

RADIAL LIMITS OF n -SUBHARMONIC FUNCTIONS IN THE POLYDISC

BY

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ABSTRACT. We prove a relation between a certain weighted radial limit of an n -subharmonic function in the polydisc U^n and the representing measure of its least n -harmonic majorant. We apply this result to functions in $N(U^n)$, the Nevanlinna class of U^n . In particular, we obtain a necessary condition for a function to belong to the component of the origin in $N(U^n)$. These results are extensions of the work of J. H. Shapiro and A. L. Shields to $n > 1$.

1. Introduction. The results of the paper were motivated by the paper of J. H. Shapiro and A. L. Shields [3] concerning functions in the Nevanlinna class N . The space N consists of all function f analytic in the open unit disc satisfying $\sup_{0 < r < 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta < \infty$. By the Canonical Factorization Theorem, every function $f \in N$ has a unique factorization

$$(1.1) \quad f = B(S_{\mu_1}/S_{\mu_2})F$$

where $B(z)$ is the Blaschke product with respect to the zeroes of $f(z)$, $F(z)$ is an outer function, and $S_{\mu_j}(z)$ are singular inner functions defined by

$$S_{\mu_j}(z) = \exp \left[- \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu_j(t) \right]$$

where the μ_j are finite positive singular Borel measures on $[0, 2\pi)$.

In their paper [3], the authors proved that if $f \in N$, then for all $t, |t| = 1$,

$$(1.2) \quad \overline{\lim}_{r \rightarrow 1^-} (1 - r) \log |f(rt)| = 2\sigma_f(\{t\})$$

where $\sigma_f = \mu_2 - \mu_1$. The key in proving this result was in showing that if B is a Blaschke product, then

$$(1.3) \quad \overline{\lim}_{r \rightarrow 1^-} (1 - r) \log |B(rt)| = 0, \quad |t| = 1.$$

These results are then used to obtain information about topological properties of the space N .

In this paper we consider analogues of the above for functions in the Nevanlinna class in the polydisc U^n in \mathbf{C}^n . Unfortunately, there is no analogue of the factorization (1.1) for functions of several variables. However, as is shown in this paper, by

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using the least n -harmonic majorant of $\log|f(z)|$ one can obtain generalizations of (1.2) and many of the other results in [3] to functions in the Nevanlinna class in the polydisc. These results, however, are the consequence of more general results concerning n -subharmonic functions in the polydisc.

The key result of this paper, given as a lemma in §3, is as follows: if $F \not\equiv -\infty$ is n -subharmonic in U^n with $F(z) \leq 0$ and $\lim_{r \rightarrow 1} \int_{T^n} F(rt) d\lambda_n(t) = 0$, then

$$\overline{\lim}_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) F(r_1 t_1, \dots, r_n t_n) = 0$$

for all t on the distinguished boundary T^n . For the case $n = 1$, this gives (1.3) since $\log|B(z)|$ satisfies the hypothesis. The above result is then used to prove that if $f \not\equiv -\infty$ is n -subharmonic in U^n with $\sup_{0 < r < 1} \int_{T^n} f^+(rt) d\lambda_n(t) < \infty$, then

$$\overline{\lim}_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) f(r_1 t_1, \dots, r_n t_n) = 2^n \sigma_f(\{t\})$$

where σ_f is the singular part of the representing measure of f .

In §4, the results of §3 are applied to obtain generalizations of the results in [3] to functions in the Nevanlinna class $N(U^n)$. In particular, we obtain a necessary condition that a function f in $N(U^n)$ lie in the component of the origin in $N(U^n)$. §2 of this paper contains the necessary background material and notation.

2. Notation and preliminary results. Let U denote the unit disc in \mathbb{C} with boundary T . For $n > 1$, U^n denotes the unit polydisc in \mathbb{C}^n with distinguished boundary T^n , that is, $T^n = \{\zeta \in \mathbb{C}^n: |\zeta_j| = 1, 1 \leq j \leq n\}$. Also, we denote by λ the normalized Lebesgue measure on T and by λ_n the normalized Lebesgue measure on T^n .

A continuous real-valued function f on an open set in \mathbb{C}^n is n -harmonic if f is harmonic in each variable z_j separately, $1 \leq j \leq n$. Also, a function f defined on an open set Ω in \mathbb{C}^n with $-\infty \leq f < \infty$ is n -subharmonic if f is upper semicontinuous in Ω and subharmonic in each variable separately.

For $z \in U, t \in T$, the Poisson kernel on $U \times T$ is given by

$$P(z, t) = \frac{1 - |z|^2}{|t - z|^2}.$$

Also, for $z \in U^n, t \in T^n$, the Poisson kernel on $U^n \times T^n$ is given by

$$P(z, t) = \prod_{j=1}^n P(z_j, t_j).$$

If μ is a finite (signed) Borel measure on T^n , its Poisson integral is the function

$$P_z[d\mu] = \int_{T^n} P(z, t) d\mu(t).$$

The function $U(z) = P_z[d\mu]$ is n -harmonic in U^n .

