NONIMMERSIONS AND NONEMBEDDINGS OF QUATERNIONIC SPHERICAL SPACE FORMS

BY

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ABSTRACT. We determine the orders of the canonical elements in KO-rings of quaternionic spherical space forms \( S^{4n+3}/Q_k \) and apply them to prove the nonexistence theorems of immersions and embeddings of \( S^{4n+3}/Q_k \) in Euclidean spaces.

1. Statements of results. Let \( Q_k = \{ x, y: x^{2k-1} = y^2, xyx = y \} \) be the generalized quaternion group of order \( 2^k \) \((k > 2)\). (Note that the relation \( x^{2k-1} = 1 \) follows from the above two relations.) Let \( d_i: Q_k \to S^3 = \text{Sp}(1) = SU(2) \) be the natural inclusion defined by \( d_i(x) = \exp(2\pi i/2^{k-1}) \), \( d_i(y) = j \). Then \( Q_k \) acts freely on the unit sphere \( S^{4n+3} \) in the quaternion \((n + 1)\)-space \( H^{n+1} \) by the diagonal action \((n + 1)d_i: Q_k \to \text{Sp}(n + 1) \). The quotient manifold \( S^{4n+3}/Q_k \) is called the quaternionic spherical space form. D. Pitt \([8]\) studied the structure of KO-rings of \( S^{4n+3}/Q_k \) and considered the problem of immersing or embedding \( S^{4n+3}/Q_k \) in Euclidean space \( R^m \) using the techniques of M. F. Atiyah \([1]\) (cf. also \([5, \text{Chapter 6}]\) and \([6, \text{Chapter 3}]\)).

The purpose of this note is to determine the orders of the canonical elements in \( K\text{-}O(S^{4n+3}/Q_k) \) and apply them to improve the nonexistence theorems of immersions and embeddings of \( S^{4n+3}/Q_k \). Let \( M \not\subset R^m \) (or \( M \not\subset R^m \)) denote nonexistence of a \( C^\infty\)-immersion (or a \( C^\infty\)-embedding) of \( M \) in \( R^m \). Let \( \nu(n) \) be the nonnegative integer such that \( n = q \cdot 2^i(n) \), where \( q \) is odd. Our main theorem is

**Theorem 1.1.** If \( \nu(2n+1+i) < 2n + k - 2i + \epsilon \), then \( S^{4n+3}/Q_k \not\subset R^{4n+2+2i} \) and \( S^{4n+3}/Q_k \not\subset R^{4n+3+2i} \), where \( \epsilon = 0 \) if \( n \) is even \( > 0 \), and \( \epsilon = 1 \) if \( n \) is odd.

Define

\[
N(n, k) = \max[i: 1 \leq i \leq n, \nu(2n+1+i) < 2n + k - 2i + \epsilon].
\]

The case \( N(n, k) = n \) was obtained by Pitt \([8, \text{Corollary 5.6}]\), and the case \( k = 3 \) was obtained by K. Fujii. It follows from Theorem 1.1, for example, that

\[
\begin{align*}
S^{15}/Q_k \not\subset R^{20}, & \quad S^{15}/Q_k \not\subset R^{21}, & \quad S^{31}/Q_3 \not\subset R^{42}, & \quad S^{31}/Q_3 \not\subset R^{43}, \\
S^{31}/Q_k \not\subset R^{44}, & \quad S^{31}/Q_k \not\subset R^{45} & \text{for } k \geq 4.
\end{align*}
\]

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The complex representation ring $R_{C}(Q_{k})$ of $Q_{k}$ is generated as a free abelian group by $1, a, b, c$ and $d_{r}$ ($r = 1, 2, \ldots, 2^{k-2} - 1$) defined below (cf. [2, §47.15; 8, §1 and 3, §3]):

\[
\begin{align*}
1(x) &= 1, & a(x) &= 1, & b(x) &= -1, & c(x) &= -1, \\
1(y) &= 1, & a(y) &= -1, & b(y) &= 1, & c(y) &= -1,
\end{align*}
\]

\[
d_r(x) = \begin{bmatrix} \omega^r & 0 \\ 0 & \omega^{-r} \end{bmatrix}, \quad d_r(y) = \begin{bmatrix} 0 & (1)^{r} \\ 1 & 0 \end{bmatrix},
\]

where $\omega$ is a primitive $2^{k-1}$th root exp($2\pi i/2^{k-1}$) of unity. The multiplicative structure is given by

\[
a^2 = b^2 = c^2 = 1, \quad ab = c, \quad d_r d_s = d_{r+s} + d_{r-s}, \quad bd_r = d_{2^{k-2}+r},
\]

where we define

\[
d_0 = 1 + a, \quad d_{2^{k-2}+r} = b + c, \quad d_{-r} = d_r, \quad d_{2^{k-2}+r} = d_{2^{k-2}-r}.
\]

