})^K_b$ be the localization of $S'(\mathfrak{g})^K$ by b (i.e. the ring of all rational functions on $\mathfrak{g}$ of the form f/b^k where $f \in S'(\mathfrak{g})^K$ and $k \in Z$) and let $S'(\mathfrak{k}+\mathfrak{a})^{M'}_{b_0}$ be the localization of $S'(\mathfrak{k}+\mathfrak{a})^{M'}$ by $b_0 = b_{|\mathfrak{k}+\mathfrak{a}}$, M' being the normalizer of $\mathfrak{a}$ in K; then the restriction from $\mathfrak{g}$ to $\mathfrak{k}+\mathfrak{a}$ induces an isomorphism of $S'(\mathfrak{g})^K_b$ onto $S'(\mathfrak{k}+\mathfrak{a})^{M'}_{b_0}$.*

Starting from this result we are able to characterize the image of $S'(\mathfrak{g})^K$ in $S'(\mathfrak{k}+\mathfrak{a})^{M'}$ in the split rank one case. Thus, from now on we assume that $\mathfrak{a}_{\mathbf{R}} \cap [\mathfrak{g}_{\mathbf{R}}, \mathfrak{g}_{\mathbf{R}}]$ is one dimensional. If M is the centralizer of $\mathfrak{a}$ in K then $M'/M \simeq W_\Delta$. Let T denote the set of equivalence classes of irreducible, finite dimensional holomorphic representations of K admitting an M-fixed vector. Using results in [2] we prove that any $\tau \in \mathrm{T}$ can be realized as a K-submodule of homogeneous harmonic elements in $S(\mathfrak{p})$. The degree of these elements, $d(\tau)$, is uniquely determined by τ. Let $S'_n(\mathfrak{a})$ denote the homogeneous subspace of $S'(\mathfrak{a})$ of degree n, and let $S'(\mathfrak{k})_\tau$ denote the primary component of $S'(\mathfrak{k})$ of type τ. Our main result is the following.

THEOREM. *The operation of restriction from $\mathfrak{g}$ to $\mathfrak{k}+\mathfrak{a}$ induces an isomorphism of $S'(\mathfrak{g})^K$ onto*

$$\bigoplus_{n \geq 0} \bigoplus_{\substack{\tau \in \mathrm{T} \\ d(\tau) \leq n}} \left(S'(\mathfrak{k})^M_\tau \otimes S'_n(\mathfrak{a}) \right)^{W_\Delta}.$$

Let $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{Q}$ denote the universal enveloping algebras over $\mathbf{C}$, of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{a}$, respectively. Also let $\mathcal{G}^K$ and $\mathcal{K}^M$ be the centralizers of K in $\mathcal{G}$ and of M in $\mathcal{K}$, respectively. In many fundamental questions concerning the infinite dimensional representation theory of a Lie group with Lie algebra $\mathfrak{g}_R$ it would be very important to know the image of an injective antihomomorphism $P: \mathcal{G}^K \to \mathcal{K}^M \otimes \mathcal{Q}$, due to Lepowsky and Rader (see [5 or 6]), which replaces the famous Harish-Chandra homomorphism $\gamma: \mathcal{G}^K \to \mathcal{Q}$ (see [3]). Our main theorem should prove useful in this respect.

2. We use much of the notation in [2]. Thus $\mathfrak{g}_{\mathbf{R}} = \mathfrak{k}_{\mathbf{R}} + \mathfrak{p}_{\mathbf{R}}$ is a Cartan decomposition of a real reductive Lie algebra $\mathfrak{g}_{\mathbf{R}}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the corresponding complexification. The associated Cartan involution θ is 1 on $\mathfrak{k}$ and -1 on $\mathfrak{p}$. Also, $\mathfrak{a}_{\mathbf{R}}$ is a maximal abelian subspace of $\mathfrak{p}_{\mathbf{R}}$, so that its complexification $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{p}$. Let G be the adjoint group of $\mathfrak{g}$ and let K_θ be the subgroup of all elements in G which commute with θ. Clearly $\mathfrak{k}$ and $\mathfrak{p}$ are stable under the action of K_θ. Now if K denotes the analytic subgroup of G with Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$, then K is the identity component of K_θ. Moreover, if F is the set of all elements of order 2 in the connected Lie subgroup A of G corresponding to $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{a})$, then F is a finite abelian group of order $2^{\dim_{\mathbf{C}}(\mathfrak{a})}$ which normalizes K and such that $K_\theta = KF$ (see Proposition 1, p. 761 in [2]).

For any vector space V let $S'(V)$ denote the ring of all polynomial functions on V, and for every nonnegative integer i let $S'_i(V)$ denote the homogeneous subspace of

$S^i(V)$ of degree i. Then $S'(\mathfrak{g})$ is a K_θ-module: if $f \in S'(\mathfrak{g})$ and $a \in K$ then $af \in S'(\mathfrak{g})$ is given by $(af)(x) = f(a^{-1} \cdot x)$, $x \in \mathfrak{g}$. Let $S'(\mathfrak{g})^K$ be the ring of K-invariant polynomials.

