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THE PROPERTIES *-REGULARITY AND UNIQUENESS OF C^*-NORM IN A GENERAL *-ALGEBRA

BY

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ABSTRACT. In this paper two properties of a *-algebra A are considered which are concerned with the relationship between the algebra and its C^*-enveloping algebra. These properties are that A have a unique C^*-norm, and that A be *-regular. Both of these concepts are closely involved with the representation theory of the algebra.

Introduction. The C^*-enveloping algebra of a Banach *-algebra plays a fundamental role in the theory of *-representations of the algebra on Hilbert space. For example in abstract harmonic analysis the algebra $C^*(G)$ is of central importance (here G is a locally compact group, and $C^*(G)$ is the C^*-enveloping algebra of $L^1(G)$). In this paper we consider two properties that concern the relationship between a *-algebra A and its C^*-enveloping algebra $C^*(A)$. A collection $\mathfrak{R}$ of *-representations of A is separating for A if whenever $\pi(f) = 0$ for all $\pi \in \mathfrak{R}$, then $f = 0$. If A has a separating set of *-representations, then we call A reduced.

When is every separating collection of *-representations of the reduced Banach *-algebra A also separating for $C^*(A)$? It is not difficult to verify that this question is exactly equivalent to the following one: When does A have a unique C^*-norm? Besides occurring naturally in the representation theory of a *-algebra, this question occurs and is of interest in various other contexts. In C^*-algebra theory this question has been asked of the tensor product algebra of two C^*-algebras, and has received considerable attention; see for example [**12, 20, 21**]. In harmonic analysis it is well known that if $L^1(G)$ has a unique C^*-norm, then G is amenable. The question of whether the converse of this statement holds has been of interest and has only recently been settled in the negative by D. Poguntke (personal communication). Another basic property of a *-algebra which we consider is called *-regularity. It is being actively used and studied in harmonic analysis for the *-algebra $L^1(G)$ by J. Biodol [**4**], D. Poguntke [**17**], J. Biodol et al. [**5**], Barnes [**3**], and others. Let Π_A denote the space of primitive ideals of $C^*(A)$ equipped with the usual hull-kernel topology [**8**, 3.1]. Equivalently, Π_A is exactly the set of all kernels of irreducible *-representations of $C^*(A)$. A is *-regular if for any closed set $\Gamma \subset \Pi_A$ and $P \notin \Gamma$, there exists $f \in A$ such that $f \in Q$ for all $Q \in \Gamma$ and $f \notin P$. This concept is involved in an important way in the question of how the representation theory of $C^*(A)$

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relates to that of A. In this paper we develop the basic theory of the two properties *-regularity and uniqueness of C^*-norm. These two concepts are not unrelated. In fact we prove an elementary but important result relating the two (we assume that A is reduced). A is *-regular if and only if for every closed ideal I of $C^*(A)$ the quotient algebra $A/(A \cap I)$ has a unique C^*-norm. This theorem is used as a tool throughout this paper.

In §§1–4 we develop the fundamental properties of the two concepts, in §5 we investigate when they are preserved with respect to tensor products of *-algebras, and in §§6–7 we consider examples from general Banach algebra theory where they hold.

1. Preliminaries. Throughout this paper A will be a *-algebra with a C^*-norm. When in addition A is a Banach *-algebra, A is called an A^*-algebra (this is the terminology in C. Rickart's book [**18**] where the basic properties of A^*-algebras are derived). In this paper we often deal with *-algebras that are not necessarily Banach algebras. One reason for this generality is to have a theory which applies to the algebraic tensor product of two Banach *-algebras. In this work with general *-algebras, some ideas and results of Theodore Palmer [**16**] are very useful. Palmer defines a *-algebra A to be a BG^*-algebra if for every *-representation π of A as linear operators on a pre-Hilbert space K, it is true that $\pi(f) \in B(K)$ for all $f \in A$, where $B(K)$ denotes the algebra of all bounded linear operators on K. We use the following results from [**16**].

1.1. Assume that A is a BG^*-algebra. Then

(1) $\gamma_A(f) = \sup\{\|\pi(f)\|$: all *-representations π of $A\}$ is finite for all $f \in A$;

(2) a *-ideal in a BG^*-algebra is itself a BG^*-algebra;

(3) if I is a *-ideal of A, then A/I is a BG^*-algebra;

(4) if I is a *-ideal of A and π is a *-representation of I, then π has an extension to a *-representation of A.

