

ON A QUESTION OF QUILLEN

BY

S. M. BHATWADEKAR AND R. A. RAO

ABSTRACT. Let R be a regular local ring, and f a regular parameter of R . Quillen asked whether every projective R_f -module is free. We settle this question when R is a regular local ring of an affine algebra over a field k . Further, if R has infinite residue field, we show that projective modules over Laurent polynomial extensions of R_f are also free.

Introduction. In [Q] Quillen posed the following

Question. Let R be a regular local ring and f a regular parameter of R . Are all finitely generated projective R_f -modules free?

An affirmative answer implies the

CONJECTURE (BASS-QUILLEN). Let R be a regular local ring. Then every finitely generated projective $R[T]$ -module is free.

Lindel [L, Theorem] has proved the Bass-Quillen conjecture when R is the local ring of an affine algebra over a field k at a regular point (not necessarily closed). However, it is not clear whether a positive solution to the Bass-Quillen conjecture implies the truth of Quillen's question. Therefore the latter is, apart from its application to the Bass-Quillen conjecture, of some independent interest. In this paper we settle the Quillen question affirmatively when R is a regular local ring of an affine algebra over a field k .

Curiously, in this case we are able to reduce the Quillen question to the Bass-Quillen conjecture via the following interesting result (see Theorem 2.4).

THEOREM A. Let R be any local ring. Then every stably free $R[T]$ -module is free if and only if every stably free $R(T)$ -module is free.

Swan has given an example of a four-dimensional regular affine complex algebra A and a projective module P over $A[Y, Y^{-1}]$ which is not extended from A [Sw, §2]. Moreover he has shown that $P_{\mathfrak{p}}$ is free for every prime ideal \mathfrak{p} of A . Theorem A shows the fact that $P_{\mathfrak{p}}$ is free is not accidental.

Swan's example leads us to consider projective modules over Laurent polynomial extensions of R_f . The constraints in the proof of Theorem 2.4 force us to reconstruct a different approach in this context. However in this approach we need to assume

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that R has infinite residue field. In Theorem 3.2, we prove

THEOREM B. *Let R be a regular local ring of an affine algebra over a field k with infinite residue field, and let f be a regular parameter of R . Then every finitely generated projective $R_f[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module is free.*

Consequently, all finitely generated projective $R[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -modules are free.

Mohan Kumar has answered the Quillen question affirmatively when R is a power series ring over a field [Mo, Corollary 2]. In the last section of this paper we extend his arguments to show that every finitely generated projective $R_f[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module is free. As an interesting application of this result we prove that every finitely generated projective $R(T)[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module is free when R is a power series ring over a field.

1. Preliminaries. Throughout this paper all rings will be commutative noetherian and all modules will be finitely generated.

(A) *Patching technique.* Let $\psi: B \rightarrow A$ be a homomorphism of rings and let s be an element of B such that:

- (i) s is a non-zero-divisor in B .
- (ii) $\psi(s)$ is a non-zero-divisor in A .
- (iii) ψ induces an isomorphism $B/sB \simeq A/\psi(s)A$.

The commutative diagram

$$(1.1) \quad \begin{array}{ccc} B & \xrightarrow{\psi} & A \\ \downarrow & & \downarrow \\ B_s & \xrightarrow{\psi_s} & A_s \end{array}$$

resulting from a situation as above will be called a *patching diagram*.

We shall sometimes describe (1.1) as $B \xrightarrow{\psi} A$ is *analytically isomorphic along s* .

It is easy to see that diagram (1.1) is *cartesian* (i.e. B is the fibre product of B_s and A over A_s).

Let $\mathbf{P}(R)$ denote the category of all finitely generated projective R -modules.

Given a patching diagram (1.1), the corresponding square

$$\begin{array}{ccc} \mathbf{P}(B) & \rightarrow & \mathbf{P}(A) \\ \downarrow & & \downarrow \\ \mathbf{P}(B_s) & \rightarrow & \mathbf{P}(A_s) \end{array}$$

is cartesian. This is a special case of a classical result of Milnor as shown in [Ry].

EXAMPLES. (1) *Covering diagrams.* Let s and t be elements of a ring B such that $Bs + Bt = B$. Assume s is a non-zero-divisor in B . Then $B \rightarrow B_t$ is analytically isomorphic along s .