The injection map $\mathfrak{k} + \mathfrak{a} \to \mathfrak{k} + \mathfrak{p} = \mathfrak{g}$ induces contravariantly the restriction homomorphism $S'(\mathfrak{g}) \to S'(\mathfrak{k} + \mathfrak{a})$. This homomorphism restricted to $S'(\mathfrak{g})^K$ induces a homomorphism

$$\pi\colon S'(\mathfrak{g})^K \to S'(\mathfrak{k} + \mathfrak{a}). \tag{2.1}$$

In [3] a homogeneous polynomial $b \in S'(\mathfrak{g})^K$ is defined such that $b(x + y) = b(y)$ for all $x \in \mathfrak{k}$ and $y \in \mathfrak{p}$. If M' is the normalizer of $\mathfrak{a}$ in K then M' operates on $\mathfrak{k} + \mathfrak{a}$ and we may consider $S'(\mathfrak{k} + \mathfrak{a})^{M'}$, the ring of M'-invariant polynomials. Let $S'(\mathfrak{g})_b^K$ be the localization of $S'(\mathfrak{g})^K$ by b, so that $S'(\mathfrak{g})_b^K$ is the ring of all rational functions on $\mathfrak{g}$ of the form f/b^q where $f \in S'(\mathfrak{g})^K$ and $q \in \mathbf{Z}$. If $S'(\mathfrak{k} + \mathfrak{a})_{b_0}^{M'}$ denotes the corresponding localization of $S'(\mathfrak{k} + \mathfrak{a})^{M'}$ by $b_0 = \pi(b)$, then we know that (2.1) extends to an isomorphism of algebras

$$\pi\colon S'(\mathfrak{g})_b^K \to S'(\mathfrak{k} + \mathfrak{a})_{b_0}^{M'} \tag{2.2}$$

(see Theorem 6.1, p. 147 in [3]). In particular (2.1) is injective. In this paper we describe its image when $\dim \mathfrak{a} = 1$.

First, we recall a few basic facts about S-triples in $\mathfrak{g}$. An *S-triple* is a set of 3 linearly independent elements (x, e, f) in $\mathfrak{g}$ where $[x, e] = 2e$, $[x, f] = -2f$ and $[e, f] = x$. It is called *normal* in case $e, f \in \mathfrak{p}$ and $x \in \mathfrak{k}$. For any $y \in \mathfrak{p}$ let $\mathfrak{p}^y$ denote the centralizer of y in $\mathfrak{p}$. Then $y \in \mathfrak{p}$ is called *regular* if $\dim \mathfrak{p}^y \leqslant \dim \mathfrak{p}^u$ for any $u \in \mathfrak{p}$. Also $y \in \mathfrak{p}$ is regular if and only if $\dim \mathfrak{p}^y = \dim \mathfrak{a}$ (see Propositions 7 and 8, p. 770 in [2]). A normal S-triple (x, e, f) is called *principal* if e (and hence f) is regular. Theorem 3, p. 773 in [2] guarantee that they exist. Now fix a closed Weyl chamber $D \subset \mathfrak{a}_{\mathbf{R}}$. A normal S-triple (x, e, f) is called *standard* if $e + f \in D$. Let Δ be the set of roots of $(\mathfrak{g}, \mathfrak{a})$ and let $\Pi \subset \Delta$ be the set of simple positive roots corresponding to D. Also let w be the unique element in $\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}]$ such that $\langle \lambda, w \rangle = 2$ for all $\lambda \in \Pi$ (obviously $w \in D$). From [2] we know: any normal S-triple is K-conjugate to a standard S-triple (Lemma 6, p. 776); a standard normal S-triple (x, e, f) is principal if and only if $e + f = w$ (Proposition 13, p. 776); any two principal normal S-triples are K_θ-conjugate (Theorem 6, p. 778).

From now on, we shall assume that $\mathfrak{a}_{\mathbf{R}} \cap [\mathfrak{g}_{\mathbf{R}}, \mathfrak{g}_{\mathbf{R}}]$ is *one dimensional*, that is, that $\mathfrak{g}_{\mathbf{R}}$ is of *split rank one*. Also (x, e, f) will be a principal normal S-triple and $z = x/2$.

Proposition 1. *ad* $z\colon \mathfrak{p} \to \mathfrak{p}$ *is diagonalizable with eigenvalues* 1, −1 *and possibly* 0.