A Banach *-algebra is a BG^*-algebra, and more generally, a U^*-algebra is a BG^*-algebra [**16, 14**].

Let A be a BG^*-algebra which is reduced (the set of *-representations of A separate the points of A). Then γ_A as defined above in 1.1 (1) is a C^*-norm, and in fact the largest C^*-norm, on A. The C^*-enveloping algebra of A, denoted by $C^*(A)$, is the completion of A with respect to the C^*-norm γ_A. We consider A as a *-subalgebra of $C^*(A)$ via the natural embedding of A in this completion, $C^*(A)$. Often we suppress the subscript A, using the notation γ for both the largest C^*-norm on A and the C^*-norm on $C^*(A)$. *Throughout this paper we assume that A is a reduced BG^*-algebra*, and thus, that A has a largest C^*-norm and that $C^*(A)$ exists.

Let Π_A be the primitive ideal space of $C^*(A)$ equipped with the Jacobson hull-kernel topology [**8**, 3.1]. When $\Gamma \subset \Pi_A$, we denote the complement of Γ in Π_A by Γ'. For a subset B of $C^*(A)$ define

$$h(B) = \{P \in \Pi_A : B \subset P\}.$$

If I is an ideal of A, then

$$A/I = \{f + I : f \in A\}$$

is the usual quotient algebra. If $P \in \Pi_A$ and $f \in C^*(A)$, then let $\hat{f}(P) = f + P \in A/P$. Thus $\hat{f}(P) = 0$ if and only if $f \in P$.

Now we define *-regularity.

DEFINITION 1.2. A is *-regular if given any closed set $\Gamma \subset \Pi_A$ and $P \notin \Gamma$, there exists $f \in A$ such that $\hat{f}(\Gamma) = \{0\}$ and $\hat{f}(P) \neq 0$.

Uniqueness of C^*-norm can be described in terms similar to those in the definition above as follows.

PROPOSITION 1.3. *A has a unique C^*-norm if and only if for every proper closed set $\Gamma \subset \Pi_A$ there exists $f \in A$, $f \neq 0$, such that $\hat{f}(\Gamma) = \{0\}$.*

It is clear from Proposition 1.3 that when A is *-regular, then A has a unique C^*-norm. The proof of the proposition follows in a straightforward manner from the next result. If $P \in \Pi_A$, then $C^*(A)/P$ has a unique C^*-norm $\|\cdot\|$, and we use the notation $\|\hat{f}(P)\| = \|f + P\|$, $f \in C^*(A)$.

PROPOSITION 1.4. *For $\Gamma \subset \Pi_A$ let*

$$\tau_\Gamma(f) = \sup\{\|\hat{f}(P)\|: P \in \Gamma\}, \qquad f \in C^*(A)$$

(for Γ empty let τ_Γ be the C^*-norm on $C^*(A)$). Then every C^*-seminorm on A is of the form τ_Γ for some closed subset Γ of Π_A.

PROOF. The fact that τ_Γ is a C^*-seminorm is obvious. Now let δ be a C^*-seminorm on A. Since $\delta(f) \leqslant \gamma_A(f)$ for all $f \in A$, δ can be extended to a C^*-seminorm $\bar{\delta}$ on $C^*(A)$. Let

$$I = \{f \in C^*(A): \bar{\delta}(f) = 0\},$$

and let $\Gamma = h(I)$. Now both $\tau(f + I) = \sup\{\|\hat{f}(P)\|: P \in \Gamma\}$ and $\nu(f + I) = \bar{\delta}(f)$ are C^*-norms on $C^*(A)/I$. Therefore they are equal by [**18**, Corollary (4.8.6)], and thus for $f \in A$,

$$\delta(f) = \sup\{\|\hat{f}(P)\|: P \in \Gamma\}.$$

It is informative to consider the concept of *-regularity in the special case when A is commutative. We do this now. Let Φ_A denote the space of all nonzero multiplicative linear functionals on A equipped with the w^*-topology. We use the usual notation $\hat{f}(\varphi) = \varphi(f)$, $f \in A$, $\varphi \in \Phi_A$ (there should be no confusion with the notation $\hat{f}$ used above). Let

$$\Delta_A = \{\varphi \in \Phi_A: \varphi(f^*) = \overline{\varphi(f)} \text{ for all } f \in A\}.$$

We call A hermitian if for all $f = f^* \in A$ the function $\hat{f}$ is real valued on Φ_A. This is equivalent to the property that $\Delta_A = \Phi_A$. When A is a Banach *-algebra it is also equivalent to: (1) the spectrum of f in A, denoted $\sigma_A(f)$, is real for all $f = f^* \in A$; (2) $\sigma_A(f^*f)$ is nonnegative for all $f \in A$; see [**6**, §41]. In this context we also use the terminology that A is symmetric.