(2) Let $B = k[[Z_1, \dots, Z_{p-1}]][[Z_p]]$ and $A = k[[Z_1, \dots, Z_p]]$, where k is a field. Let f be an element of B which is a *distinguished monic* in Z_p , i.e. it is a monic polynomial

in Z_p with its lower degree coefficients belonging to the maximal ideal of $k[[Z_1, \dots, Z_{p-1}]]$. As a consequence of the Weierstrass Preparation Theorem we see that

$$\begin{array}{ccc} B & \hookrightarrow & A \\ \downarrow & & \downarrow \\ B_f & \hookrightarrow & A_f \end{array}$$

is a patching diagram.

(3) Let (R, m) be a local ring. A monic polynomial $f \in R[T]$ is called a *Weierstrass polynomial* if $f = T^n + a_1 T^{n-1} + \dots + a_n$, $a_i \in m$ for $i = 1, 2, \dots, n$.

Let $f \in R[T]$ be a Weierstrass polynomial. Then we have a patching diagram:

$$\begin{array}{ccc} R[T] & \hookrightarrow & R[T]_{(m,T)} \\ \downarrow & & \downarrow \\ R[T]_f & \hookrightarrow & R[T]_{(m,T)}[1/f] \end{array}$$

PROOF. Undoubtedly, we do have an inclusion map $R[T] \hookrightarrow R[T]_{(m,T)}$. Since f is monic, $R[T]/(f(T))$ is semilocal, and any maximal ideal n of it "sits" over m . But since $f(T) \in n$ and f is a Weierstrass polynomial, we have $T \in n$. Therefore $n = (m, T)$. Thus $R[T]/(f(T))$ is local, and so

$$R[T]/(f(T)) = R[T]/(f(T))_{(m,T)} = R[T]_{(m,T)}/(f(T)).$$

(4) Let Λ be a flat \mathbf{Z} -algebra. Then applying $\otimes_{\mathbf{Z}} \Lambda$ to the patching diagram (1.1), we get a new patching diagram

$$\begin{array}{ccc} B \otimes \Lambda & \rightarrow & A \otimes \Lambda \\ \downarrow & & \downarrow \\ B_s \otimes \Lambda & \rightarrow & A_s \otimes \Lambda \end{array}$$

In applications here we shall take $\Lambda = \mathbf{Z}[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$, a Laurent polynomial ring over \mathbf{Z} .

(B) *Regular k-spots*. Let k be a field. By a *regular spot over a field k* we mean a localisation C_p of a finitely generated k -algebra C at a regular prime $p \in \text{Spec } C$.

Lindel [L, Proposition 2] analysed regular k -spots over perfect fields as étale extensions of rings of the type $K[Z_1, \dots, Z_n]_{(\varphi(Z_1), Z_2, \dots, Z_n)}$. We shall need the following finer analysis (see proof of [N, Theorem 2.8]):

PROPOSITION. *Let (R, m) be a regular k -spot over a perfect field k . Let $g \in m$ and f be any regular parameter of R with (g, f) a sequence. Then there exist a field $K \supset k$ and a regular K -spot R' such that:*

(i) $R' = K[Z_1, \dots, Z_d]_{(\varphi(Z_1), \dots, Z_d)}$, where $\varphi(Z_1) \in K[Z_1]$ is an irreducible monic. Moreover, we may assume $Z_d = f$.

(ii) $R' \hookrightarrow R$ is an analytic isomorphism along h for some $h \in gR \cap R'$.

REMARK. If R above has infinite residue field then the field K is also infinite.

2. The Quillen question for regular k -spots. We begin this section with a lemma.

LEMMA 2.1. *Let R be a semilocal ring and let $R[T]$ be a polynomial algebra in one variable over R . Let J be an ideal of $R[T]$ containing a monic polynomial. Let $\mu(J/J^2) = d \geq 2$, where $\mu(J/J^2)$ denotes the minimal number of generators of J/J^2 . Then there exist $g_1, \dots, g_d \in J$ such that $(g_1, \dots, g_d) = J$. Moreover, g_1 can be chosen to be monic.*

PROOF. Let h_1, \dots, h_d be elements of J such that $(h_1, \dots, h_d) + J^2 = J$. Let $g \in J$ be a monic polynomial and let $g_1 = h_1 + g^N$. Then for $N \gg 0$, g_1 is monic. Moreover, $\mu(J/J^2 + (g_1)) = d - 1$. Let $R' = R[T]_{(g_1)}$ and $J' = J/(g_1)$. Then J' is an ideal of R' with $\mu(J'/J'^2) = d - 1 \geq 1$. Since g_1 is monic, R' is semilocal. Therefore there exist $g'_2, \dots, g'_d \in J'$ such that $(g'_2, \dots, g'_d) = J'$. Let g_i be a lift of g'_i in J for $2 \leq i \leq d$. Then obviously $(g_1, g_2, \dots, g_d) = J$.