For any function f in $U^n, f_r(t) = f(rt)$ if $0 \leq r < 1$ and $t \in T^n; f^*(t) = \lim_{r \rightarrow 1} f_r(t)$ for those $t \in T^n$ at which the limit exists; for real $f, f^+ = \max\{f, 0\}$.

The following lemma will be needed in the sequel.

LEMMA 2.1. Suppose $f \not\equiv -\infty$ is n -subharmonic in U^n with $\sup_{0 < r < 1} \int_{T^n} f_r^+ d\lambda_n < \infty$. Then there exists a finite Borel measure μ_f on T^n such that

$$(2.1) \quad f(z) \leq \int_{T^n} P(z, t) d\mu_f(t)$$

and μ_f is minimal among the Borel measures on T^n satisfying (2.1). Furthermore,

$$(2.2) \quad \lim_{r \rightarrow 1} \int_{T^n} f(rt)\phi(t) d\lambda_n(t) = \int_{T^n} \phi(t) d\mu_f(t)$$

for all ϕ continuous on T^n . If

$$(2.3) \quad d\mu_f = h d\lambda_n + d\sigma_f$$

is the Lebesgue decomposition of μ_f with $h \in L^1(T^n)$ and σ_f is singular, then $h(t) = \lim_{r \rightarrow 1} P_{rt}[d\mu_f]$ a.e. on T^n . Finally, if the family $\{f_r^+ : 0 < r < 1\}$ is uniformly integrable on T^n , then $\sigma_f \leq 0$ and

$$(2.4) \quad f(z) \leq P_z[h d\lambda_n].$$

The proof of the lemma may be found in [2]. An analogous result for plurisubharmonic functions on more general domains also holds [4].

The function $P_z[d\mu_f]$ in (2.1) is the least n -harmonic majorant of f . The measure μ_f will be referred to as the representing measure of the least n -harmonic majorant of f . From the above, if f satisfies the hypothesis of the lemma, then

$$(2.5) \quad f^+(z) \leq P_z[h^+ d\lambda_n + d\sigma_f^+].$$

We will also need the following version of the maximum principle for n -subharmonic functions.

LEMMA 2.2. Suppose $D_j, 1 \leq j \leq n$, are bounded open connected subsets of \mathbb{C} with boundary B_j . Let $D = \times_{j=1}^n D_j$ and $B = \times_{j=1}^n B_j$. If f is n -subharmonic in D and satisfies $\limsup_{z \rightarrow \zeta} f(z) \leq 0$ for every $\zeta \in B$, then $f(z) \leq 0$ for all $z \in D$.

The proof is an immediate consequence of the maximum principle for subharmonic functions and hence is omitted.

3. Radial limits of n -subharmonic functions. In this section we investigate radial limits of n -subharmonic functions. The first result is a generalization of (1.3) to n -subharmonic functions in U^n .

LEMMA 3.1. If $F(z)$ is n -subharmonic in U^n ($F \not\equiv -\infty$) satisfying

- (i) $F(z) \leq 0$ for all $z \in U^n$, and
- (ii) $\lim_{r \rightarrow 1} \int_{T^n} F(rt) d\lambda_n(t) = 0$,

then for all $t \in T^n$,

$$\overline{\lim}_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) F(r_1 t_1, \dots, r_n t_n) = 0,$$

where $(r) = (r_1, \dots, r_n), 0 < r_j < 1$.

PROOF. Without loss of generality we may assume that $t_j = 1$ for all $j = 1, \dots, n$. Also, for convenience we give the proof only for $n = 2$ and indicate the modifications which need to be made for arbitrary n . Suppose that

$$\overline{\lim}_{(r_1, r_2) \rightarrow (1, 1)} (1 - r_1)(1 - r_2)F(r_1, r_2) < 0.$$

Then there exist constants $A > 0$ and $R, 0 < R < 1$, such that

$$(3.1) \quad F(r_1, r_2) \leq -AP(r_1, 1)P(r_2, 1)$$

for all $r_1, r_2 \geq R$. Let C_R denote the semicircle in the upper half of the unit disc passing through R and 1 with diameter $1 - R$. By considering the image of this semicircle under the conformal map $w = (1 + z)/(1 - z)$ it is clear that $P(\zeta, 1) = (1 + R)/(1 - R)$ for all $\zeta \in C_R, \zeta \neq 1$. Finally let D_R denote the open subset of U bounded by the semicircle C_R and the line segment from R to 1 .

We will show that for all $(z_1, z_2) \in D_R \times D_R$

$$(3.2) \quad F(z_1, z_2) \leq -AP(z_1, 1)P(z_2, 1) + ABP(z_1, 1) + ABP(z_2, 1) + AB^2$$

where $B = (1 + R)/(1 - R)$.