In general Δ_A is a closed subset of Φ_A. Also, for $f \in A$

$$\gamma_A(f) = \sup\{|\hat{f}(\varphi)| : \varphi \in \Delta_A\},$$

and Δ_A can be identified with $\Phi_{C^*(A)}$ ($\varphi \in \Phi_A$ is in Δ_A exactly when φ has an extension to a multiplicative linear functional on $C^*(A)$). A is regular if given any closed subset $\Gamma \subset \Phi_A$ and $\varphi \notin \Gamma$, there exists $f \in A$ such that $\hat{f}(\Gamma) = \{0\}$ and $\hat{f}(\varphi) \neq 0$. The concept of regularity is very important in the theory of commutative Banach algebras and in harmonic analysis. We have the following elementary result.

THEOREM 1.5. *Assume that A is commutative (recall that under our standing assumption, A is a reduced BG^*-algebra). A is regular if and only if A is hermitian and $*$-regular.*

PROOF. If $\varphi \in \Delta_A$, then $\bar{\varphi}$ denotes the unique extension of φ to $C^*(A)$. We use the fact without proof that $\varphi \to \ker(\bar{\varphi})$ is a homeomorphism of Δ_A onto Π_A. Also, note that for $f \in A$, $\hat{f}(\varphi) = 0$ if and only if $f \in \ker(\bar{\varphi})$, if and only if $\hat{f}(\ker(\bar{\varphi})) = 0$.

First assume A is regular. Now Δ_A is a closed subset of Φ_A, and if $\psi \in \Phi_A \backslash \Delta_A$ ($\backslash$ denoting set difference), then there exists $f \in A$ such that $\hat{f}(\Delta_A) = \{0\}$, $\hat{f}(\psi) \neq 0$. But then $\gamma_A(f) = \sup\{|\hat{f}(\varphi)| : \varphi \in \Delta_A\} = 0$. This contradiction proves that $\Delta_A = \Phi_A$, that is, A is hermitian. Then the $*$-regularity of A follows from the homeomorphic identification of Δ_A and Π_A.

Conversely, if A is $*$-regular and hermitian, then again $\Delta_A = \Phi_A$, and it follows that A is regular, again from the identification of Δ_A and Π_A.

2. Ideals and quotients. This section contains many of the basic results of the paper. For I a γ-closed ideal of A, we investigate the relationships among A, I, and A/I with respect to the properties of uniqueness of C^*-norm and $*$-regularity. These relationships are not completely predictable. For example, if A is $*$-regular, then A/I is $*$-regular, but the analogous result for the uniqueness of C^*-norm property does not hold. In fact we prove that A/I has a unique C^*-norm for all γ-closed ideals I of A if and only if A is $*$-regular (Theorem 2.3). This central result is used extensively in the remainder of the paper.

At the end of this section we give an example of a commutative, semisimple, symmetric Banach $*$-algebra A with a unique C^*-norm such that A is not $*$-regular.

When E is a subset of A, we denote by $\bar{E}$ the closure of E in $C^*(A)$.

PROPOSITION 2.1. *Assume that I is a closed ideal of $C^*(A)$ and $A/A \cap I$ has a unique C^*-norm. Then $\overline{A \cap I} = I$.*

PROOF. Let $J = \overline{A \cap I} \subset I$. Consider the maps

$$\varphi: A/A \cap I \to C^*(A)/I, \qquad \psi: A/A \cap I \to C^*(A)/J$$

given by

$$\varphi(f + A \cap I) = f + I, \qquad \psi(f + A \cap I) = f + J.$$

Both φ and ψ are $*$-maps and are one-to-one. Hence

$$f \to \gamma_1(\varphi(f + I \cap A)) \quad \text{and} \quad f \to \gamma_2(\psi(f + I \cap A))$$

are C^*-norms on $A/A \cap I$ where γ_1 and γ_2 are the natural C^*-norms on $C^*(A)/I$ and $C^*(A)/J$ respectively. Thus $\gamma_1(f + I) = \gamma_2(f + J)$ for all $f \in A$.