In this paper $R(T)$ will denote the localisation of the polynomial algebra $R[T]$ by the multiplicatively closed subset of all monic polynomials in $R[T]$.

We use Lemma 2.1 in the proof of the following important theorem.

THEOREM 2.2. *Let R be a local ring and let P be a projective $R(T)$ -module such that $P \oplus R(T) \simeq R(T)^d$. Then there exists a projective $R[T]$ -module Q such that $Q \oplus R[T] \simeq R[T]^d$ and $Q \otimes_{R[T]} R(T) \simeq P$.*

PROOF. Let $Y = T^{-1}$ and $\tilde{R} = R[Y]_{(\mathfrak{m}, Y)}$ where \mathfrak{m} denotes the maximal ideal of R . Then $R[Y] \hookrightarrow \tilde{R}$ is analytically isomorphic along Y and we have the patching diagram

$$\begin{array}{ccc} R[Y] & \hookrightarrow & \tilde{R} \\ \downarrow & & \downarrow \\ R[Y, Y^{-1}] & \hookrightarrow & \tilde{R}_Y (= R(T)) \end{array}$$

Let $[a_1, \dots, a_d]$ denote a unimodular row of $R(T)^d$ defining the projective module P over $R(T)$. Since $\tilde{R}_Y = R(T)$, without loss of generality we can assume that $a_i \in \tilde{R}$ for $1 \leq i \leq d$. Let I be the ideal of \tilde{R} generated by a_1, \dots, a_d . If $\mu(I) \leq d - 1$ then, since \tilde{R} is local, one of the generators, say a_d , belongs to the ideal generated by the rest of the generators a_1, \dots, a_{d-1} . But then $P \simeq R(T)^{d-1}$ and, taking $Q \simeq R[T]^{d-1}$, we are through. Therefore we assume that $d = \mu(I)$.

Since $[a_1, \dots, a_d]$ is a unimodular row over \tilde{R}_Y we have $Y^n \in I$ for some positive integer n . Let $J = I \cap R[Y]$. Then, since $R[Y] \hookrightarrow \tilde{R}$ is analytically isomorphic along Y and $Y^n \in I$, it follows that $J\tilde{R} = I$ and $\mu(J/J^2) = d \geq 2$. Therefore by Lemma 2.1, $J = (g_1, \dots, g_d)$ and g_1 is monic in Y . Since \tilde{R} is local and $\mu(I) = d$, there exists an element σ of $\text{GL}_d(\tilde{R})$ such that $[g_1, \dots, g_d]\sigma = [a_1, \dots, a_d]$. Now consider the following short exact sequence:

$$\begin{array}{ccccccc} 0 & \rightarrow & R[Y, Y^{-1}] & \rightarrow & R[Y, Y^{-1}]^d & \rightarrow & Q' \rightarrow 0 \\ & & & & \downarrow & & \\ & & & & 1 & \mapsto & [g_1, \dots, g_d] \end{array}$$

Since $Y^n \in J = (g_1, \dots, g_d), [g_1, \dots, g_d]$ is a unimodular row of $R[Y, Y^{-1}]^d$. Therefore Q' is a projective $R[Y, Y^{-1}]$ -module and $P \simeq Q' \otimes_{R[Y, Y^{-1}]} \tilde{R}_Y$. But Q'_{g_1} is free and g_1 is monic in Y . Hence by [Sw, Lemma 1.3] there exists a projective module Q over $R[Y^{-1}]$ such that $Q' \simeq Q \otimes_{R[Y^{-1}]} R[Y, Y^{-1}]$. Now $Q' \oplus R[Y, Y^{-1}] \simeq R[Y, Y^{-1}]^d$ and therefore

$$(Q \oplus R[Y^{-1}])_{Y^{-1}} \simeq R[Y, Y^{-1}]^d.$$

Hence, by [Q, Theorem 3 and Su, Theorem 1], $Q \oplus R[Y^{-1}] \simeq R[Y^{-1}]^d$, and we are through.