Proof. We may assume that (x, e, f) is a standard principal normal S-triple. One knows that $\Pi = \{\lambda\}$ and that the root space decomposition of $\mathfrak{g}$ is of the form $\mathfrak{g} = \mathfrak{g}^0 + \mathfrak{g}^\lambda + \mathfrak{g}^{-\lambda} + \mathfrak{g}^{2\lambda} + \mathfrak{g}^{-2\lambda}$; here $\mathfrak{g}^{2\lambda}$ and $\mathfrak{g}^{-2\lambda}$ can be zero and $\mathfrak{g}^0 = \mathfrak{m} + \mathfrak{a}$, where $\mathfrak{m}$, as usual, is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Therefore, the eigenvalues of ad w, $w = e + f$, in $\mathfrak{g}$ are 2, 0, −2 and possibly 4 and −4. Now x and w are G-conjugate, so the eigenvalues of ad x in $\mathfrak{g}$ are the same as those of ad w. Since $\dim(p^e \cap [\mathfrak{g}, \mathfrak{g}]) = 1$, $\mathfrak{p}^e \cap [\mathfrak{g}, \mathfrak{g}] = Ce$. Therefore, up to a scalar, e is the unique highest weight vector in

$\mathfrak{p}^e \cap [\mathfrak{g}, \mathfrak{g}]$ of the TDS (three dimensional simple Lie algebra over **C**) spanned by (x, e, f). From the representation theory of a TDS now follows that 4 cannot be an eigenvalue of ad x in $\mathfrak{p}$. For the same reason -4 cannot be a lowest weight of x in $\mathfrak{p}$. Q.E.D.

REMARK. The multiplicity of ± 1 as eigenvalues of ad z in $\mathfrak{p}$ is $1 + \dim \mathfrak{g}^{2\lambda}$. This follows immediately from the representation theory of a TDS and the fact that $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$.

Since the polynomial b is K-invariant the closed algebraic set $V(b) = \{y \in \mathfrak{p}: b(y) = 0\}$ is stable under the action of K. Regarding its K-structure in the rank one case we have the following facts.

PROPOSITION 2. *The algebraic set $V(b)$ is irreducible and of codimension 1 in $\mathfrak{p}$. Moreover there are only a finite number of K orbits in $V(b)$. Furthermore if (x, e, f) is a principal normal S-triple in $\mathfrak{g}$, then $K \cdot e$ and $K \cdot f$ are the unique orbits in $V(b)$ of maximal dimension so that $K \cdot e \cup K \cdot f$ is Zariski open and dense in $V(b)$.*

PROOF. We know that the ring $S'(\mathfrak{p})^K$ of K-invariant polynomial functions in $\mathfrak{p}$ is a polynomial ring $C[u]$ in one homogeneous polynomial u (because $\dim \mathfrak{a} = 1$). Since b is also homogeneous we may assume that $b = u^r$ for some positive integer r. Therefore $V(b) = u^{-1}(0)$. On the other hand we have a homomorphism of K into the group of permutations of the set of irreducible factors of u. Since K is connected each irreducible factor is K-invariant. This implies the irreducibility of u and hence of $V(b)$. According to Theorem 9, p. 785 in **[2]** there are only a finite number of K_θ-orbits, hence of K-orbits, in $u^{-1}(0)$, and the set of regular elements in $u^{-1}(0)$ is the unique K_θ-orbit of maximal dimension in $u^{-1}(0)$.

Since e and f are nilpotent elements $b(e) = b(f) = 0$ (see Corollary 5.5, p. 147 in **[3]**), therefore $K_\theta \cdot e = K_\theta \cdot f$ is the orbit of regular elements in $V(b)$. Without loss of generality we may assume that (x, e, f) is a standard principal normal S-triple, so that $w = e + f$. In the rank one case under consideration, the abelian group F of all elements of order 2 in A is of order 2 and $F = \{1, a\}$ where $a = \mathrm{Ad}(\exp \pi i w/2)$. If we put

$$u = \tfrac{1}{2}(e - f - x) \quad \text{and} \quad v = \tfrac{1}{2}(f - e - x)$$

then (w, u, v) is a new S-triple. But $e = (u - v + w)/2$, thus $a \cdot e = f$. Therefore $K_\theta \cdot e = KF \cdot e = K \cdot e \cup K \cdot f$. Q.E.D.

Note that all eigenvalues of z in $S'(\mathfrak{k})$ are integers and that if $S'(\mathfrak{k})_j$ is the eigenspace of $S'(\mathfrak{k})$ corresponding to the eigenvalue j then

$$S'(\mathfrak{k}) = \bigoplus_{j=-\infty}^{\infty} S'(\mathfrak{k})_j.$$

Moreover $S'(\mathfrak{g}) = S'(\mathfrak{k} + \mathfrak{p}) = S'(\mathfrak{k}) \otimes S'(\mathfrak{p})$, in other words an element $u \in S'(\mathfrak{g})$ can be viewed as a polynomial on $\mathfrak{p}$ with values in $S'(\mathfrak{k})$. In particular b is a polynomial on $\mathfrak{p}$ with values in C and u/b^q is a rational function on $\mathfrak{p}$ with values in $S'(\mathfrak{k})$.

THEOREM 3. *Let* u/b^q $(u \in S'(\mathfrak{g})^K)$ *be a rational function on* $\mathfrak{p}$ *homogeneous of degree* $n \geqslant 0$. *Then* u/b^q *is a polynomial if and only if*

$$\frac{u}{b^q}(y) \in \bigoplus_{j=-n}^{n} S'(\mathfrak{k})_j \tag{2.3}$$

for all $y \in \mathfrak{p}$ *such that* $b(y) \neq 0$.

