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NONIMMERSIONS AND NONEMBEDDINGS OF QUATERNIONIC SPHERICAL SPACE FORMS

BY

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ABSTRACT. We determine the orders of the canonical elements in KO-rings of quaternionic spherical space forms S^{4n+3}/Q_k and apply them to prove the nonexistence theorems of immersions and embeddings of S^{4n+3}/Q_k in Euclidean spaces.

1. Statements of results. Let $Q_k = [x, y\colon x^{2^{k-2}} = y^2,\ xyx = y]$ be the generalized quaternion group of order 2^k ($k > 2$). (Note that the relation $x^{2^{k-1}} = 1$ follows from the above two relations.) Let $d_1\colon Q_k \to S^3 = \mathrm{Sp}(1) = \mathrm{SU}(2)$ be the natural inclusion defined by $d_1(x) = \exp(2\pi i/2^{k-1})$, $d_1(y) = j$. Then Q_k acts freely on the unit sphere S^{4n+3} in the quaternion $(n+1)$-space H^{n+1} by the diagonal action $(n+1)d_1\colon Q_k \to \mathrm{Sp}(n+1)$. The quotient manifold S^{4n+3}/Q_k is called the quaternionic spherical space form. D. Pitt [**8**] studied the structure of K- and KO-rings of S^{4n+3}/Q_k and considered the problem of immersing or embedding S^{4n+3}/Q_k in Euclidean space R^m using the techniques of M. F. Atiyah [**1**] (cf. also [**5**, Chapter 6] and [**6**, Chapter 3]).

The purpose of this note is to determine the orders of the canonical elements in $\widetilde{KO}(S^{4n+3}/Q_k)$ and apply them to improve the nonexistence theorems of immersions and embeddings of S^{4n+3}/Q_k. Let $M \not\subseteq R^m$ (or $M \not\subset R^m$) denote nonexistence of a C^∞-immersion (or a C^∞-embedding) of M in R^m. Let $\nu(n)$ be the nonnegative integer such that $n = q \cdot 2^{\nu(n)}$, where q is odd. Our main theorem is

THEOREM 1.1. *If $\nu\binom{2n+1+i}{i} < 2n + k - 2i + \varepsilon$, then $S^{4n+3}/Q_k \not\subseteq R^{4n+2+2i}$ and $S^{4n+3}/Q_k \not\subset R^{4n+3+2i}$, where $\varepsilon = 0$ if n is even > 0, and $\varepsilon = 1$ if n is odd.*

Define

$$N(n, k) = \max\left[i\colon 1 \leq i \leq n,\ \nu\binom{2n+1+i}{i} < 2n + k - 2i + \varepsilon\right].$$

The case $N(n, k) = n$ was obtained by Pitt [**8**, Corollary 5.6], and the case $k = 3$ was obtained by K. Fujii. It follows from Theorem 1.1, for example, that

$$S^{15}/Q_k \not\subseteq R^{20},\quad S^{15}/Q_k \not\subset R^{21};\quad S^{31}/Q_3 \not\subseteq R^{42},\quad S^{31}/Q_3 \not\subset R^{43},$$
$$S^{31}/Q_k \not\subseteq R^{44},\quad S^{31}/Q_k \not\subset R^{45}\qquad \text{for } k \geq 4.$$

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The complex representation ring $R_C(Q_k)$ of Q_k is generated as a free abelian group by $1, a, b, c$ and d_r $(r = 1, 2, \ldots, 2^{k-2} - 1)$ defined below (cf. [**2**, §47.15; **8**, §1 and **3**, §3]):

$$\begin{cases} 1(x) = 1, \\ 1(y) = 1, \end{cases} \quad \begin{cases} a(x) = 1, \\ a(y) = -1, \end{cases} \quad \begin{cases} b(x) = -1, \\ b(y) = 1, \end{cases} \quad \begin{cases} c(x) = -1, \\ c(y) = -1, \end{cases}$$

$$d_r(x) = \begin{bmatrix} \omega^r & 0 \\ 0 & \omega^{-r} \end{bmatrix}, \qquad d_r(y) = \begin{bmatrix} 0 & (-1)^r \\ 1 & 0 \end{bmatrix},$$

where ω is a primitive 2^{k-1}th root $\exp(2\pi i/2^{k-1})$ of unity. The multiplicative structure is given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = c, \qquad d_r d_s = d_{r+s} + d_{r-s}, \qquad bd_r = d_{2^{k-2}-r},$$

where we define

$$d_0 = 1 + a, \qquad d_{2^{k-2}} = b + c, \qquad d_{-r} = d_r, \qquad d_{2^{k-2}+r} = d_{2^{k-2}-r}.$$