The reduced representation ring $\tilde{R}_{C}(Q_{k})$ is generated as a free abelian group by $a = a-1, b = b-1, \gamma = a + b + c - 3, \delta_r = d_r - 2 (r = 1, 2, \ldots, 2^{k-2} - 1)$ with relations (cf. [3, Proposition 3.3]):

\[
a^2 = -2a, \quad \beta^2 = -2\beta, \quad \gamma = \alpha\beta + 2a + 2\beta, \quad \alpha\delta_1 = -2\alpha,
\]

\[
\beta\delta_1 = -2\beta + \delta_{2^{k-2}-1} - \delta_1, \quad \delta_{r+1} = \delta_r + 2\delta_{1} + 2\delta_r - \delta_{r-1}
\]

where $\delta_{2^{k-2}} = \gamma - \alpha, \delta_0 = \alpha$. Thus $\tilde{R}_{C}(Q_{k})$ is generated by $a, \beta$ and $\delta_1$ as a ring.

Let $c_{R}: R_{R}(Q_{k}) \to R_{C}(Q_{k})$ be the complexification. The real representation ring $R_{R}(Q_{k})$, considered as the subring $c_{R}(R_{R}(Q_{k}))$ of $R_{C}(Q_{k})$, is generated by $1, a, b, c$, $d_2$, and $2d_{2r+1}$ ($r = 0, 1, \ldots, 2^{k-3} - 1$) (cf. [8, Proposition 1.5]).

Define elements $v$ and $z$ in $R_{R}(Q_{k})$ by

\[
(1.2) \quad c_{R}(2\delta_1) = v, \quad c_{R}^{-1}(\delta_2^2) = z.
\]

Let $\lambda$ be the canonical complex plane bundle over the quaternion projective space $HP^n = S^{4n+3}/S^3$, and let $\pi: S^{4n+3}/Q_k \to HP^n$ be the natural projection. Let $\xi_{C}: \tilde{R}_{C}(Q_{k}) \to \tilde{K}(S^{4n+3}/Q_{k})$ be the projection defined in [3, §4] and put $\delta = \xi_{C}(\delta_1)$. Then we have $\delta = \pi^*\lambda - 2$ (cf. [3, Lemma 4.4]). The order $\#\delta_i$ of $\delta_i \in \tilde{K}(S^{4n+3}/Q_{k})$ is determined by H. Oshima in [7, Proposition 5.2] and T. Mormann in [6, Chapter 2, Theorem 4.52] as follows.

**Proposition 1.3.** $\#\delta_i = 2^{2n+k-2} (1 \leq i \leq n)$.

Let $r_{C}: \tilde{K}(X) \to \tilde{KO}(X)$ and $c_{R}: \tilde{KO}(X) \to \tilde{K}(X)$ be the realification and the complexification, respectively. Let $\xi_{R}: \tilde{R}_{R}(Q_{k}) \to \tilde{KO}(S^{4n+3}/Q_{k})$ be the projection defined in [4, (3.9)] (or in [8, Theorem 2.5]). Then, by (1.2),

\[
\xi_{R}v = r_{C}(\pi^*\lambda - 2) \quad \text{and} \quad \xi_{R}z = c_{R}^{-1}((\pi^*\lambda - 2)^2)
\]

(cf. [4, Lemma 3.10]), because $\delta_1$ is self-conjugate and $c_{R}r_{C} = 1 + \text{conjugation}$. For simplicity we write $v$ and $z$ instead of $\xi_{R}v$ and $\xi_{R}z$. Then, for the complexification $c_{R}: \tilde{KO}(S^{4n+3}/Q_{k}) \to \tilde{K}(S^{4n+3}/Q_{k})$, we have

\[
(1.4) \quad c_{R}(v) = 2\delta, \quad c_{R}(z) = \delta^2.
\]
Let \( \#a \) (or \( \#A \)) denote the order of an element \( a \) (or a group \( A \)). The orders of the canonical elements in \( \widetilde{KO}(S^{4n+3}/Q_k) \) are determined as follows.