Assume that $g \in I$, but $g \notin J$. Choose $\{g_n\} \subset A$ such that $\gamma(g_n - g) \to 0$. Then $\gamma_1(g_n + I) \to 0$ and $\gamma_2(g_n + I) \to \gamma_2(g + I) \neq 0$. This contradiction proves that $J = I$.

Let A be a BG^*-algebra with largest C^*-norm γ_A. If I is a *-ideal of A, then as Palmer shows in [**16**], 1.1 (4) implies that

$$\gamma_I(f) = \gamma_A(f), \qquad f \in I.$$

From this it follows that Π_I can be identified with the open subset $h(\bar{I})'$ of Π_A. Now let I be a γ-closed ideal of A. Since $\bar{I} \cap A = I$, $C^*(A/I)$ is isometrically *-isomorphic to $C^*(A)/(\bar{I})$. It follows that $\Pi_{A/I}$ can be identified with the closed subset $h(\bar{I})$ of Π_A [**8**, Proposition 3.2.1]. We use this information in what follows.

THEOREM 2.2. *Let I be a *-ideal in A.*

(1) *If A has a unique C^*-norm, then I has a unique C^*-norm.*

(2) *If A is *-regular, then I is *-regular.*

(3) *Assume that I is γ-closed. If A is *-regular, then A/I is *-regular.*

PROOF. First we establish (1). Denote by $\bar{I}$ the closure of I in $C^*(A)$. Now $\Pi_I = h(\bar{I})'$ is an open subset of Π_A. Assume W is a nonempty open set in Π_I. If A has a unique C^*-norm, then there exists $f \in A$, $f \neq 0$, such that $\hat{f}(W') = \{0\}$. Fix $Q \in W$ such that $\hat{f}(Q) \neq 0$. Since $\hat{f}(P) = 0$ for all $P \in h(\bar{I})$, then $f \in \bar{I} \cap A$. We may assume $\|(f^*f)\hat{}(Q)\| = 1$. Choose $g \in I$ such that $\gamma(f^* - g)\gamma(f) < 1$. If $(gf)\hat{}(Q) = 0$, then

$$1 = \|(f^*f)\hat{}(Q)\| = \|(f^* - g)\hat{}(Q)\hat{f}(Q)\| \leq \gamma(f^* - g)\gamma(f) < 1.$$

Thus $gf \in I$, $(gf)\hat{}(Q) \neq 0$ and $(gf)\hat{}(P) = 0$ for all $P \notin W$. This proves (1), and essentially the same argument proves (2).

Now assume that A is *-regular and I is γ-closed. Let U be an open set in Π_A such that $U \cap h(\bar{I})$ is nonempty (where $\bar{I}$ denotes as before the closure of I in $C^*(A)$). If $Q \in U \cap h(\bar{I})$, then there exists $g \in A$ such that $\hat{g}(Q) \neq 0$ and $\hat{g}(U') = \{0\}$. Thus $(g + I)\hat{}(Q) \neq 0$ and $(g + I)\hat{}(P) = 0$ for all $P \in h(\bar{I})\backslash U$.

REMARK. If A has a unique C^*-norm and I is a γ-closed ideal, then in general A/I need not have a unique C^*-norm. An example is given at the end of this section, Example 2.9.

For $\Gamma \subset \Pi_A$, we set $k(\Gamma) = \bigcap\{P\colon P \in \Gamma\}$.

THEOREM 2.3. *A is *-regular if and only if for every γ-closed ideal I of A, A/I has a unique C^*-norm.*

PROOF. If A is *-regular, then for any γ-closed ideal I of A, A/I is *-regular by Theorem 2.2. Thus A/I has a unique C^*-norm.

Conversely, assume that A/I has a unique C^*-norm for every γ-closed ideal I. Let Γ_0 be a closed subset of Π_A. Let

$$\Gamma = \left\{P \in \Pi_A\colon \hat{f}(P) = 0 \text{ for all } f \in A \cap k(\Gamma_0)\right\}.$$

We prove that $\Gamma_0 = \Gamma$. Suppose $\Gamma_0 \neq \Gamma$. Let $I = k(\Gamma) \cap A$. We have that A/I has a unique C^*-norm and $k(\Gamma_0) \cap A = I$. But by Proposition 2.1, I is γ-dense in $k(\Gamma_0)$.