COROLLARY 2.3. *Let R be a two-dimensional local ring with $\frac{1}{2} \in R$. Then every stably free $R(T)$ -module is free.*

PROOF. By Theorem 2.2 it suffices to show that every stably free $R[T]$ -module is free. A proof of this can be found in [BR, 2.7].

As an easy consequence of Theorem 2.2 we get Theorem A.

THEOREM 2.4. *Let R be a local ring. Then every stably free $R[T]$ -module is free if and only if every stably free $R(T)$ -module is free.*

As an application of Theorem 2.4 we prove

THEOREM 2.5. *Let R be a regular spot of dimension d over a field k , and let f be a regular parameter of R . Then every finitely generated projective R_f -module is free.*

PROOF. Let P be a projective R_f -module. If $d = 1$ then R_f is a field and there is nothing to prove. So we assume $d \geq 2$.

We first assume that k is perfect.

Let g be an element of R such that P_g is free. Without loss of generality we may assume that g and f have no common factors in R . Hence (g, f) is a sequence in R .

Now by [N, Theorem 2.8], as stated in the preliminaries, there exist a field $K \supset k$ and a K -spot $R' = K[Z_1, \dots, Z_d]_{(\varphi(Z_1), \dots, Z_d)}$ such that $R' \hookrightarrow R$ is analytically isomorphic along h for some $h \in gR \cap R'$. Moreover $Z_d = f$.

Therefore $R'_{Z_d} \hookrightarrow R_f$ is analytically isomorphic along h . Hence, since P_h is free, by (1.1) there exists a projective R'_{Z_d} -module Q such that $P \simeq Q \otimes_{R'_{Z_d}} R_f$. Therefore it is enough to prove that Q is free.

Let $S = K[Z_1, \dots, Z_{d-1}]_{(\varphi(Z_1), \dots, Z_{d-1})}$ and $T = Z_d^{-1}$. Then $R'_{Z_d} = S(T)$. Now we are through in view of Theorem 2.4 and [L, Theorem].

In general we can reduce the problem to the case when the ground field k is perfect as follows:

Let k_0 be the prime subfield of k . If k is not perfect then $\text{tr deg}_{k_0} k \geq 1$. By the argument of Swan (see [L]) there exist a function field k' of k_0 contained in k and a regular k' -spot R' containing f such that:

- (1) $R' \hookrightarrow R$;
- (2) f is a regular parameter of R' ;
- (3) projective module P extends from R'_f ;
- (4) $\text{tr deg}_{k_0} k' \geq 1$.

Since k' is a function field of k_0 , R' is a spot over k_0 also. Moreover, as R' contains k' , by virtue of (4), R' has infinite residue field. This observation will be needed later in Theorem 3.2.

This completes the proof of Theorem 2.5.

3. Laurent polynomial extensions of R_f . We begin with a proposition which we shall use in the sequel.

PROPOSITION 3.1. *Let B be a one-dimensional noetherian domain, and let A be an overring of $B[T]$ which is contained in its quotient field. Assume that A is a unique factorization domain. Then every projective $A[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module is free.*

PROOF. We induct on $n + m$. If $n + m = 0$, then by [R, Theorem 1.1(A)] every projective A -module is a direct sum of a free module and a rank one projective module. Since A is a U.F.D., $\text{Pic } A = 0$. Thus every projective A -module is free.

Assume $n + m > 0$. Let P be a projective $A[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module.

Case (i). Let $n > 0$. Let S denote the multiplicatively closed subset of $A[X_1]$ consisting of all monic polynomials in X_1 with coefficients in B . Then $B[T, X_1]_S = B(X_1)[T] \hookrightarrow A(X_1)$.

Therefore, by induction, $P \otimes A(X_1)[X_2, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ is free. Therefore, there exists a monic polynomial $f \in A[X_1]$ such that P_f is free. By [Su, Theorem 1], P is free.

Case (ii). Let $n = 0$. Let S' denote the multiplicatively closed subset of $A[Y_1]$ consisting of monic polynomials in Y_1 with coefficients in B . Then $B[T, Y_1]_{S'} = B(Y_1)[T] \hookrightarrow A(Y_1)$.