PROOF. We can write uniquely $u = \Sigma u_j$ (finite sum) where u_j is a polynomial function on $\mathfrak{p}$ with values in $S'(\mathfrak{k})_j$. Clearly u/b^q is a polynomial if and only if u_j/b^q is a polynomial for all j.

Since the eigenvalues of z in $\mathfrak{p}$ are 1, 0, -1 (Proposition 1) there exist

$$\lim_{t \to +\infty} e^{-t}\operatorname{Ad}(\exp tz)\cdot y = y_+$$

and

$$\lim_{t \to -\infty} e^{t}\operatorname{Ad}(\exp tz)\cdot y = y_-$$

for all $y \in \mathfrak{p}$.

If u/b^q is a polynomial on $\mathfrak{p}$ we have

$$\frac{u_j}{b^q}(y_+) = \lim_{t \to +\infty}\left(e^{-t}\operatorname{Ad}(\exp tz)\cdot y\right) = \lim_{t \to +\infty} e^{(j-n)t}\frac{u_j}{b^q}(y). \tag{2.4}$$

If $u_j \neq 0$ we can certainly choose $y \in \mathfrak{p}$ such that $u_j(y) \neq 0$, then (2.4) implies that $j \leqslant n$. Similarly, letting $t \to -\infty$ we obtain that $j \geqslant -n$.

Conversely if (2.3) holds and $b(y) \neq 0$, $y \in \mathfrak{p}$, then

$$\lim_{t \to +\infty} \frac{u}{b^q}\left(e^{-t}\operatorname{Ad}(\exp tz)\cdot y\right) = \frac{u_n}{b^q}(y).$$

But

$$b(y_+) = \lim_{t \to +\infty}\left(e^{-t}\operatorname{Ad}(\exp tz)\cdot y\right) = \lim_{t \to +\infty} b\left(e^{-t}y\right) = 0,$$

therefore $u(y_+) = 0$ whenever $q > 0$. Similarly we obtain that $u(y_-) = 0$, for all $y \in \mathfrak{p}$ such that $b(y) \neq 0$. The element $e + f$ is K-conjugate to w, thus $e + f$ is regular and semisimple, hence $b(e + f) \neq 0$ (see §5 in [3]). Now $(e + f)_+ = e$ and $(e + f)_- = f$, thus $u(e) = u(f) = 0$. Therefore u vanishes on $K \cdot e \cup K \cdot f$ and hence u is zero on $V(b)$ (see Proposition 2). By Hilbert's Nullstellensatz b divides u^m for some m. Since b is a power of a prime polynomial (see the proof of Proposition 2), it follows that u/b^q is a polynomial. Q.E.D.

If V denotes a K-submodule of $S'(\mathfrak{k})$ then V_j is the eigenspace of z in V corresponding to the eigenvalue j.

LEMMA 4. *Let* $f \neq 0$ *be a* K*-invariant rational function on* $\mathfrak{p}$ *with values in an irreducible* K*-submodule* V *of* $S'(\mathfrak{k})$. *Then the following conditions are equivalent*:

(i) $f(y) \in \bigoplus_{j=-n}^{n} S'(\mathfrak{k})_j$ *for all* $y \in \mathfrak{p}$ *where* f *is defined*;

(ii) $V = \bigoplus_{j=-n}^{n} V_j$.

PROOF. That (ii) implies (i) is obvious. So assume (i) and take $v = f(y) \neq 0$, $y \in \mathfrak{p}$. Then

$$k \cdot v = k \cdot f(y) = f(k \cdot y) \in \bigoplus_{j=-n}^{n} V_j \quad \text{for all } k \in K.$$

Thus the cyclic K-submodule generated by v is contained in $\oplus_{j=-n}^{n} V_j$. By irreducibility V coincides with this cyclic module, hence $V = \oplus_{j=-n}^{n} V_j$. Q.E.D.

Now let M be the connected Lie subgroup of G corresponding to $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{m})$, where $\mathfrak{m}$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. If the rational function $f = u/b^q$ on $\mathfrak{p}$, $0 \neq u \in S'(\mathfrak{g})^K$, takes its values in a K-submodule of $S'(\mathfrak{k})$ then the space V^M of M-invariants in V is different from zero. In fact, since the restriction homomorphism (2.2) is injective, there exists $y \in \mathfrak{a}$ such that $(u/b^q)(y) = v \neq 0$. But $k \cdot v = (u/b^q)(k \cdot y) = (u/b^q)(y) = v$ for all $k \in M \subset K$.

This observation and the statements of Theorem 3 and Lemma 4 lead us to consider K-irreducible submodules V of $S'(\mathfrak{k})$ such that $V^M \neq 0$ and $V = \oplus_{j=-n}^{n} V_j$. Let T denote the set of all equivalence classes of irreducible holomorphic finite dimensional K-modules V_τ such that $V_\tau^M \neq 0$.