**Theorem 1.5.** For \( z^i \) and \( vz^i \) \( \in \widetilde{KO}(S^{8m+7}/Q_k) \),
\[
\#z^i = 2^{4m+k-4i+3} \quad (0 < i \leq m),
\]
\[
\#vz^i = 2^{4m+k-4i} \quad (0 \leq i \leq m).
\]

**Theorem 1.6.** For \( z^i \) and \( vz^i \) \( \in \widetilde{KO}(S^{8m+3}/Q_k) \),
\[
\#z^i = 2^{4m+k-4i} \quad (0 < i \leq m),
\]
\[
\#vz^i = 2^{4m+k-4i-3} \quad (0 \leq i \leq m).
\]

**Corollary 1.7.** For \( v \in \widetilde{KO}(S^{4n+3}/Q_k) \),
\[
\#v = \begin{cases} 2^{2n+k-2} & \text{if } n \text{ is odd}, \\ 2^{2n+k-3} & \text{if } n \text{ is even} > 0. \end{cases}
\]


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**2. Proofs of Theorems 1.5 and 1.6.** First we prepare a lemma.

**Lemma 2.1.** In \( \widetilde{KO}(S^{4n+3}/Q_k) \):
\[
2^k v + \sum_{j=1}^{2^{k-3}} (2c_{2j} + c_{2j+1} v) z^i = 0,
\]
\[
2^k z + \sum_{j=1}^{2^{k-3}} (2^{-1}c_{2j} v + c_{2j+1} z) vz^i = 0,
\]
where \( c_j \) are integers satisfying
\[
(*) \quad c_2 = 2^{k-2}(2^{2k-3} + 1)/3, \quad c_{2^s-2+1} = 1, \\
\nu(c_s) \geq \max(1, k - s) \quad \text{for } 0 < s \leq 2^{k-2}.
\]

**Proof.** It was proved in [6] that \( \sum_{s=1}^{2k-2+1} c_s \delta^i_1 = 0 \) in \( \tilde{R}_c(Q_k) \), where
\[
c_s = \binom{2^{k-2} + s}{2s - 1} + 2 \binom{2^{k-2} + s - 1}{2s - 1} + \binom{2^{k-2} + s - 2}{2s - 1}.
\]

\( (*) \) follows easily from
\[
\nu\left(\frac{m}{n}\right) = \alpha(n) + \alpha(m - n) - \alpha(m),
\]
where \( \alpha(n) \) denotes the number of nonzero terms in the dyadic expansion of \( n \). (a) and (b) are proved by multiplying this by 2 and \( \delta_1 \), respectively, and applying \( c_1 \tilde{c}_1 \).
Lemma 2.2. In $\widetilde{KO}(S^{8m+7}/Q_k)$:

(i) $z^{m+1} = 0$,

(ii) $2^{4i+k}vz^{m-i} = 0 \quad (0 \leq i \leq m)$,

(iii) $2^{4i+k+3}z^{m-i} = 0 \quad (0 \leq i \leq m)$,

(iv) $2^{4i+k-1}vz^{m-i} + 2^{4i+k+2}z^{m-i} = 0 \quad (0 \leq i \leq m)$,

(v) $2^{4i+k+2}z^{m-i} + 2^{4i+k+3}vz^{m-i-1} = 0 \quad (0 \leq i \leq m)$.

Proof. (i) follows from [8, Theorem 2.5].

(ii) and (iii) are proved by induction on $i$. (ii) for $i = 0$ follows from (i) and (a) $\times z^m$. (iii) for $i = 0$ follows from (b) $\times 2^iz^{m-1}$ and (ii) for $i = 0$. (iii) for any $i \leq j - 1$, (ii) for any $j \leq i$ and (b) $\times 2^{4i+3}z^{m-i-1}$ imply (iii) for $j = i$. (ii) for any $j \leq i$, (iii) for any $j \leq i$ and (a) $\times 2^{4i+4}z^{m-i-1}$ imply (ii) for $j = i + 1$.

Using (b) $\times 2^{4i+2}z^{m-i-1}$ (resp. (a) $\times 2^{4i+3}z^{m-i-1}$) and (i) $\sim$ (iii), we obtain (iv) (resp. (v)).

Lemma 2.3. In $\widetilde{KO}(S^{8m+3}/Q_k)$:

(i) $z^{m+1} = 0$,

(ii) $vz^m = 0$,

(iii) $2^{4i+k}z^{m-i} = 0 \quad (0 \leq i \leq m)$,

(iv) $2^{4i+k+1}vz^{m-i-1} = 0 \quad (0 \leq i \leq m)$,

(v) $2^{4i+k-1}z^{m-i} + 2^{4i+k}vz^{m-i-1} = 0 \quad (0 \leq i \leq m)$,

(vi) $2^{4i+k}vz^{m-i-1} + 2^{4i+k+3}z^{m-i-1} = 0 \quad (0 \leq i \leq m - 1)$.