This is impossible since

$$\bar{I} \subset k(\Gamma) \subsetneq k(\Gamma_0).$$

PROPOSITION 2.4. (1) *A has a unique C^*-norm if and only if for every nonzero closed ideal I of $C^*(A)$, $I \cap A$ is nonzero.*

(2) *A is *-regular if and only if for every closed ideal I of $C^*(A)$, $I \cap A$ is dense in I.*

PROOF. Let I be a nonzero closed ideal of $C^*(A)$. If $I \cap A = \{0\}$, then

$$\tau(f) = \inf\{\gamma(f-g) : g \in I\}$$

is a C^*-norm on A, and furthermore $\tau \neq \gamma$ on A. Thus if A has a unique C^*-norm, then $I \cap A \neq \{0\}$.

Conversely, assume that τ is a C^*-norm on A and $\tau \neq \gamma$ on A. Let $\bar{\tau}$ be the extension of τ to $C^*(A)$, and let

$$I = \{g \in C^*(A) : \bar{\tau}(g) = 0\}$$

Then $I \neq \{0\}$, but $I \cap A = \{0\}$.

Now assume A is *-regular. By Theorem 2.3 if I is a closed ideal of $C^*(A)$, then $A/A \cap I$ has a unique C^*-norm. Thus $A \cap I$ is dense in I by Proposition 2.1.

Assume that A has the property that for every closed ideal I of $C^*(A)$, $I \cap A$ is dense in I. If $\Gamma \subset \Pi_A$ is closed and $P \notin \Gamma$, let $I = k(\Gamma)$. Then $I \cap A$ is dense in I. Choose $g \in k(\Gamma)$ such that $\|\hat{g}(P)\| = 1$. Let $f \in A \cap I$ be such that $\gamma(g-f) < 1$. Then $\hat{f}(\Gamma) = \{0\}$, but $\hat{f}(P) \neq 0$. Thus A is *-regular.

PROPOSITION 2.5. *Let B be a reduced GB^*-algebra, and let A be a γ_B-dense *-subalgebra of B. Assume that A is a GB^*-algebra.*

(1) *If A has a unique C^*-norm, then B has a unique C^*-norm.*

(2) *If A is *-regular, then B is *-regular.*

PROOF. Let τ be a C^*-norm on B. Given $f \in B$ choose $\{f_n\} \subset A$ such that $\gamma_B(f_n - f) \to 0$. Then $\tau(f_n - f) \to 0$, and assuming that A has a unique C^*-norm, $\tau(f_n) = \gamma_B(f_n)$ for $n \geqslant 1$. Thus $\gamma_B(f) = \tau(f)$. This proves (1).

Now assume that I is a γ_B-closed ideal of B. Then $A \cap I$ is γ_A-closed in A. If A is *-regular, then by Theorem 2.3, $A/A \cap I$ has a unique C^*-norm. Since the map

$$f + A \cap I \to f + I, \qquad f \in A,$$

is a *-embedding of $A/A \cap I$ into a $\gamma_{B/I}$-dense *-subalgebra of B/I, it follows from part (1) that B/I has a unique C^*-norm. Thus the *-regularity of A implies the *-regularity of B by Theorem 2.3.

PROPOSITION 2.6. *Let I be a *-ideal of A such that I has a unique C^*-norm and $h(\bar{I})$ is nowhere dense in Π_A. Then A has a unique C^*-norm.*

PROOF. Π_I is homeomorphic to the open set $U = h(\bar{I})'$. Let W be a nonempty open subset of Π_A. Since $h(\bar{I})$ is closed and nowhere dense, $U \cap W$ must be nonempty. Therefore there exists $g \in I$, $g \neq 0$, such that $\hat{g}(P) = 0$ for all $P \in U \setminus (U \cap W)$. But also $\hat{g}(P) = 0$ for all $P \in h(\bar{I})$. Therefore, $\hat{g}(P) = 0$ for all $P \in W'$. This proves that A has a unique C^*-norm.

THEOREM 2.7. *Let A be a reduced BG^*-algebra.*

(1) *There exists a γ-closed *-ideal M_0 of A such that M_0 has a unique C^*-norm and M_0 contains every *-ideal of A that has a unique C^*-norm.*

(2) *There exists a γ-closed *-ideal M_1 of A such that M_1 is *-regular and M_1 contains every *-ideal of A that is *-regular.*

PROOF. We prove (2); the proof of (1) is similar. First assume that I and J are *-regular *-ideals of A. We prove that

(3) $K = I + J$ is *-regular.