Let H be the subspace of the symmetric algebra $S(\mathfrak{p})$ over $\mathfrak{p}$ spanned by all powers e^k, $k = 0, 1, 2, \ldots$, of all nilpotent elements e in $\mathfrak{p}$. It is clear that H is a homogeneous subspace of $S(\mathfrak{p})$ and that H is stable under the action of K_θ. The elements in H are called the harmonic elements in $S(\mathfrak{p})$. Now let M_θ be the centralizer of $\mathfrak{a}$ in K_θ. Then $M_\theta = MF$ (see Lemma 20, p. 803 in [**2**]). Let Γ be the set of all equivalence classes of irreducible holomorphic finite dimensional K_θ-modules V_γ such that the space $V_\gamma^{M_\theta}$ of M_θ-invariants in V_γ is different from zero. If H_γ is the set of all $h \in H$ which transform under K_θ according to γ, then $H = \oplus_{\gamma \in \Gamma} H_\gamma$. Moreover, in the rank one case, each H_γ is K_θ-irreducible (see Theorem 2.2.9 in [**1**]). Since H_γ is unique there exists a nonnegative integer $d(\gamma)$ such that H_γ is pure homogeneous of degree $d(\gamma)$. That is, $d(\gamma)$ gives the degree in which γ occurs harmonically in $S(\mathfrak{p})$. The remarkable fact is that $d(\gamma)$ can be obtained from the abstract K_θ-module V_γ. Indeed, if (x, e, f) is a principal normal S-triple and $z = x/2$ then $d(\gamma)$ equals the highest eigenvalue of z in V_γ (Corollary 2.2.5. in [**1**]).

PROPOSITION 5. *Each irreducible K-module V_τ, $\tau \in$ T, is isomorphic to a K-submodule of a K_θ-module V_γ of type $\gamma \in \Gamma$.*

PROOF. We may assume that $K_\theta \neq K$; in this case the nontrivial element $a \in F$ is not in K. Given a K-module V of type $\tau \in$ T we define a structure of K-module on $V \times V$. Let φ denote the automorphism of K_θ defined by conjugation by a. Let

$$\gamma(k)(x, y) = (kx, \varphi(k)y) \quad \text{and} \quad \gamma(ka)(x, y) = (ky, \varphi(k)x)$$

for all $(x, y) \in V \times V$ and all $k \in K$. Since $K_\theta = KF$, $\varphi(K) \subset K$ and $K \cap F = \{1\}$, γ is well defined on K_θ. Moreover it is easy to check that γ is a representation of K_θ on $V \times V$. By hypothesis there exists $0 \neq x \in V^M$; then $(x, x) \in (V \times V)^{M_\theta}$ since $M_\theta = MF$ and $\varphi(M) \subset M$. Thus $(V \times V)^{M_\theta} \neq 0$. If $V \times V$ is K_θ-irreducible we are done since V is isomorphic to $V \times \{0\}$ as K-modules. If not $V \times V = W \oplus W'$ where W and W' are irreducible K_θ-submodules (K_θ is a reductive group) and we may assume that $W \simeq V \times \{0\}$ and $W' \simeq \{0\} \times V$ as K-modules. If $W^{M_\theta} \neq 0$, W is a K_θ-module of type $\gamma \in \Gamma$ containing a K-submodule of type τ. If not, $W'^{M_\theta} \neq 0$. Then we define $\gamma(u)w' = \varphi(u)w'$ for all $w' \in W'$ and all $u \in K_\theta$. The K_θ-module (W', γ) belongs to Γ and as a K-module is isomorphic to V. Q.E.D.

The following result is of independent interest and it will be used later in this paper.

PROPOSITION 6. *When $\mathfrak{k}$ is not abelian (this means essentially $\mathfrak{g}_{\mathbf{R}} \neq \mathfrak{sl}(2, \mathbf{R})$) given a principal normal S-triple (x, e, f) in $\mathfrak{g}$ there exist elements E and F in $\mathfrak{k}$ and a real number c such that (cx, E, F) is an S-triple in $\mathfrak{k}$.*