The reduced representation ring $\tilde{R}_C(Q_k)$ is generated as a free abelian group by $\alpha = a - 1$, $\beta = b - 1$, $\gamma = a + b + c - 3$, $\delta_r = d_r - 2$ $(r = 1, 2, \ldots, 2^{k-2} - 1)$ with relations (cf. [**3**, Proposition 3.3]):

$$\alpha^2 = -2\alpha, \quad \beta^2 = -2\beta, \quad \gamma = \alpha\beta + 2\alpha + 2\beta, \quad \alpha\delta_1 = -2\alpha,$$
$$\beta\delta_1 = -2\beta + \delta_{2^{k-2}-1} - \delta_1, \quad \delta_{r+1} = \delta_1\delta_r + 2\delta_1 + 2\delta_r - \delta_{r-1}$$

where $\delta_{2^{k-2}} = \gamma - \alpha$, $\delta_0 = \alpha$. Thus $\tilde{R}_C(Q_k)$ is generated by α, β and δ_1 as a ring.

Let c_R: $R_R(Q_k) \to R_C(Q_k)$ be the complexification. The real representation ring $R_R(Q_k)$, considered as the subring $c_R(R_R(Q_k))$ of $R_C(Q_k)$, is generated by $1, a, b, c, d_{2r}$ and $2d_{2r+1}$ $(r = 0, 1, \ldots, 2^{k-3} - 1)$ (cf. [**8**, Proposition 1.5]).

Define elements v and z in $\tilde{R}_R(Q_k)$ by

$$(1.2) \qquad c_R^{-1}(2\delta_1) = v, \qquad c_R^{-1}(\delta_1^2) = z.$$

Let λ be the canonical complex plane bundle over the quaternion projective space $HP^n = S^{4n+3}/S^3$, and let π: $S^{4n+3}/Q_k \to HP^n$ be the natural projection. Let ξ_C: $\tilde{R}_C(Q_k) \to \tilde{K}(S^{4n+3}/Q_k)$ be the projection defined in [**3**, §4] and put $\delta = \xi_C(\delta_1)$. Then we have $\delta = \pi^*\lambda - 2$ (cf. [**3**, Lemma 4.4]). The order $\#\delta^i$ of $\delta^i \in \tilde{K}(S^{4n+3}/Q_k)$ is determined by H. Ōshima in [7, Proposition 5.2] and T. Mormann in [**6**, Chapter 2, Theorem 4.52] as follows.

PROPOSITION 1.3. $\#\delta^i = 2^{2n+k-2i}$ $(1 \leq i \leq n)$.

Let r_C: $\tilde{K}(X) \to \widetilde{KO}(X)$ and c_R: $\widetilde{KO}(X) \to \tilde{K}(X)$ be the realification and the complexification, respectively. Let ξ_R: $\tilde{R}_R(Q_K) \to \widetilde{KO}(S^{4n+3}/Q_k)$ be the projection defined in [**4**, (3.9)] (or in [**8**, Theorem 2.5]). Then, by (1.2),

$$\xi_R v = r_C(\pi^*\lambda - 2) \quad \text{and} \quad \xi_R z = c_R^{-1}\left((\pi^*\lambda - 2)^2\right)$$

(cf. [**4**, Lemma 3.10]), because δ_1 is self-conjugate and $c_R r_C = 1 +$ conjugation. For simplicity we write v and z instead of $\xi_R v$ and $\xi_R z$. Then, for the complexification c_R: $\widetilde{KO}(S^{4n+3}/Q_k) \to \tilde{K}(S^{4n+3}/Q_k)$, we have

$$(1.4) \qquad c_R(v) = 2\delta, \qquad c_R(z) = \delta^2.$$

Let $\#a$ (or $\#A$) denote the order of an element a (or a group A). The orders of the canonical elements in $\widetilde{KO}(S^{4n+3}/Q_k)$ are determined as follows.