Before proving inequality (3.2), we show how this is used to obtain that

$$\lim_{r \rightarrow 1} \int_{T^n} F(rt) d\lambda_n(t) \leq -Ab$$

for some positive constant b , which is a contradiction. Fix a Stolz region S in U with vertex at 1 and let S^+ denote the intersection of S with the upper half of the disc. Then for r sufficiently close to $1, S^+ \cap \{z \mid |z| = r\} \subset D_R$. Finally, let

$$(3.3) \quad I_r = \{e^{i\theta} | re^{i\theta} \in S^+\}$$

Then (3.2) gives

$$\begin{aligned} \int_{T^2} F(rt) d\lambda_2(t) &\leq \int_{I_r^2} F(rt) d\lambda_2(t) \leq -A \left[\frac{1}{2\pi} \int_{I_r} P(re^{i\theta}, 1) d\theta \right]^2 \\ &\quad + 2AB|I_r| \left[\frac{1}{2\pi} \int_{I_r} P(re^{i\theta}, 1) d\theta \right] + AB^2|I_r|^2, \end{aligned}$$

where $|I_r|$ denotes the length of I_r . A routine computation shows that $|I_r| \sim (1 - r) \tan(\beta/2)$ and

$$\lim_{r \rightarrow 1} \frac{1}{2\pi} \int_{I_r} P(re^{i\theta}, 1) d\theta = \frac{\beta}{2\pi}$$

where β is the angle of the Stolz domain at the vertex 1 . Hence

$$\lim_{r \rightarrow 1} \int_{T^2} F(rt) d\lambda_2(t) \leq -A \left(\frac{\beta}{2\pi} \right)^2.$$

To complete the proof of the lemma we need to establish inequality (3.2). For $\epsilon > 0$, consider the 2-subharmonic function H_ϵ given by

$$(3.4) \quad H_\epsilon(z_1, z_2) = F(z_1, z_2) + AP(z_1, 1)P(z_2, 1) - \epsilon Aw(z_1)P(z_2, 1) \\ - \epsilon AP(z_1, 1)w(z_2) - ABP(z_1, 1) - ABP(z_2, 1) - AB^2,$$

where for $z = re^{i\theta}$, $0 \leq \theta \leq \pi$, and fixed δ , $0 < \delta < 1$,

$$w(z) = \operatorname{Re} \left[e^{-i\pi/4} \frac{1+z}{1-z} \right]^{1+\delta} = \left| \frac{1+z}{1-z} \right|^{1+\delta} \cos(1+\delta) \left(\phi(z) - \frac{\pi}{4} \right).$$

Here $\phi(z) = \arg((1+z)/(1-z))$. For z in the upper half of the unit disc, $0 \leq \phi(z) \leq \pi/2$. Thus

$$(3.5) \quad w(z) \geq \left| \frac{1+z}{1-z} \right|^{1+\delta} \rho, \quad z \in D_R,$$

where $\rho = \cos(1+\delta)(\pi/4)$ which is positive. We now show that for all $\zeta = (\zeta_1, \zeta_2) \in \partial D_R \times \partial D_R$

$$(3.6) \quad \overline{\lim}_{z \rightarrow \zeta} H_\epsilon(z) \leq 0.$$

As the boundary of D_R consists of the semicircle C_R , the line segment $\Gamma_R = [R, 1]$ and the point $\{1\}$, there are nine cases to be considered, which by symmetry can be reduced to four cases. Note, $H_\epsilon(\zeta)$ is defined on $\partial D_R \times \partial D_R$ for all ζ except when $\zeta_j = 1$ for some j .

(i) $(\zeta_1, \zeta_2) \in \Gamma_R \times \Gamma_R$.

Then by (3.1), $F(\zeta_1, \zeta_2) + AP(\zeta_1, 1)P(\zeta_2, 1) \leq 0$. Since all the remaining terms are negative, $H_\epsilon(\zeta_1, \zeta_2) \leq 0$.

(ii) $(\zeta_1, \zeta_2) \in C_R \times C_R$, $\zeta_j \neq 1, j = 1, 2$.

Then since $P(\zeta_j, 1) = (1+R)/(1-R) = B$,

$$H_\epsilon(\zeta_1, \zeta_2) \leq AP(\zeta_1, 1)P(\zeta_2, 1) - AB^2 = 0.$$

(iii) $(\zeta_1, \zeta_2) \in C_R \times \Gamma_R$, $\zeta_1 \neq 1$.

Then

$$H_\epsilon(\zeta_1, \zeta_2) \leq AP(\zeta_1, 1)P(\zeta_2, 1) - ABP(\zeta_2, 1) = 0.$$

Similarly, if $(\zeta_1, \zeta_2) \in \Gamma_R \times C_R$, $\zeta_2 \neq 1$, $H_\epsilon(\zeta_1, \zeta_2) \leq 0$.