Therefore, by induction, $P \otimes A(Y_1)[Y_2^{\pm 1}, \dots, Y_m^{\pm 1}]$ is free. By [Sw, Lemma 1.3], P "extends" from $A[Y_1^{-1}, Y_2^{\pm 1}, \dots, Y_m^{\pm 1}]$.

By case (i) above, P is free.

THEOREM 3.2. *Let R be a regular spot of dimension d over a field k with infinite residue field. Let f be a regular parameter of R . Then every finitely generated projective $R_f[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module is free.*

PROOF. Let P be a projective $R_f[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module.

As in Theorem 2.5 we may assume that k is perfect.

In this case we prove the result by induction on d . If $d \leq 2$ then $\dim R_f \leq 1$, and by [Sw, Corollary 1.4], P is free. So let $d > 2$. In general, by the corollary just mentioned, $P \otimes L[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ is free, where L denotes the quotient field of R . Therefore, for some $g \in R$, P_g is free. Without loss of generality we may assume that g and f have no common factors. Thus, (g, f) is a sequence in R .

Now by [N, Theorem 2.8] as stated in the preliminaries, there exist an infinite field $K \supset k$ and a K -spot $R' = K[Z_1, \dots, Z_d]_{(\varphi(Z_1), Z_2, \dots, Z_d)}$ such that $R' \hookrightarrow R$ is analytically isomorphic along h for some $h \in gR \cap R'$. Moreover $Z_d = f$. Therefore $R'_{Z_d} \hookrightarrow R_f$ is analytically isomorphic along h . Hence, as before, we may assume that P extends from $R'_{Z_d}[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$. Call it P still. Thus it suffices to prove that

every projective module over $R'_{Z_d}[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ is free. We note here that $d > 2$. To prove this we shall use the following simple lemma (see, for example, [N, Proposition 1.11]).

LEMMA 3.3. *Let I be an ideal of a polynomial ring $B[T_1, \dots, T_n]$ of height ≥ 2 . Then I contains a nonzero homogeneous polynomial.*

Let C denote $K[Z_2, \dots, Z_d]$ and let S denote the multiplicatively closed subset of C consisting of all nonzero homogeneous polynomials in C . By Lemma 3.3, $\dim C_S \leq 1$.

Now $R'_{Z_d S}$ is a localisation of $C_S[Z_1]$. By Proposition 3.1, P_S is free. Hence for some $F \in S$, P_F is free. We may assume that Z_d does not divide F in C .

Let $F = F_1 + Z_d F_2$, $0 \neq F_1 \in K[Z_2, \dots, Z_{d-1}]$.

Since K is infinite, we may change Z_i to $Z_i + \alpha_i Z_2$, for $3 \leq i \leq d - 1$, for suitable $\alpha_i \in K$ and assume that $F(1, 0, \dots, 0) \neq 0$ with respect to the new set of variables, i.e. upto a unit F is a monic in Z_2 with coefficients in $K[Z_3, \dots, Z_d]$. Note that in view of the homogeneous change of variables, F will be homogeneous with respect to the new set of variables.

Let

$$\begin{aligned} \tilde{R}' &= K[Z_1, Z_3, \dots, Z_d]_{(\varphi(Z_1, Z_3, \dots, Z_d))}, \\ B &= \tilde{R}'[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}] \quad \text{and} \\ A &= R'[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]. \end{aligned}$$

By §1(A), Example 3, we have an analytic isomorphism $B[Z_2] \xrightarrow{\sim} A$ along F . Since (F, Z_d) is a sequence in $B[Z_2]$ we get a patching diagram

$$\begin{array}{ccc} B_{Z_d}[Z_2] & \xrightarrow{\sim} & A_{Z_d} \\ \downarrow & & \downarrow \\ B_{Z_d}[Z_2]_F & \xrightarrow{\sim} & A_{Z_d F} \end{array}$$

Since P_F is free, P extends from $B_{Z_d}[Z_2]$ ($= \tilde{R}'_{Z_d}[Z_2, X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$). By induction all projective $B_{Z_d}[Z_2]$ -modules are free. Thus, P is free.

This completes the proof of Theorem 3.2.

COROLLARY 3.4. *Let C be an affine algebra over a field k . Let $R = C_{\mathfrak{p}}$, where \mathfrak{p} is a nonmaximal regular prime ideal of R . Let f be a regular parameter of R . Then every finitely generated projective $R_f[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module is free.*

In the consequences below, R will denote a regular spot over a field k with infinite residue field.