THEOREM 1.5. *For z^i and $vz^i \in \widetilde{KO}(S^{8m+7}/Q_k)$,*

$$\#z^i = 2^{4m+k-4i+3} \qquad (0 < i \leqslant m),$$
$$\#vz^i = 2^{4m+k-4i} \qquad (0 \leqslant i \leqslant m).$$

THEOREM 1.6. *For z^i and $vz^i \in \widetilde{KO}(S^{8m+3}/Q_k)$,*

$$\#z^i = 2^{4m+k-4i} \qquad (0 < i \leqslant m),$$
$$\#vz^i = 2^{4m+k-4i-3} \qquad (0 \leqslant i < m).$$

COROLLARY 1.7. *For $v \in \widetilde{KO}(S^{4n+3}/Q_k)$,*

$$\#v = \begin{cases} 2^{2n+k-2} & \text{if } n \text{ is odd}, \\ 2^{2n+k-3} & \text{if } n \text{ is even} > 0. \end{cases}$$

K. Fujii [4] proved the result for $k = 3$. H. Ōshima [7] announced it for $k = 4$ and conjectured Corollary 1.7.

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2. Proofs of Theorems 1.5 and 1.6. First we prepare a lemma.

LEMMA 2.1. *In $\widetilde{KO}(S^{4n+3}/Q_k)$:*

(a)
$$2^k v + \sum_{j=1}^{2^{k-3}} (2c_{2j} + c_{2j+1}v)z^j = 0,$$

(b)
$$2^k z + \sum_{j=1}^{2^{k-3}} \left(2^{-1}c_{2j}v + c_{2j+1}z\right)z^j = 0,$$

where c_s are integers satisfying

(*)
$$c_2 = 2^{k-2}(2^{2k-3} + 1)/3, \qquad c_{2^{k-2}+1} = 1,$$
$$\nu(c_s) \geqslant \max(1, k - s) \quad \text{for } 0 < s \leqslant 2^{k-2}.$$

PROOF. It was proved in [6] that $\Sigma_{s=1}^{2^{k-2}+1} c_s \delta_1^s = 0$ in $\tilde{R}_C(Q_k)$, where

$$c_s = \binom{2^{k-2}+s}{2s-1} + 2\binom{2^{k-2}+s-1}{2s-1} + \binom{2^{k-2}+s-2}{2s-1}.$$

(*) follows easily from

$$\nu\binom{m}{n} = \alpha(n) + \alpha(m-n) - \alpha(m),$$

where $\alpha(n)$ denotes the number of nonzero terms in the dyadic expansion of n. (a) and (b) are proved by multiplying this by 2 and δ_1, respectively, and applying $c_R^{-1}\xi_C$.

LEMMA 2.2. *In* $\widetilde{KO}(S^{8m+7}/Q_k)$:

(i) $$z^{m+1} = 0,$$

(ii) $$2^{4i+k} vz^{m-i} = 0 \qquad (0 \leqslant i \leqslant m),$$

(iii) $$2^{4i+k+3} z^{m-i} = 0 \qquad (0 \leqslant i < m),$$

(iv) $$2^{4i+k-1} vz^{m-i} + 2^{4i+k+2} z^{m-i} = 0 \qquad (0 \leqslant i < m),$$

(v) $$2^{4i+k+2} z^{m-i} + 2^{4i+k+3} vz^{m-i-1} = 0 \qquad (0 \leqslant i < m).$$

PROOF. (i) follows from [**8**, Theorem 2.5].

(ii) and (iii) are proved by induction on i. (ii) for $i = 0$ follows from (i) and (a) $\times z^m$. (iii) for $i = 0$ follows from (b) $\times 2^3 z^{m-1}$ and (ii) for $i = 0$. (iii) for any $j \leqslant i - 1$, (ii) for any $j \leqslant i$ and (b) $\times 2^{4i+3} z^{m-i-1}$ imply (iii) for $j = i$. (ii) for any $j \leqslant i$, (iii) for any $j \leqslant i$ and (a) $\times 2^{4i+4} z^{m-i-1}$ imply (ii) for $j = i + 1$.

Using (b) $\times 2^{4i+2} z^{m-i-1}$ (resp. (a) $\times 2^{4i+3} z^{m-i-1}$) and (i) $\sim$ (iii), we obtain (iv) (resp. (v)).