(iv) $\zeta_1 = 1, \zeta_2 \in C_R \cup \Gamma_R$

(we also permit the possibility that $\zeta_2 = 1$). Then for $z \in D_R \times D_R$

$$H_\epsilon(z) \leq AP(z_2, 1)[P(z_1, 1) - \epsilon w(z_1)].$$

But for $z_1 \in D_R$, by (3.5),

$$P(z_1, 1) - \epsilon w(z_1) \leq \left| \frac{1+z_1}{1-z_1} \right| \left(1 - \epsilon \rho \left| \frac{1+z_1}{1-z_1} \right|^\delta \right),$$

and as $z_1 \rightarrow 1$, the term on the right becomes negative. Thus

$$\overline{\lim}_{(z_1, z_2) \rightarrow (1, \zeta_2)} H_\epsilon(z_1, z_2) \leq 0,$$

and this holds for any $\zeta_2 \in D_R \cup \Gamma_R$. A similar argument holds for $\zeta_2 = 1, \zeta_1 \in C_R \cup \Gamma_R$. Therefore (3.6) holds for all $\zeta \in \partial D_R \times \partial D_R$ and hence by the maximum

principle for n -subharmonic functions (Lemma 2.2) $H_\epsilon(z) \leq 0$ for all $z \in D_R \times D_R$. Letting $\epsilon \rightarrow 0$ we obtain inequality (3.2).

For arbitrary n , a similar argument will show that for all $z \in D_R^n$,

$$(3.7) \quad F(z) \leq -A \prod_{j=1}^n P(z_j, 1) + A \sum_{\substack{S \subseteq \{1, \dots, n\} \\ S \neq \emptyset}} B^{n-|S|} \prod_{j \in S} P(z_j, 1).$$

As above, (3.7) is proved by considering the functions $H_\epsilon(z)$, $\epsilon > 0$, defined by

$$H_\epsilon(z) = F(z) + A \prod_{j=1}^n P(z_j, 1) - \epsilon A \prod_{j=1}^n w(z_j) \prod_{k \neq j} P(z_k, 1) - A \sum_{\substack{S \subseteq \{1, \dots, n\} \\ S \neq \emptyset}} B^{n-|S|} \prod_{j \in S} P(z_j, 1).$$

The remainder of the proof follows as above.

We now state and prove the main result of the paper.

THEOREM 3.2. *Let $f(z) \not\equiv -\infty$ be n -subharmonic in U^n satisfying*

$$\sup_{0 < r < 1} \int_{T^n} f^+(rt) \, d\lambda_n(t) < \infty.$$

Then for all $t \in T^n$,

$$\overline{\lim}_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) f(r_1 t_1, \dots, r_n t_n) = 2^n \sigma_f(\{t\}),$$

where as in Lemma 2.1, σ_f is the singular part of the representing measure μ_f of the least n -harmonic majorant of f .

PROOF. Let $H(z) = P_z[d\mu]$, where $d\mu = h d\lambda_n + d\sigma$ with $h \in L^1(T^n)$. We first show that

$$(3.8) \quad \lim_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) H(r_1 t_1, \dots, r_n t_n) = 2^n \sigma(\{t\})$$

for all $t \in T^n$. By linearity it suffices to prove the result for μ nonnegative.

Fix $t \in T^n$. We first assume that $\sigma(\{t\}) = 0$. Let $\epsilon > 0$ be given. Then there exist open intervals $I_j, j = 1, \dots, n$, on the unit circle with $t_j \in I_j$ such that

$$\sigma(I) < \epsilon \quad \text{and} \quad \int_I h \, d\lambda_n < \epsilon$$

where $I = \times_{j=1}^n I_j$. Then

$$\begin{aligned} H(r_1 t_1, \dots, r_n t_n) &= \int \prod_{j=1}^n P(r_j t_j, \zeta_j) [h \, d\lambda_n(\zeta) + d\sigma(\zeta)] \\ &\quad + \int_{T^n - I} \prod_{j=1}^n P(r_j t_j, \zeta_j) \, d\mu(\zeta) \\ &\leq \prod_{j=1}^n \left(\frac{1 + r_j}{1 - r_j} \right) 2\epsilon + \int_{T^n - I} \prod_{j=1}^n P(r_j t_j, \zeta_j) \, d\mu(\zeta). \end{aligned}$$

On $T^n - I$, $\prod_{j=1}^n (1 - r_j)P(r_j t_j, \zeta_j) \leq 2^n$. Also if $\zeta \in T^n - I$, there exists k such that $\zeta_k \in T - I_k$ and thus

$$\lim_{(r) \rightarrow (1)} \prod_{j=1}^n (1 - r_j)P(r_j t_j, \zeta_j) = 0.$$

Hence

$$\lim_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) H(r_1 t_1, \dots, r_n t_n) \leq 2^{n+1} \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we obtain

$$(3.9) \quad \lim_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) H(r_1 t_1, \dots, r_n t_n) = 0.$$