Proof. (i) follows from Lemma 2.2(i) and the naturality. (ii) is proved in [4, §4]. The proofs of (iii) $\sim$ (vi) are similar to those of Lemma 2.2(ii) $\sim$ (v), so we omit the details.

Proof of Theorem 1.5. By Lemma 2.2(iii), (ii) we have

$\#z^i \leq 2^{4m+k-4i+3} \quad (0 < i \leq m)$ and $\#vz^i \leq 2^{4m+k-4i} \quad (0 \leq i \leq m)$.

Let $j$: $S^{8m+3}/Q_k \to S^{8m+7}/Q_k$ be the natural inclusion. Then it follows from [4, §4] that Ker $j^*$, the kernel of the induced homomorphism $j^*$: $\widetilde{KO}(S^{8m+7}/Q_k) \to \widetilde{KO}(S^{8m+3}/Q_k)$, is generated by $vz^m$. According to [5, Chapter 6, Proposition 5.7],

$\#\widetilde{KO}(S^{8m+7}/Q_k) = 2^{2m}2^{4m+k+4}$ and $\#\widetilde{KO}(S^{8m+3}/Q_k) = 2^{2m}2^{4m+4}$.

Hence, by Lemma 2.2(ii), we obtain

$2^k = \#\widetilde{KO}(S^{8m+7}/Q_k)/\#\widetilde{KO}(S^{8m+3}/Q_k) \leq \#\text{Ker } j^* = \#vz^m \leq 2^k$.

Thus $\#vz^m = 2^k$. Therefore, by Lemma 2.2(iv), (v) we have

$2^{4m+k-4i+2}z^i = -2^{4m+k-4i-1}vz^i = \ldots = -2^{k-4}vz^i \neq 0$.

Proof of Theorem 1.6. By Lemma 2.3(iii), (iv) we have

$\#z^i \leq 2^{4m+k-4i} \quad (0 < i \leq m)$ and $\#vz^i \leq 2^{4m+k-4i-3} \quad (0 \leq i \leq m)$.  

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By Lemma 2.3(vi), (v) we have
\[ 2^{4m+k-4i}z = -2^{4m+k-4i-4}v = \ldots = -2^{4m+k-4}v. \]
But, by (1.4) and Proposition 1.3, \( c_R(2^{4m+k-4}v) = 2^{4m+k-3} \neq 0 \), so \( 2^{4m+k-4}v \neq 0 \).

3. Atiyah's criterion. Exterior power operation \( \lambda(\alpha), \alpha \in R_C(Q_k) \), is determined by
\[
\begin{align*}
\lambda^0(\alpha) &= 1, \quad \lambda^1(\alpha) = \alpha, \quad \lambda^i(\alpha) = 0 \quad \text{for } i > 1, \alpha = 1, a, b, c, \\
\lambda^2(d_r) &= 1 \quad \text{for } r \text{ odd}, \quad \lambda^2(d_r) = a \quad \text{for } r \text{ even}, \quad \lambda^i(d_r) = 0 \quad \text{for } i > 2.
\end{align*}
\]
Define \( \lambda_i(\alpha) = \sum_{i \geq 0} \lambda(\alpha)t^i \). Then the Grothendieck \( \gamma \)-operations \( \gamma' \) are obtained from the equality of the polynomials
\[ \lambda_{t/(1-t)}(\alpha) = \gamma_i(\alpha) = \sum_{i \geq 0} \gamma'(\alpha)t^i. \]

The following is well known (cf. [8, p. 2]).

**Lemma 3.1.** \( \gamma_i(\delta_{2r+1}) = 1 + \delta_{2r+1}(t - t^2) \), where \( \delta_{2r+1} = d_{2r+1} - 2 \in \tilde{R}_C(Q_k) \).