LEMMA 2.3. *In* $\widetilde{KO}(S^{8m+3}/Q_k)$:

(i) $$z^{m+1} = 0,$$

(ii) $$vz^m = 0,$$

(iii) $$2^{4i+k} z^{m-i} = 0 \qquad (0 \leqslant i < m),$$

(iv) $$2^{4i+k+1} vz^{m-i-1} = 0 \qquad (0 \leqslant i < m),$$

(v) $$2^{4i+k-1} z^{m-i} + 2^{4i+k} vz^{m-i-1} = 0 \qquad (0 \leqslant i < m),$$

(vi) $$2^{4i+k} vz^{m-i-1} + 2^{4i+k+3} z^{m-i-1} = 0 \qquad (0 \leqslant i < m - 1).$$

PROOF. (i) follows from Lemma 2.2(i) and the naturality. (ii) is proved in [**4**, §4]. The proofs of (iii) $\sim$ (vi) are similar to those of Lemma 2.2(ii) $\sim$ (v), so we omit the details.

PROOF OF THEOREM 1.5. By Lemma 2.2(iii), (ii) we have

$$\# z^i \leqslant 2^{4m+k-4i+3} \quad (0 < i \leqslant m) \qquad \text{and} \qquad \# vz^i \leqslant 2^{4m+k-4i} \quad (0 \leqslant i \leqslant m).$$

Let j: $S^{8m+3}/Q_k \to S^{8m+7}/Q_k$ be the natural inclusion. Then it follows from [**4**, §4] that Ker j^*, the kernel of the induced homomorphism j^*: $\widetilde{KO}(S^{8m+7}/Q_k) \to \widetilde{KO}(S^{8m+3}/Q_k)$, is generated by vz^m. According to [**5**, Chapter 6, Proposition 5.7],

$$\# \widetilde{KO}(S^{8m+7}/Q_k) = 2^{2mk+4m+k+4} \quad \text{and} \quad \# \widetilde{KO}(S^{8m+3}/Q_k) = 2^{2mk+4m+4}.$$

Hence, by Lemma 2.2(ii), we obtain

$$2^k = \# \widetilde{KO}(S^{8m+7}/Q_k) / \# \widetilde{KO}(S^{8m+3}/Q_k) \leqslant \# \operatorname{Ker} j^* = \# vz^m \leqslant 2^k.$$

Thus $\# vz^m = 2^k$. Therefore, by Lemma 2.2(iv), (v) we have

$$2^{4m+k-4i+2} z^i = -2^{4m+k-4i-1} vz^i = \cdots = -2^{k-i} vz^m \neq 0.$$

PROOF OF THEOREM 1.6. By Lemma 2.3(iii), (iv) we have

$$\# z^i \leqslant 2^{4m+k-4i} \quad (0 < i \leqslant m) \qquad \text{and} \qquad \# vz^i \leqslant 2^{4m+k-4i-3} \quad (0 \leqslant i < m).$$

By Lemma 2.3(vi), (v) we have

$$2^{4m+k-4i-1}z^i = -2^{4m+k-4i-4}vz^i = \cdots = -2^{4m+k-4}v.$$

But, by (1.4) and Proposition 1.3, $c_R(2^{4m+k-4}v) = 2^{4m+k-3}\delta \neq 0$, so $2^{4m+k-4}v \neq 0$.

3. Atiyah's criterion. Exterior power operation $\lambda^i(\alpha)$, $\alpha \in R_C(Q_k)$, is determined by

$$\lambda^0(\alpha) = 1,\quad \lambda^1(\alpha) = \alpha,\quad \lambda^i(\alpha) = 0 \quad \text{for } i > 1, \alpha = 1, a, b, c,$$
$$\lambda^2(d_r) = 1 \quad \text{for } r \text{ odd},\quad \lambda^2(d_r) = a \quad \text{for } r \text{ even},\qquad \lambda^i(d_r) = 0 \quad \text{for } i > 2.$$

Define $\lambda_t(\alpha) = \Sigma_{i\geq 0}\lambda^i(\alpha)t^i$. Then the Grothendieck γ-operations γ^i are obtained from the equality of the polynomials

$$\lambda_{t/(1-t)}(\alpha) = \gamma_t(\alpha) = \sum_{i\geq 0} \gamma^i(\alpha)t^i.$$

The following is well known (cf. [**8**, p. 2]).

LEMMA 3.1. $\gamma_t(\delta_{2r+1}) = 1 + \delta_{2r+1}(t - t^2)$, *where* $\delta_{2r+1} = d_{2r+1} - 2 \in \tilde{R}_C(Q_k)$.