PROOF. Let $\mathfrak{h}_{\mathbf{R}}$ be a maximal abelian subalgebra of $\mathfrak{g}_{\mathbf{R}}$ containing $\mathfrak{a}_{\mathbf{R}}$. The complexification $\mathfrak{h}$ of $\mathfrak{h}_{\mathbf{R}}$ is a Cartan subalgebra of $\mathfrak{g}$; let ϕ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$. Let σ and τ denote the conjugations of $\mathfrak{g}$ with respect to $\mathfrak{g}_{\mathbf{R}}$ and $\mathfrak{u} = \mathfrak{k}_{\mathbf{R}} + i\mathfrak{p}_{\mathbf{R}}$, respectively. If $\mathfrak{k}$ is not abelian then there exists $\alpha \in \phi$ such that $\alpha \neq \alpha^\sigma$ and $\alpha|_{\mathfrak{a}} = \lambda$. In fact we may assume that $\mathfrak{g}_{\mathbf{R}}$ is simple. Now if $\alpha = \alpha^\sigma$ for all $\alpha \in \phi$ such that $\alpha|_{\mathfrak{a}} = \lambda$ then $\dim \mathfrak{g}^\lambda = 1$, $\mathfrak{g}^{2\lambda} = \{0\}$ and $[\mathfrak{g}^\lambda, \mathfrak{g}^{-\lambda}] \subset \mathfrak{a}$. Hence $\mathfrak{g}_{\mathbf{R}} = \mathfrak{m}_{\mathbf{R}} + \mathfrak{a}_{\mathbf{R}} + \mathfrak{g}_{\mathbf{R}}^{\lambda} + \mathfrak{g}_{\mathbf{R}}^{-\lambda}$ where $\mathfrak{m}_{\mathbf{R}} = \mathfrak{m} \cap \mathfrak{g}_{\mathbf{R}}$ and $\mathfrak{g}_{\mathbf{R}}^{\pm\lambda} = \mathfrak{g}^{\pm\lambda} \cap \mathfrak{g}_{\mathbf{R}}$. Furthermore $\mathfrak{a}_{\mathbf{R}} + \mathfrak{g}_{\mathbf{R}}^{\lambda} + \mathfrak{g}_{\mathbf{R}}^{-\lambda}$ is an ideal in $\mathfrak{g}_{\mathbf{R}}$, thus $\mathfrak{g}_{\mathbf{R}} = \mathfrak{a}_{\mathbf{R}} + \mathfrak{g}_{\mathbf{R}}^{\lambda} + \mathfrak{g}_{\mathbf{R}}^{-\lambda}$ and $\mathfrak{k}_{\mathbf{R}}$ is one dimensional.

Take $\alpha \in \phi$ such that $\alpha^\sigma \neq \alpha$ and $\alpha|_{\mathfrak{a}} = \lambda$. Then $(\alpha, \alpha^\sigma) \leq 0$ since $\alpha^\sigma - \alpha \notin \phi$ (Lemma 1.1.3.6, p. 25 in [4]). The quadratic form $B(X, \tau X)$ is negative definite on $\mathfrak{g}$, thus we may choose $X \in \mathfrak{g}^{-\alpha}$ such that

$$B(X, \tau X) = \frac{2}{2(\alpha, \alpha^\sigma) - (\alpha, \alpha)}.$$

Let $X' = -\tau X \in \mathfrak{g}^\alpha$ and put $E = X + \theta X$, $F = X' + \theta X'$, $y = b(X' + \sigma X') \in \mathfrak{g}^\lambda$ where $b^2 = (2(\alpha, \alpha^\sigma) - (\alpha, \alpha))/((\alpha, \alpha) + (\alpha, \alpha^\sigma)) < 0$. Then

$$\begin{aligned}[y, \theta y] &= b^2[X' + \sigma X', \theta X' + \tau X'] \\ &= b^2([X', \tau X'] + \sigma[X', \tau X'])\end{aligned}$$

since $[X', \theta X'] = 0$ because $\theta X' \in \mathfrak{g}^{-\alpha^\sigma}$ and $\alpha - \alpha^\sigma \notin \phi$. But

$$[X', \tau X'] = -[X', X] = -B(X', X)H_\alpha = \frac{2H_\alpha}{2(\alpha, \alpha^\sigma) - (\alpha, \alpha)}, \tag{2.5}$$

thus

$$[y, \theta y] = \frac{2b^2}{2(\alpha, \alpha^\sigma) - (\alpha, \alpha)}(H_\alpha + \sigma H_\alpha) \in \mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}].$$

Moreover

$$\frac{2b^2}{2(\alpha, \alpha^\sigma) - (\alpha, \alpha)}\lambda(H_\alpha + \sigma H_\alpha) = \frac{2b^2((\alpha, \alpha) + (\alpha, \alpha^\sigma))}{2(\alpha, \alpha^\sigma) - (\alpha, \alpha)} = 2$$

therefore $[y, \theta y] = w$ and $(w, y, \theta y)$ is an S-triple. Now if we put $x = y + \theta y$, $e = (w - y + \theta y)/2$, $f = (w + y - \theta y)/2$ we get a standard normal principal S-triple, since $e + f = w$.

On the other hand if $T=[E, F]$ one can easily verify that (T, E, F) is an S-triple. In fact,

$$\begin{aligned} T &= [X+\theta X, X'+\theta X'] \\ &= \frac{2}{2(\alpha,\alpha^\sigma)-(\alpha,\alpha)}(H_\alpha+\theta H_\alpha)+[X,\theta X']+\theta[X,\theta X']. \end{aligned}$$

Then

$$\begin{aligned} [T,E] = {} & \frac{2(-(\alpha,\alpha)+(\alpha,\alpha^\sigma))}{2(\alpha,\alpha^\sigma)-(\alpha,\alpha)}(X+\theta X) \\ & +[[X,\theta X'],\theta X]+\theta[[X,\theta X'],\theta X] \end{aligned}$$

since $[[X, \theta X'], X]=0$, because -3λ is not a restricted root. But

$$\begin{aligned} [[X,\theta X'],\theta X] &= -[[\theta X',\theta X],X]-[[\theta X,X],\theta X'] \\ &= \frac{2(\alpha,\alpha^\sigma)}{2(\alpha,\alpha^\sigma)-(\alpha,\alpha)}X \end{aligned}$$

where we used (2.5) and $[\theta X, X]=0$ since $-\alpha^\theta=\alpha^\sigma$ and $\alpha^\sigma-\alpha\notin\phi$. Therefore

$$\begin{aligned} [T,E] = \frac{2}{2(\alpha,\alpha^\sigma)-(\alpha,\alpha)} & (-(\alpha,\alpha)+(\alpha,\alpha^\sigma))(X+\theta X) \\ & +(\alpha,\alpha^\sigma)X+(\alpha,\alpha^\sigma)\theta X)=2E. \end{aligned}$$

In a similar way we obtain that $[T, F]=-2F$.