COROLLARY 3.5. *All projective $R(T)[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -modules are free.*

PROOF. Since R is local, $R(T) = R[T^{-1}]_{(m, T^{-1})}[1/T^{-1}]$. Thus, this is a particular case of Theorem 3.2.

COROLLARY 3.6. *All projective $R[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -modules are free.*

PROOF. Immediate from Corollary 3.5 by using Suslin's monic inversion theorem [Su, Theorem 1].

We have not been able to resolve whether projective $R_f[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -modules are free when R has finite residue field. However, we believe it to be true, and towards this end we prove that all projective $R_f[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -modules of rank $\geq \dim R_f$ are free.

Before this:

PROPOSITION 3.7. *Let B be a reduced noetherian ring of dimension d and let A be an overring of $B[X]$ which is contained in its total quotient ring. Then any stably free projective $A[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module P of rank $\geq d + 1$ is free.*

PROOF. We induct on $n + m$. If $n + m = 0$, this is a consequence of [R, Theorem 1.1(B)].

The general proof can be argued as in Proposition 3.1.

We now prove

THEOREM 3.8. *Let R be a regular spot of dimension d over a field k and f be a regular parameter of R . Then every projective $R_f[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module of rank $\geq d - 1$ is free. In particular if $d = 3$, then every projective $R_f[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module is free.*

PROOF. Since all projective modules over $R_f[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ are stably free, by Swan's theorem [Sw, Theorem 1.1] every projective $R_f[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module of rank $\geq d$ is free. Our additional claim is that projective modules of rank $d - 1$ are also free.

Via Swan's argument (see [L]) we may assume k is perfect. As before, we may reduce the problem to the case when $R = K[Z_1, \dots, Z_d]_{(\varphi(Z_1), Z_2, \dots, Z_d)}$ for some field extension K of k and $f = Z_d$. Of course, K may be finite now.

Let P be a projective $R_f[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module of rank $\geq d - 1$. We prove that P is free by induction on d . We know that if $d \leq 2$, then P is free. Therefore we assume that $d > 2$.

Let $\tilde{R} = K[Z_1, \dots, Z_{d-1}]_{(\varphi(Z_1), Z_2, \dots, Z_{d-1})}$. Now $R_{Z_d Z_{d-1}}$ is a localisation of $\tilde{R}_{Z_{d-1}}[Z_d]$. Since $\dim \tilde{R}_{Z_{d-1}} \leq d - 2$, by Proposition 3.7 $P_{Z_{d-1}}$ is free.

Let $R' = K[Z_1, \dots, Z_{d-2}, Z_d]_{(\varphi(Z_1), Z_2, \dots, Z_{d-2}, Z_d)}$. Then $R'_{Z_d}[Z_{d-1}] \hookrightarrow R_{Z_d}$ is an analytic isomorphism along Z_{d-1} . Therefore, since $P_{Z_{d-1}}$ is free, by §1(A), Example 4, P extends from $R'_{Z_d}[Z_{d-1}, X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$. By induction, P is free.

COROLLARY 3.9. *Let R be a regular k -spot of dimension d . Then:*

- (i) *Every projective $R(T)[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module of rank $\geq d$ is free.*
- (ii) *Every projective $R[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module of rank $\geq d$ is free.*

We discuss some examples:

EXAMPLES. \mathbf{R} will denote the field of real numbers.

(1) This example shows that Theorem 2.5 is not true for any arbitrary $f \in R$. Let $R = \mathbf{R}[X, Y, Z]_{(X, Y, Z)}$, $f = X^2 + Y^2 + Z^2$. It is easy to see that the projective module P over R_f , given by the unimodular row (X, Y, Z) , is *not* free.

(2) This example shows that Corollary 3.6 is not valid if we replace a regular k -spot by a nonsingular affine k -algebra. For another example see [Sw]. Let $R = \mathbf{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$ be the coordinate ring of the real 2-sphere S^2 . Let P be the projective $R[T, T^{-1}]$ -module defined by the unimodular row $((1-x)T + (1+x), y, z)$, where x, y, z denote the images in R of X, Y, Z , respectively. Then P is *not* extended from R . This is because $P/(T-1)P$ is free, whereas $P/(T+1)P$ is isomorphic to the tangent bundle of S^2 and so is nontrivial.