If $\sigma(\{t\}) = a > 0$, we consider the measure $\nu = \mu - a\delta_t$, where δ_t is point mass at t . Letting $V(z) = P_z[d\nu]$ we have

$$H(z) = V(z) + aP(z, t).$$

Since $\nu \geq 0$ and $\nu(\{t\}) = 0$, by (3.9),

$$\lim_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) V(r_1 t_1, \dots, r_n t_n) = 0.$$

Thus since $\lim_{(r) \rightarrow (1)} \prod_{j=1}^n (1 - r_j)P(r t_j, t_j) = 2^n$, (3.8) follows.

Let $F(z) = f(z) - H(z)$. Then F is n -subharmonic, less than or equal to zero, and by (2.2) satisfies $\lim_{r \rightarrow 1} \int_{T^n} F(rt) d\lambda_n(t) = 0$. Therefore by Lemma 3.1 and (3.8) we obtain that

$$\overline{\lim}_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) f(r_1 t_1, \dots, r_n t_n) = 2^n \sigma(\{t\}),$$

thus proving the result.

An immediate consequence of Theorem 3.2 is the following.

COROLLARY 1. *If $f \not\equiv -\infty$ is n -subharmonic in U^n with $\sup_{0 < r < 1} \int_{T^n} f^+(rt) d\lambda_n(t) < \infty$, then the representing measure of the least harmonic majorant of f is continuous at every $t \in T^n$ if and only if*

$$\overline{\lim}_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) f(r_1 t_1, \dots, r_n t_n) = 0$$

for all $t \in T^n$.

Recall that a measure μ is continuous at $t \in T^n$ if $\mu(\{t\}) = 0$.

For f n -subharmonic on T^n , $0 < r_j < 1$, let

$$(3.10) \quad M_\infty(f; (r)) = M_\infty(f; r_1, \dots, r_n) = \sup_{t \in T^n} f(r_1 t_1, \dots, r_n t_n).$$

Then the following holds.

COROLLARY 2. Suppose $f \not\equiv -\infty$ is n -subharmonic with

$$\sup_{0 < r < 1} \int_{T^n} f^+(rt) d\lambda_n(t) < \infty \quad \text{and} \quad \overline{\lim}_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) f(r_1 t_1, \dots, r_n t_n) = 0$$

for all $t \in T^n$. Then

$$(3.11) \quad \overline{\lim}_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) M_\infty(f; r_1, \dots, r_n) = 0.$$

PROOF. Let A denote the limit superior in (3.11). Then there exist sequences $\{r_{j(k)}\}_{k=1}^\infty, j = 1, \dots, n$, such that

$$\left(\prod_{j=1}^n (1 - r_{j(k)}) \right) M_\infty(f; (r_{j(k)})) \rightarrow A.$$

Let $H(z) = P_z[d\mu]$ be the least n -harmonic majorant of f , where μ by hypothesis is continuous on T^n . By the maximum principle, for each $(r_{j(k)})$, there exists a point $(t_{j(k)}) \in T^n$ such that

$$\begin{aligned} M_\infty(f; (r_{j(k)})) &= f(r_{1(k)} t_{1(k)}, \dots, r_{n(k)} t_{n(k)}) \\ &\leq \int_{T^n} \prod_{j=1}^n P(r_{j(k)} t_{j(k)}, \zeta_j) d\mu(\zeta). \end{aligned}$$

Since T^n is compact, by choosing subsequences if necessary, we may assume that $(t_{j(k)})$ converges to some $t \in T^n$. Since μ is continuous at t , as in the proof of Theorem 3.2,

$$\lim_{k \rightarrow \infty} \left(\prod_{j=1}^n (1 - r_{j(k)}) \right) \int_{T^n} \prod_{j=1}^n P(r_{j(k)} t_{j(k)}, \zeta_j) d\mu(\zeta) = 0.$$

Thus

$$\overline{\lim}_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) M_\infty(f; r_1, \dots, r_n) \leq 0.$$

However, since $f(r_1 t_1, \dots, r_n t_n) \leq M_\infty(f; (r))$, we obtain equality.