Now

$$x=y+\theta y=b(X'+\sigma X'+\theta X'+\tau X')=b(F-E).$$

But $F-E$ is K-conjugate to $i[E, F]=iT$, thus $(-i/b)x$ is the semisimple element of an S-triple in $\mathfrak{k}$. Since any two principal normal S-triples are K_θ-conjugate the proposition is proved with $c=-i/b$. Q.E.D.

THEOREM 7. *Let (x, e, f) be a principal normal S-triple in $\mathfrak{g}$.*

(i) *If $\mathfrak{g}^{2\lambda}=0$ and $\dim \mathfrak{g}^\lambda>1$ there exist $E, F\in\mathfrak{k}$ such that (x, E, F) is an S-triple in $\mathfrak{k}$.*

(ii) *When $\mathfrak{g}^{2\lambda}\neq 0$, $x/2$ is the semisimple element of an S-triple in $\mathfrak{g}$.*

PROOF. By Proposition 6 we just need to compute

$$b^2=\frac{2(\alpha,\alpha^\sigma)-(\alpha,\alpha)}{(\alpha,\alpha)+(\alpha,\alpha^\sigma)}=\frac{2(\alpha,\alpha^\sigma)/(\alpha,\alpha)-1}{1+(\alpha,\alpha^\sigma)/(\alpha,\alpha)}.$$

When $\mathfrak{g}^{2\lambda}=0$ Lemma 3 in Appendix 2, p. 33 in **[4]** gives $(\alpha,\alpha^\sigma)=0$, hence $b^2=-1$ and we may choose $b=-i$ to get $c=1$.

If $\mathfrak{g}^{2\lambda}\neq 0$ the same lemma tells that $(\alpha,\alpha^\sigma)<0$, therefore $2(\alpha,\alpha^\sigma)/(\alpha,\alpha)=-1$, -2 or -3. But

$$0<\frac{4(\lambda,\lambda)}{(\alpha,\alpha)}=\frac{2\alpha(H_\alpha+\sigma H_\alpha)}{(\alpha,\alpha)}=2+2\frac{(\alpha,\alpha^\sigma)}{(\alpha,\alpha)}$$

thus $2(\alpha,\alpha^\sigma)/(\alpha,\alpha)=-1$. Hence $b^2=-4$ and we may take $b=-2i$ to obtain $c=1/2$. Q.E.D.

PROPOSITION 8. *If V_γ and $V_{\gamma'}$ are K_θ-modules of type $\gamma, \gamma' \in \Gamma$ both of which contain a K-submodule of the same type, then $d(\gamma) = d(\gamma')$.*

PROOF. Let W be a K-irreducible submodule of V_γ. Then an aW is also a K-submodule, hence $W \cap aW$ is equal to W or to $\{0\}$ and correspondingly $V_\gamma = W$ or $V_\gamma = W \oplus aW$ since $W + aW$ is a K_θ-submodule of V_γ.

Let (x, e, f) be a standard principal normal S-triple and let $z = x/2$. If $V_\gamma = W \oplus aW$ then $d(\gamma)$ is the highest eigenvalue of z in W or in aW. If $z(aw) = d(\gamma)aw$ for $0 \neq w \in W$ then $(az)w = d(\gamma)w$. But $az = -z$ thus $zw = -d(\gamma)w$. If $[\mathfrak{g}_{\mathbf{R}}, \mathfrak{g}_{\mathbf{R}}] \neq \mathfrak{sl}(2, \mathbf{R})$ then z or $2z$ is the semisimple element of an S-triple in $\mathfrak{k}$ and hence the eigenvalues of z in a K-module are symmetric. Thus $d(\gamma)$ is in any case the highest eigenvalue of z in W. When $[\mathfrak{g}_{\mathbf{R}}, \mathfrak{g}_{\mathbf{R}}] = \mathfrak{sl}(2, \mathbf{R})$ $\mathfrak{k}$ is abelian, W is one dimensional and $d(\gamma)$ equals the absolute value of the eigenvalue of z in W. In both cases the proposition follows. Q.E.D.

Propositions 5 and 8 enable us to define the degree of a K-module V_τ, $\tau \in \mathrm{T}$.