Recall that if M is a *-ideal of A then Π_M may be identified with $h(M)'$. We have Π_K identified with $h(K)' = h(I)' \cup h(J)'$. Let V be an open subset of $h(K)'$ with $P \in V$. We may assume that $P \in h(I)'$. Then since I is *-regular, there exists $f \in I$ such that $\hat{f}(P) \neq 0$ and $\hat{f}(h(I)'\backslash V) = \{0\}$. But also $\hat{f}(h(I)) = \{0\}$, and thus $\hat{f}(V') = \{0\}$. This proves (3).

Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be the collection of all *-ideals of A that are *-regular. Let

$$M_1 = \bigcup \{I_\lambda : \lambda \in \Lambda\}.$$

From (3) it follows that M_1 is again a *-ideal of A. Now Π_{M_1} is identified with

$$h(M_1)' = \bigcup \{h(I_\lambda)' : \lambda \in \Lambda\}.$$

The same argument as above shows that if V is open in $h(M_1)'$ and $P \in V$, there exists $f \in I_\mu$ for some $\mu \in \Lambda$ such that $\hat{f}(P) \neq 0$ and $\hat{f}(V') = \{0\}$.

Finally, the γ-closure of M_1 in A is *-regular (since M_1 is), and therefore M_1 is γ-closed. This proves (2).

EXAMPLE 2.8. Denote the set of integers by $\mathbf{Z}$. Let $w = \{w_n\}_{n \in \mathbf{Z}}$ be a sequence of positive real numbers such that

$$1 \leqslant w_{n+m} \leqslant w_n w_m, \qquad n, m \in \mathbf{Z}.$$

Let $l^1(w)$ be the linear space of complex sequences $\{a_n\}_{n \in \mathbf{Z}}$ such that $\|\{a_n\}\|_1 = \sum_{n=-\infty}^{+\infty} |a_n| w_n < +\infty$. Define an involution * on a sequence $\{a_n\}$ by $\{a_n\}^*$ is the sequence $\{b_n\}_{n \in \mathbf{Z}}$ where $b_n = \bar{a}_{(-n)}$, $n \in \mathbf{Z}$. With convolution multiplication, this involution *, and the norm $\|\cdot\|_1$, $l^1(w)$ is a commutative Banach *-algebra. These algebras are discussed in [**11**, §19] where the notation $W[w]$ is used in place of $l^1(w)$.

Define

$$\alpha_0 = 1, \qquad \alpha_n = \frac{|n|}{\log(|n|) + 1}, \qquad n \in \mathbf{Z}, n \neq 0.$$

Then

(i) $\alpha_{n+m} \leqslant \alpha_n + \alpha_m$, $n, m \in \mathbf{Z}$;

(ii) $\lim_{n \to \infty}(\alpha_n / n) = 0$; and

(iii) $\sum_{n=-\infty}^{+\infty} \alpha_n n^{-2} = +\infty$.

Let $w = \{w_n\} = \{e^{\alpha_n}\}$.

Then by (i), $1 \leqslant w_{n+m} \leqslant w_n w_m$, $n, m \in \mathbf{Z}$. Note that by (ii), $(w_n)^{1/n} = e^{(\alpha_n/n)} \to 1$ as $n \to \infty$. It follows from [**11**, p. 120] that the carrier space of $l^1(w)$ can be identified with the circle $\{z \in C : |z| = 1\}$. It can be directly verified that $l^1(w)$ is symmetric. But by [**9**, Theorem 2.11] (iii) implies that $l^1(w)$ is not regular. Thus Theorem 1.5

together with a result proved later, Theorem 6.3, imply that $l^1(w)$ does not have a unique C^*-norm. This provides an example of a commutative symmetric semisimple Banach *-algebra that does not have a unique C^*-norm.