For a Borel measure μ on T^n , we denote by μ_j the projection of the measure μ onto the j th coordinate T , that is, if $E \subset T$ is a Borel set,

$$(3.12) \quad \mu_j(E) = \mu \left(\bigtimes_{k=1}^n T_k \right)$$

where $T_k = T$ for all $k \neq j$ and $T_j = E$. Also, if f is n -subharmonic, $0 < r < 1$, we define $f_r^j(w), w \in U$, by

$$(3.13) \quad f_r^j(w) = \int_{T^{n-1}} f(\underbrace{rt_1, \dots, w, \dots, rt_n}_{j \text{th coordinate}}) d\lambda_{n-1}(t).$$

Then for each $j = 1, \dots, n$, and each $r, 0 < r < 1$, f_r^j is subharmonic in U . Furthermore, if $0 < r_1 < r_2 < 1$, then

$$f_{r_1}^j(w) \leq f_{r_2}^j(w), \quad w \in U.$$

This follows from the fact that for fixed z_j , the function $\zeta \rightarrow f(\zeta_1, \dots, z_j, \dots, \zeta_n)$ is $(n - 1)$ -subharmonic in U^{n-1} .

THEOREM 3.3. *Let $f \not\equiv -\infty$ be n -subharmonic in U^n satisfying*

$$\sup_{0 < r < 1} \int_{T^n} f^+(rt) d\lambda_n(t) < \infty,$$

and let μ be the representing measure of the least n -harmonic majorant of f . Then

$$(3.14) \quad \overline{\lim}_{r \rightarrow 1} (1 - r)f_r^j(rt) = 2\mu_j(\{t\}), \quad t \in T.$$

PROOF. Let $H(z) = P_z[d\mu]$, $z \in U^n$. Also, for $w \in U$, $0 < r < 1$, set $F_r(w) = f_r^j(w) - H^j(w)$ where

$$H^j(w) = H(\underbrace{0, \dots, w, \dots, 0}_{j \text{th coordinate}}).$$

Then $F_r(w)$ is subharmonic in U , $F_r(w) \leq 0$, and

$$\begin{aligned} \int_T F_r(rt_j) d\lambda(t_j) &= \int_T f_r^j(rt_j) d\lambda(t_j) - H(0) \\ &= \int_{T^n} f(rt) d\lambda_n(t) - \mu(T^n). \end{aligned}$$

Thus by Lemma 2.1, $\lim_{r \rightarrow 1} \int_T F_r(rt_j) d\lambda(t_j) = 0$.

We shall prove the following two statements.

- (i) $\lim_{r \rightarrow 1} (1 - r)H^j(rt_j) = 2\mu_j(\{t_j\})$, and
- (ii) $\overline{\lim}_{r \rightarrow 1} (1 - r)F_r(rt_j) = 0$.

By definition of μ_j , $\int_T \phi(t_j) d\mu_j(t_j) = \int_{T^n} \phi(t_j) d\mu(t)$, for any ϕ continuous on T . Thus

$$H^j(w) = H(0, \dots, w, \dots, 0) = \int_{T^n} P(w, t_j) d\mu(t) = \int_T P(w, t_j) d\mu_j(t_j).$$

Hence by (3.8), for the case $n = 1$, $\lim_{r \rightarrow 1} (1 - r)H^j(rt_j) = 2\mu_j(\{t_j\})$, thus proving (i).

Suppose that $\overline{\lim}_{r \rightarrow 1} (1 - r)F_r(rt_j) < 0$. Again, without loss of generality we may assume $t_j = 1$. Then there exist constants $A > 0$ and $r_0, 0 < r_0 < 1$, such that

$$F_r(r) < -AP(r, 1), \quad r \geq r_0.$$

Let $\varepsilon > 0$ be given. Since $\int_T F_r(rt) d\lambda(t) \rightarrow 0$, we can choose $R \geq r_0$ such that

$$-\varepsilon < \int_T F_r(rt) d\lambda(t)$$

for all $r \geq R$. Since $F_R \leq F_r$ where $r \geq R$, we have

$$F_R(r) \leq -AP(r, 1) \quad \text{for } r \geq R.$$

As in the proof of Lemma 3.1,

$$F_R(z) < -AP(z, 1) + AP(R, 1)$$

for all $z \in D_R$. Now for $r > R$,

$$\begin{aligned}
 -\varepsilon &< \int_T F_R(Rt) d\lambda(t) \leq \int_T F_R(rt) d\lambda(t) \leq \int_{I_r} F_R(rt) d\lambda(t) \\
 &< -A \int_{I_r} P(rt, 1) d\lambda(t) + AP(R, 1)\lambda(I_r),
 \end{aligned}$$

where I_r is defined as in (3.3). Letting $r \rightarrow 1$ we obtain $-\varepsilon < -A\beta/2\pi$ where β is the angle of the Stolz domain at the vertex. But ε can be chosen arbitrarily small. This contradiction proves (ii). Thus

$$\overline{\lim}_{r \rightarrow 1} (1 - r)f'_r(rt) = 2\mu_j(\{t\}) \quad \text{for all } t \in T.$$