4. Laurent polynomial extensions of $k[[Z_1, \dots, Z_d]]_f$ and $k[[Z_1, \dots, Z_d]](T)$.

PROPOSITION 4.1. *Let k be a field, and let $R = k[[Z_1, \dots, Z_d]]$. Let $f \in R$ be a regular parameter of R . Then every finitely generated projective $R_f[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module is free.*

PROOF. The case when $n = m = 0$ was covered in [Mo]. Our proof covers this case too.

Since R is complete and $f \in R$ is a regular parameter of R , we may, without any loss of generality, assume that $f = Z_1$.

We prove the result by induction on d . If $d = 1$, R_{Z_1} is a field and the result is due to Swan [Sw, Corollary 1.4]. Let $d \geq 2$. Let P be a projective $R_{Z_1}[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module. In general, by the above-mentioned corollary of Swan, there exists $g (\neq 0)$ in R such that P_g is free. We may assume that $Z_1 \nmid g$.

Let $g = g_1 + Z_1 g_2$, $g_1 (\neq 0)$ being a power series in Z_2, \dots, Z_d .

After a change of variables involving Z_2, \dots, Z_d only, one may assume that g_1 , and so g , is regular in Z_d . By the Weierstrass Preparation Theorem we can assume that g is a Weierstrass polynomial in $k[[Z_1, \dots, Z_{d-1}]][[Z_d]]$ up to a unit.

Let $S = k[[Z_1, \dots, Z_{d-1}]]$. Then, by §1(A), Examples 2 and 4, we have an analytic isomorphism

$$S[Z_d, X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}] \cong R[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$$

along g . Since (g, Z_1) is a sequence we have an analytic isomorphism

$$S_{Z_1}[Z_d, X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}] \cong R_{Z_1}[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$$

along g .

Since P_g is free, P extends from $S_{Z_1}[Z_d, X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$. Induction prevails.

Now we prove the main result of this section, which is the analogue of Corollary 3.5 when R is complete.

THEOREM 4.2. *Let k be a field. Then every finitely generated projective $k[[Z_1, \dots, Z_d]](T)[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module is free.*

PROOF. Let $R = k[[Z_1, \dots, Z_d]]$, $A = R[T^{-1}]_{(Z_1, \dots, Z_d, T^{-1})}$. It is easy to see that the natural inclusion map $A \hookrightarrow R[[T^{-1}]]$ is analytically isomorphic along T^{-1} .

Observe that since R is local, $A_{T^{-1}} = R(T)$.

By §1(A), Example 4, we have a patching diagram

$$\begin{array}{ccc}
 A[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}] & \Leftrightarrow & R[[T^{-1}]] [X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}] \\
 \downarrow & & \downarrow \\
 R(T)[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}] & \Leftrightarrow & R[[T^{-1}]]_{T^{-1}} [X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]
 \end{array}$$

Since by Proposition 4.1, all projective modules over $R[[T^{-1}]]_{T^{-1}}[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ are free, P extends from $A[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$.

Therefore, it suffices to prove that every projective $A[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ -module is free.

We can find a $g \in R$ such that P_g is free. We may, without loss of generality, assume that g is a Weierstrass polynomial in Z_d with coefficients in $S = k[[Z_1, \dots, Z_{d-1}]]$. Then $S[Z_d] \hookrightarrow R$ is an analytic isomorphism along g . Consequently $B[Z_d] \hookrightarrow A$ is an analytic isomorphism along g as by §1(A), Examples 2 and 3, where $B = S[[T^{-1}]]_{(Z_1, \dots, Z_{d-1}, T^{-1})}$.

By §1(A), Example 4, we have a patching diagram

$$\begin{array}{ccc}
 B[Z_d, X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}] & \Leftrightarrow & A[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}] \\
 \downarrow & & \downarrow \\
 B[Z_d]_g[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}] & \Leftrightarrow & A_g[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]
 \end{array}$$

Patch P on $A[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ and a suitable free module F on $B[Z_d]_g[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ via an isomorphism over $A_g[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ to get a projective module P^* over $B[Z_d, X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ such that

$$P \simeq P^* \otimes A[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}].$$

Since $P_g^* \simeq F$, by [Su, Theorem 1], P^* is free. Thus, P is free.

NOTE ADDED IN PROOF. Later, the second author has extended Theorem B in a preprint titled *On Projective R_{f_1, \dots, f_r} -Modules*.

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