Let v and z be the elements in $\tilde{R}_R(Q_k)$ defined in (1.2). Then we prove

LEMMA 3.2. $\gamma_t(v) = 1 + v(t - t^2) + z(t - t^2)^2$.

PROOF. Since γ_t is natural with respect to the complexification c_R, we have, by Lemma 3.1 and (1.2),

$$\gamma_t(v) = \gamma_t c_R^{-1}(2\delta_1) = c_R^{-1}\{\gamma_t(\delta_1)\}^2 = c_R^{-1}\{1 + \delta_1(t - t^2)\}^2$$
$$= c_R^{-1}\left\{1 + 2\delta_1(t - t^2) + \delta_1^2(t - t^2)^2\right\} = 1 + v(t - t^2) + z(t - t^2)^2.$$

As an application of Grothendieck γ-operations in KO-theory, M. F. Atiyah [**1**] obtained the following

THEOREM 3.3. *Let M be an n-dimensional compact smooth manifold and $\tau_0 \in \widetilde{KO}(M)$ the stable class of the tangent bundle of M. Then, if M is immersible* (*resp. embeddable*) *in R^{n+r}, $\gamma^i(-\tau_0) = 0$ for all $i > r$* (*resp. $i \geq r$*).

LEMMA 3.4. *Let τ_0 be the stable class of the tangent bundle $\tau = \tau(S^{4n+3}/Q_k)$ of S^{4n+3}/Q_k. Then*

$$\gamma_t(-\tau_0) = \sum_{i\geq 0}\binom{2n+1+2i}{2i}z^i(t - t^2)^{2i} - \sum_{i\geq 0}\frac{1}{2}\binom{2n+2+2i}{2i+1}vz^i(t - t^2)^{2i+1}.$$

PROOF. According to [**9**, Corollary 3.3],

$$-\tau_0 = 4n + 3 - \tau = 4(n+1) - (n+1)(r_C\pi^*\lambda) = -(n+1)v.$$

By Lemma 3.2 and (1.2), we have

$$\begin{aligned}
\gamma_t(-\tau_0) &= \gamma_t(-(n+1)v) = (\gamma_t(v))^{-n-1} = \left\{1 + v(t-t^2) + z(t-t^2)^2\right\}^{-n-1} \\
&= c_R^{-1}\left\{1 + \delta_1(t-t^2)\right\}^{-2n-2} \\
&= c_R^{-1}\left\{\sum_{j\geqslant 0} (-1)^j \binom{2n+1+j}{j} \delta_1^j (t-t^2)^j\right\} \\
&= c_R^{-1}\left\{\sum_{i\geqslant 0} \binom{2n+1+2i}{2i} \delta_1^{2i}(t-t^2)^{2i}\right. \\
&\qquad \left. - \sum_{i\geqslant 0} \frac{1}{2}\binom{2n+2+2i}{2i+1} 2\delta_1^{2i+1}(t-t^2)^{2i+1}\right\} \\
&= \sum_{i\geqslant 0} \binom{2n+1+2i}{2i} z^i (t-t^2)^{2i} - \sum_{i\geqslant 0} \frac{1}{2}\binom{2n+2+2i}{2i+1} vz^i(t-t^2)^{2i+1}.
\end{aligned}$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let n be even. Let $y_i = z^{i/2}$ if i is even and $y_i = -2^{-1}vz^{(i-1)/2}$ if i is odd. Then, by Theorem 1.6, $\# y_i = 2^{2n+k-2i}$ for $i \leqslant n$ and $y_i = 0$ for $i > n$. Also, by Lemma 3.4,

$$\begin{aligned}
\gamma_t(-\tau_0) &= \sum_l \binom{2n+1+l}{l} y_l t^l (1-t)^l \\
&= \sum_s (-1)^s t^s \sum_{2^{-1}s \leqslant l \leqslant s} (-1)^l \binom{l}{s-l}\binom{2n+1+l}{l} y_l.
\end{aligned}$$

Hence, if S^{4n+3}/Q_k is immersed in $R^{4n+2+2i}$, then for all $s \geqslant 2i$,

$$\sum_{2^{-1}s \leqslant l \leqslant s} (-1)^l \binom{l}{s-l}\binom{2n+1+l}{l} y_l = 0.$$

The desired equalities

$$\binom{2n+1+l}{l} y_l = 0$$

are obtained by a downward induction on s, beginning with $s = 2n$.

The other cases are similar.

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