4. Functions of bounded characteristic. In this section we apply the results of the previous section to obtain some results on functions of bounded characteristic in U^n . As in the case for one variable, the Nevanlinna class $N(U^n)$ is the algebra of functions f holomorphic in U^n for which

$$\sup_{0 < r < 1} \int_{T^n} \log^+ |f(rt)| d\lambda_n(t) < \infty.$$

Also, we consider the subspace $N_*(U^n)$ consisting of all functions $f \in N(U^n)$ for which the family $\{\log^+ |f_r| : 0 < r < 1\}$ is uniformly integrable on T^n . By Lemma 2.1, if $f \in N(U^n)$, then there exists a minimal Borel measure μ on T^n such that

$$(4.1) \quad \log |f(z)| \leq P_z[d\mu]$$

with $d\mu = \log |f^*| d\lambda_n + d\sigma$, where σ is singular with respect to λ_n and $f^*(t) = \lim_{r \rightarrow 1} f(rt)$ a.e. on T^n . Functions in $N_*(U^n)$ are characterized by the condition that $\sigma \leq 0$ and that

$$\lim_{r \rightarrow 1} \int_{T^n} \log^+ |f(rt)| d\lambda_n(t) = \int_{T^n} \log^+ |f^*| d\lambda_n.$$

For $f \in N(U^n)$, we define $\|f\|$ by

$$(4.2) \quad \|f\| = \lim_{r \rightarrow 1} \int_{T^n} \log(1 + |f(rt)|) d\lambda_n(t).$$

As in the one variable case, by setting $d(f, g) = \|f - g\|$ one obtains a complete translation invariant metric on $N(U^n)$ which induces a topology stronger than that of uniform convergence on compact sets. For $f, g \in N_*(U^n)$,

$$d(f, g) = \int_{T^n} \log(1 + |f^* - g^*|) d\lambda_n$$

and (N^*, d) is a complete topological vector space in which multiplication is continuous [5].

Let $f \in N(U^n)$. For convenience, we will refer to the representing measure μ of the least n -harmonic majorant of $\log |f|$ as the *boundary measure of f* . Also, if σ denotes the singular part of μ , we denote by σ^+ and σ^- the positive and negative variations of σ .

PROPOSITION 4.1. *Let $f \in N(U^n)$. Then*

$$(4.3) \quad \|f\| = \int_{T^n} \log(1 + |f^*|) d\lambda_n + \sigma^+(T^n)$$

and

$$(4.4) \quad \lim_{a \rightarrow 0} \|af\| = \sigma^+(T^n).$$

The proof of Proposition 4.1 is the same as the proofs of the corresponding results for the case $n = 1$ as given in [3], and hence is omitted.

An immediate consequence of (4.4) is that scalar multiplication is not continuous. Also, as was mentioned in [3], every finite-dimensional linear subspace of N/N_* has the discrete topology and, as a consequence, every finite-dimensional linear subspace of N which intersects N_* only at the origin also has the discrete topology.

One immediate consequence of Theorem 3.2 is the following

PROPOSITION 4.2. *Let $f \in N(U^n)$ and let σ_f be the singular part of the boundary measure of f . Then*

$$(4.5) \quad \overline{\lim}_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) \log |f(r_1 t_1, \dots, r_n t_n)| = 2^n \sigma_f(\{t\})$$

and

$$(4.6) \quad \overline{\lim}_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) \log^+ |f(r_1 t_1, \dots, r_n t_n)| = 2^n \sigma_f^+(\{t\}).$$

In [3], the authors defined the following subadditive, continuous, “nonarchimedean” functional λ_t on $N(U)$, $t \in T$, by

$$\lambda_t(f) = \overline{\lim}_{r \rightarrow 1} (1 - r) \log^+ |f(rt)|.$$

By (4.6), $\lambda_t(f) = 2\sigma_f^+(\{t\})$. The functional λ_t was used to show that $N(U)$ was disconnected. In [3, Theorem 2.1], the authors showed that if $\lambda_t(f) \neq \lambda_t(g)$ for some t , then f and g lie in different components of $N(U)$. In [1], J. W. Roberts showed that the component of the origin in $N(U)$ consists of all functions f for which $\lambda_t(f) = 0$ for all $t \in T$. This corresponds to the set of all functions f in $N(U)$ for which σ^+ is continuous on T .

Suppose $f \in N(U^n)$. As in [3], for $t \in T^n$ we define $\lambda_t(f)$ by

$$(4.7) \quad \lambda_t(f) = \overline{\lim}_{(r) \rightarrow (1)} \left(\prod_{j=1}^n (1 - r_j) \right) \log^+ |f(r_1 t_1, \dots, r_n t_n)|.$$

By (4.6), $\lambda_t(f) = 2^n \sigma^+(\{t\})$, where σ^+ is the positive variation of the singular part of the boundary measure of f . By (4.3) we have

$$(4.8) \quad \lambda_t(f) \leq 2^n \|f\|$$

for all $t \in T^n$. Thus λ_t is continuous.