Now let B be such an algebra. Let $\Omega = [0, 1] \times \Phi_B$. Define A to be the algebra of all continuous functions $f(t, \varphi)$ on Ω such that $f(0, \varphi) = \hat{g}(\varphi)$, $\varphi \in \Phi_B$, for some $g \in B$. Here $\hat{g}$ designates the usual Gelfand transform of $g \in B$. Let A have the involution $f^*(\omega) = \overline{f(\omega)}$, $\omega \in \Omega$. For $g \in B$, set $\|\hat{g}\|_B = \|g\|_B$. For $f \in A$ let

$$\|f\|_u = \sup\{|f(\omega)| : \omega \in \Omega\}.$$

We have $\gamma_A(f) = \|f\|_u$, $f \in A$. Then

$$\|f\| = \|f\|_u + \|f(0, \cdot)\|_B$$

defines a complete algebra norm on A, so A is a Banach *-algebra. It is easy to verify that Φ_A is homeomorphic to Ω. Let

$$I = \{f \in A : f(0, \varphi) = 0 \text{ for all } \varphi \in \Phi_B\}.$$

Then I is a γ-closed ideal of A, and A/I is isomorphic to B. In particular, A/I does not have a unique C^*-norm, so by Theorem 2.3, A is not *-regular. If U is a nonempty open subset of Ω, then

$$V = U \cap ((0, 1] \times \Phi_B)$$

is open and nonempty, and therefore we can choose $f \in A$, $f \neq 0$, such that $f(V') \equiv 0$. Thus A does have a unique C^*-norm. Note that A is symmetric. It is also easy to see that A is not regular (since $B \approx A/I$ is not regular).

3. Weak containment. Let B be a C^*-algebra. Given (π, H) a *-representation of B, a positive functional associated with π is one of the form

$$f \to (\pi(f)\xi, \xi), \qquad \xi \in H, f \in B.$$

Let $\mathcal{S}$ be a family of *-representations of B. Then π is weakly contained in $\mathcal{S}$ if every positive functional associated with π is the w^*-limit of a net of finite linear combinations of positive functionals associated with *-representations in $\mathcal{S}$. Fell proves in [**10**, Theorem 1.2] that π is weakly contained in $\mathcal{S}$ exactly when

$$\bigcap_{\tau \in \mathcal{S}} \ker(\tau) \subset \ker(\pi).$$

Now assume $B = C^*(A)$ where A is as usual a reduced BG^*-algebra. Let $\mathcal{R}$ denote the collection of all *-representations of A. When $\pi \in \mathcal{R}$, then denote by $\bar{\pi}$ the unique extension of π to $C^*(A)$. If $\mathcal{S} \subset \mathcal{R}$ and $\pi \in \mathcal{R}$, then we use the terminology π is weakly contained in $\mathcal{S}$ to mean $\bar{\pi}$ is weakly contained in $\bar{\mathcal{S}} = \{\bar{\tau} : \tau \in \mathcal{S}\}$.

Certain properties involving weak containment of *-representations of A are closely related to uniqueness of C^*-norm or *-regularity of A. In fact, we have the following two characterizations of these properties.

THEOREM 3.1. *Let A be a reduced BG^*-algebra.*

(1) *A has a unique C^*-norm if and only if whenever $\mathcal{S} \subset \mathcal{R}$ separates the points of A, then every $\pi \in \mathcal{R}$ is weakly contained in $\mathcal{S}$.*

(2) *A is *-regular if and only if for any $\mathcal{S} \subset \mathcal{R}$ and any $\pi \in \mathcal{R}$, π is weakly contained in $\mathcal{S}$ is equivalent to $\bigcap_{\tau \in \mathcal{S}} \ker(\tau) \subset \ker(\pi)$.*

PROOF. Assume $\mathcal{S} \subset \mathcal{R}$ and $\mathcal{S}$ separates the points of A. For $f \in C^*(A)$ let

$$\tau(f) = \sup\{\|\bar{\pi}(f)\|: \bar{\pi} \in \bar{\mathcal{S}}\}.$$

τ is a C^*-seminorm on $C^*(A)$. Let $I = \ker(\tau) = \{f \in C^*(A): \tau(f) = 0\}$. Since $\mathcal{S}$ separates points of A, we have $I \cap A = \{0\}$. Thus if A has a unique C^*-norm, then Proposition 2.4 implies that $I = \{0\}$. It follows that $\bar{\mathcal{S}}$ separates points on $C^*(A)$. Therefore by Fell's basic result [**10**, Theorem 2.1] any *-representation of $C^*(A)$ is weakly contained in $\bar{\mathcal{S}}$.