DEFINITION. *The degree $d(\tau)$ of a K-module V_τ, $\tau \in \mathrm{T}$, is the degree of any K_θ-module V_γ, $\gamma \in \mathrm{T}$, which contains a K-submodule isomorphic to V_τ.*

COROLLARY 9. *Let V_τ be a K-module of type $\tau \in \mathrm{T}$ and let (x, e, f) be a principal normal S-triple in $\mathfrak{g}$; put $z = x/2$. Then $d(\tau)$ equals the highest eigenvalue of z in V_τ, when $[\mathfrak{g}_{\mathbf{R}}, \mathfrak{g}_{\mathbf{R}}] \neq \mathfrak{sl}(2, \mathbf{R})$. If $[\mathfrak{g}_{\mathbf{R}}, \mathfrak{g}_{\mathbf{R}}] = \mathfrak{sl}(2, \mathbf{R})$ then $d(\tau)$ equals the absolute value of the eigenvalue of z in V_τ.*

COROLLARY 10 (TO PROPOSITION 5). *Let V_τ be a K-module of type $\tau \in \mathrm{T}$. Then $d(\tau)$ gives the degree in which τ occurs harmonically in $S(\mathfrak{p})$.*

If τ denotes an equivalence class of irreducible holomorphic finite dimensional K-modules let $S'(\mathfrak{k})_\tau$ be the set of all $f \in S'(\mathfrak{k})$ which transform under K according to the representation τ. Since $S'(\mathfrak{k})$ is a completely reducible K module we have $S'(\mathfrak{k}) = \oplus_\tau S'(\mathfrak{k})_\tau$.

THEOREM 11. *Let u/b^q $(u \in S'(\mathfrak{g})^K)$ be a rational function on $\mathfrak{p}$ homogeneous of degree $n \geqslant 0$. Then u/b^q is a polynomial if and only if*

$$\frac{u}{b^q}(y) \in \bigoplus_{\substack{\tau \in \mathrm{T} \\ d(\tau) \leqslant n}} S'(\mathfrak{k})_\tau \tag{2.6}$$

for all $y \in \mathfrak{p}$ where $b(y) \neq 0$.

PROOF. According to Theorem 3 we have to prove that (2.3) is satisfied if and only if (2.6) is true. Assume (2.3). Right after Lemma 4 we observed that

$$\frac{u}{b^q}(y) \in \bigoplus_{\tau \in \mathrm{T}} S'(\mathfrak{k})_\tau.$$

Now Lemma 4 and Corollary 9 imply that

$$\frac{u}{b^q}(y) \in \bigoplus_{\substack{\tau \in \mathrm{T} \\ d(\tau) \leqslant n}} S'(\mathfrak{k})_\tau.$$

Conversely if (2.6) is verified then (2.3) follows immediately. Q.E.D.

The group M' leaves $\mathfrak{k} + \mathfrak{a}$ invariant, thus the Weyl group $W = M'/M$ operates canonically on the ring $S'(\mathfrak{k} + \mathfrak{a})^M = S'(\mathfrak{k})^M \otimes S'(\mathfrak{a})$ of M invariants in $S'(\mathfrak{k} + \mathfrak{a})$. Let $(S'(\mathfrak{k})^M \otimes S'(\mathfrak{a}))^W$ denote the ring of Weyl group invariant elements in $S'(\mathfrak{k})^M \otimes S'(\mathfrak{a})$. We are ready to state and prove our main theorem.

THEOREM 12. *The operation of restriction from* $\mathfrak{g}$ *to* $\mathfrak{k} + \mathfrak{a}$ *induces an isomorphism of* $S'(\mathfrak{g})^K$ *onto*

$$\bigoplus_{n \geq 0} \bigoplus_{\substack{\tau \in \mathrm{T} \\ d(\tau) \leq n}} \left(S'(\mathfrak{k})_\tau^M \otimes S_n'(\mathfrak{a})\right)^W. \tag{2.7}$$

PROOF. We already know that the restriction homomorphism (2.1) is injective and that its image is contained in $S'(\mathfrak{k} + \mathfrak{a})^{M'} = (S'(\mathfrak{k})^M \otimes S'(\mathfrak{a}))^W$. Theorem 11 now shows, more precisely, that the image is contained in (2.7). Let $v \in \bigoplus_{\tau \in \mathrm{T},\, d(\tau) \leq n} (S'(\mathfrak{k})_\tau^M \otimes S_n'(\mathfrak{a}))^W$. Since (2.2) is an isomorphism of algebras there exist $u \in S'(\mathfrak{g})^K$ and $q \geq 0$ such that $\pi(u/b^q) = v$. By the K invariance of u/b^q we have that

$$\frac{u}{b^q}(y) \in \bigoplus_{\substack{\tau \in \mathrm{T} \\ d(\tau) \leq n}} S'(\mathfrak{k})_\tau$$

for all $y \in K \cdot \mathfrak{a}$ ($K \cdot \mathfrak{a}$ contains the set of all $y \in \mathfrak{p}$ where $b(y) \neq 0$). On the other hand since $K \cdot \mathfrak{a}$ is dense in $\mathfrak{p}$, u/b^q is homogeneous of degree n on $\mathfrak{p}$. Then Theorem 11 tells us that $u/b^q \in S'(\mathfrak{g})^K$. This completes the proof of the theorem. Q.E.D.

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