From the inequalities,

$$\begin{aligned}\log^+(x+y) &\leq \log^+ x + \log^+ y + \log 2 \quad (x, y \geq 0), \\ \log^+(x+y) &\leq \max\{\log^+ x, \log^+ y\} + \log 2,\end{aligned}$$

it follows from (4.7) that λ_t is subadditive and

$$\lambda_t(f+g) \leq \max\{\lambda_t(f), \lambda_t(g)\}.$$

Hence if $\lambda_t(f) \leq \lambda_t(g)$,

$$\lambda_t(g) = \lambda_t(g+f-f) \leq \max\{\lambda_t(f+g), \lambda_t(f)\} \leq \lambda_t(g).$$

Therefore

$$(4.9) \quad \lambda_t(f+g) = \max\{\lambda_t(f), \lambda_t(g)\}.$$

This is referred to as the nonarchimedean property of λ . As in [3] we obtain the following

PROPOSITION 4.3. *If $f, g \in N(U^n)$ and $\lambda_t(f) \neq \lambda_t(g)$ for some $t \in T^n$, then f and g lie in different components of $N(U^n)$.*

PROOF. The proof is similar to the proof of Theorem 2.1 in [3]. For completeness we include a sketch of the proof. Suppose $\lambda_t(g) < a < \lambda_t(f)$. Since λ_t is continuous, the set

$$V = \{h \in N(U^n) : \lambda_t(h) > a\}$$

is open, contains f but not g . Furthermore, by the nonarchimedean property of λ_t and the inequality $2^n \|h\| \geq \lambda_t(h)$, it follows that for every $h_0 \in N(U^n) - V$,

$$\{h \in N(U^n) : \|h_0 - h\| < a/2^n\} \cap V = \emptyset.$$

Thus V is closed.

If we let $C_n(N)$ denote the component of the origin in $N(U^n)$, then as a consequence of Proposition 4.3 we obtain the following

COROLLARY. $C_n(N) \subset \{f \in N(U^n) \mid \lambda_t(f) = 0 \text{ for all } t \in T^n\}$.

For the case $n = 1$, the result due to J. W. Roberts [1] shows that $f \in C_1(N)$ if and only if $\lambda_t(f) = 0$ for all $t \in T^n$. At this time we have been unable to obtain a characterization of $C_n(N)$ for $n > 1$ and we suspect that the analogue of the case $n = 1$ does not hold for $n > 1$.

We conclude this section with the following result concerning products of functions in $C_n(N)$.

PROPOSITION 4.4. *If $f \in C_n(N)$ and $g \in N_*(U^n)$, then $fg \in C_n(N)$.*

PROOF. Fix $g \in N_*(U^n)$ and consider the map $\phi_g: N(U^n) \rightarrow N(U^n)$ by $\phi_g(f)(z) = f(z)g(z)$. We will show that ϕ_g is continuous. Suppose $f_j \rightarrow f$ in $N(U^n)$. Then from (4.3) we obtain that both $\int_{T^n} \log(1 + |f_j^* - f^*|) d\lambda_n$ and $\sigma_{f_j-f}^+(T^n)$ converge to zero as $j \rightarrow \infty$. Here $\sigma_{f_j-f}^+$ denotes the positive variation of the singular part of the

boundary measure of $f_j - f$. Consider $\|\phi_g(f_j) - \phi_g(f)\| = \|\phi_g(f_j - f)\|$. By (4.3),

$$\|\phi_g(f_j - f)\| = \int_{T^n} \log(1 + |g^*| |f_j^* - f^*|) d\lambda_n + \sigma_{g(f_j-f)}^+(T^n).$$

Since $\int_{T^n} \log(1 + |f_j^* - f^*|) d\lambda_n \rightarrow 0$ and $\log(1 + |g^*|)$ is integrable, it follows that

$$\lim_{j \rightarrow \infty} \int_{T^n} \log(1 + |g^*| |f_j^* - f^*|) d\lambda_n = 0.$$

Also, since $\log^+ |g(f_j - f)| \leq \log^+ |f_j - f| + \log^+ |g|$ and since the representing measure of the least n -harmonic majorant of $\log^+ |g(z)|$ is absolutely continuous, $\sigma_{g(f_j-f)}^+ \leq \sigma_{f_j-f}^+$. Therefore $\sigma_{g(f_j-f)}^+(T^n) \rightarrow 0$ and thus ϕ_g is continuous. But then $\phi_g(C_n)$ is connected and contains zero. Hence $\phi_g(C_n) \subset C_n$, proving the result.

REMARK. In the case $n = 1$, if $f, g \in C_1(N)$, then $fg \in C_1(N)$. This however is a consequence of the characterization of $C_1(N)$ given in [1] and does not follow from any continuity property. Even for fixed $g \in C_1(N)$, $g \notin N_*$, the mapping $f \rightarrow fg$ is no longer continuous. One is tempted to conjecture that $C_n(N)$ is also closed under products if $n > 1$. Unfortunately we have been unable to prove this.

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