Let \( v \) and \( z \) be the elements in \( \tilde{R}_R(Q_k) \) defined in (1.2). Then we prove

**Lemma 3.2.** \( \gamma_i(v) = 1 + v(t - t^2) + z(t - t^2)^2 \).

**Proof.** Since \( \gamma_i \) is natural with respect to the complexification \( c_R \), we have, by Lemma 3.1 and (1.2),
\[
\gamma_i(v) = \gamma_i c_R(2\delta_1) = c_R^{-1}\{\gamma_i(\delta_1)\}^2 = c_R^{-1}\{1 + \delta_1(t - t^2)\}^2
\]
\[= c_R^{-1}\{1 + 2\delta_1(t - t^2) + \delta_1^2(t - t^2)^2\} = 1 + v(t - t^2) + z(t - t^2)^2. \]

As an application of Grothendieck \( \gamma \)-operations in \( KO \)-theory, M. F. Atiyah [1] obtained the following

**Theorem 3.3.** Let \( M \) be an \( n \)-dimensional compact smooth manifold and \( \tau_0 \in \overline{KO}(M) \) the stable class of the tangent bundle of \( M \). Then, if \( M \) is immersible (resp. embeddable) in \( R^{n+r} \), \( \gamma'(-\tau_0) = 0 \) for all \( i > r \) (resp. \( i \geq r \)).

**Lemma 3.4.** Let \( \tau_0 \) be the stable class of the tangent bundle \( \tau = \tau(S^{4n+3}/Q_k) \) of \( S^{4n+3}/Q_k \). Then
\[ \gamma_i(-\tau_0) = \sum_{i \geq 0} \left(2n + 1 + 2i\right)z^i(t - t^2)^{2i} - \sum_{i \geq 0} \frac{1}{2i} \left(2n + 2 + 2i\right) vz^i(t - t^2)^{2i+1}. \]

**Proof.** According to [9, Corollary 3.3],
\[ -\tau_0 = 4n + 3 - \tau = 4(n + 1) - (n + 1)(r_C \pi^* \lambda) = -(n + 1)v. \]
By Lemma 3.2 and (1.2), we have
\[ y_i(-\tau_0) = y_i\left(-\left(n + 1\right)v\right) = \left(y_i\left(v\right)\right)^{-n-1} = \left(1 + v(t - t^2) + z(t - t^2)^2\right)^{-n-1} \]
\[ = c_R\left\{1 + \delta^1(t - t^2)\right\}^{-2n-2} \]
\[ = c_R\left\{\sum_{i \geq 0} (-1)^i\left(\begin{array}{c} 2n + 1 + j \\ j \end{array}\right)\delta^i(t - t^2)^i\right\} \]
\[ = c_R\left\{\sum_{i \geq 0} \left(\begin{array}{c} 2n + 1 + 2i \\ 2i \end{array}\right)\delta_{2i}^i(t - t^2)^{2i} \right\} \]
\[ - \sum_{i \geq 0} \frac{1}{2} \left(\begin{array}{c} 2n + 2 + 2i \\ 2i + 1 \end{array}\right)2\delta_{2i+1}^{2i+1}(t - t^2)^{2i+1} \]
\[ = \sum_{i \geq 0} \left(\begin{array}{c} 2n + 1 + 2i \\ 2i \end{array}\right)z^i(t - t^2)^{2i} - \sum_{i \geq 0} \frac{1}{2} \left(\begin{array}{c} 2n + 2 + 2i \\ 2i + 1 \end{array}\right)vz^i(t - t^2)^{2i+1}. \]

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( n \) be even. Let \( y_i = z^{i/2} \) if \( i \) is even and \( y_i = -2^{-1}vz^{(i-1)/2} \) if \( i \) is odd. Then, by Theorem 1.6, \( \#y_i = 2^{2n+k-2i} \) for \( i < n \) and \( y_i = 0 \) for \( i > n \). Also, by Lemma 3.4,
\[ y_i(-\tau_0) = \sum_s \left(\begin{array}{c} 2n + 1 + l \\ l \end{array}\right)y_lt^l(1 - t)^l \]
\[ = \sum_s (-1)^s t^s \sum_{2^{-1}s \leq l \leq s} (-1)^l \left(\begin{array}{c} l \\ s - l \end{array}\right)\left(\begin{array}{c} 2n + 1 + l \\ l \end{array}\right)y_l. \]

Hence, if \( S^{4n+3}/Q_k \) is immersed in \( R^{4n+2+2i} \), then for all \( s \geq 2i \),
\[ \sum_{2^{-1}s \leq l \leq s} (-1)^l \left(\begin{array}{c} l \\ s - l \end{array}\right)\left(\begin{array}{c} 2n + 1 + l \\ l \end{array}\right)y_l = 0. \]

The desired equalities
\[ \left(\begin{array}{c} 2n + 1 + l \\ l \end{array}\right)y_l = 0 \]
are obtained by a downward induction on \( s \), beginning with \( s = 2n \).

The other cases are similar.

**References**


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