Conversely, let τ be a C^*-norm on A. Choose $\pi \in \mathcal{R}$ such that

$$\tau(f) = \|\pi(f)\|, \qquad f \in A.$$

Now $\{\pi\}$ separates points of A (τ being a C^*-norm on A), and therefore by hypothesis every representation of A is weakly contained in $\{\pi\}$. Again using Fell's Theorem, $\bar{\pi}$ must be faithful on $C^*(A)$. Thus $\|\bar{\pi}(f)\| = \gamma(f)$ for all $f \in C^*(A)$, so that $\tau = \gamma$. This proves that A has a unique C^*-norm.

To establish (2), if $\mathcal{S} \subset \mathcal{R}$, let $I(\mathcal{S})$ be the ideal

$$I(\mathcal{S}) = \bigcap_{\tau \in \mathcal{S}} \ker(\tau).$$

By Theorem 2.3 A is *-regular if and only if $A/I(\mathcal{S})$ has a unique C^*-norm for all choices of $\mathcal{S} \subset \mathcal{R}$. Then (2) follows from this fact plus (1).

4. Adjoining a unit. Many basic properties of a Banach *-algebra A are preserved when a unit is adjoined to A. For example, if A is symmetric or regular, then A_1 also has these properties (here A_1 is the usual algebra formed from A by adjoining a unit; see [**18**, pp. 2–3]). Thus it is perhaps surprising to find that the uniqueness of C^*-norm property does not necessarily extend from A to A_1. An example is given at the end of this section in which A has a unique C^*-norm, but A_1 does not. Also, it follows from this example that another desirable extension property fails in this case: A is an ideal in A_1 with a unique C^*-norm, the quotient algebra A_1/A being one dimensional has a unique C^*-norm, but A_1 does not have a unique C^*-norm.

First we prove a result characterizing when A_1 has a unique C^*-norm, given that A does.

THEOREM 4.1. *Let A be a BG*-algebra with a unique C^*-norm, and assume that A does not have a unit. Then A_1 has a unique C^*-norm if and only if $C^*(A)$ does not have a unit.*

PROOF. It is easy to see that $C^*(A)$ can be identified with a closed maximal ideal of $C^*(A_1)$ of codimension one. First assume that $C^*(A)$ has a unit e. Then $(1-e)C^*(A) = \{0\}$, so that $M = \{\lambda(1-e): \lambda \in \mathbf{C}\}$ is an ideal of $C^*(A_1)$. But $M \cap A_1 = \{0\}$ since $(1-e) \notin A_1$. Thus A_1 does not have a unique C^*-norm by Proposition 2.4.

Now assume that $C^*(A)$ does not have a unit. Let

$$K = \{f \in C^*(A_1): fC^*(A) = \{0\}\}.$$

If $K \neq \{0\}$, then $C^*(A_1) = K \oplus C^*(A)$. In particular, $1 = u + e$ where $u \in K$ and $e \in C^*(A)$. But then $eg = g$ for all $g \in C^*(A)$, a contradiction. Thus $K = \{0\}$.

Assume M is a nonzero closed ideal of $C^*(A_1)$. Since $K = \{0\}$ and $MC^*(A) \subset M \cap C^*(A)$, $M \cap C^*(A)$ is a nonzero ideal of $C^*(A)$. By Proposition 2.4, $M \cap A \neq \{0\}$. This proves that A_1 has a unique C^*-norm by Proposition 2.4.

COROLLARY 4.2. *If A is a symmetric Banach *-algebra with a unique C^*-norm, then A_1 has a unique C^*-norm.*

The corollary follows from the fact that if A is symmetric and $C^*(A)$ has a unit 1, then $1 \in A$ (one could use Lemma 7.3 to prove this).

COROLLARY 4.3. *Let $A = L^1(G)$ where G is a nondiscrete locally compact group. If A has a unique C^*-norm, then A_1 has a unique C^*-norm.*

PROOF. By [7, Corollary 1] when G is nondiscrete, then $C^*(G)$ does not have a unit. Thus Theorem 4.1 implies the result.

EXAMPLE 4.4. We give an example of a Banach *-algebra A with unique C^*-norm such that the algebra A_1 formed from A by the adjunction of a unit does not have a unique C^*-norm. Let

$$D = \{(x, y, 0) \in \mathbf{R}^3 : x^2 + y^2 \leqslant 1\}$$

and

$$R = \{(x, 0, t) \in \mathbf{R}^3 : 0 \leqslant t \leqslant 1, -1 \leqslant x \leqslant 1\}.$$

Let $\Omega = D \cup R$ (Ω is the unit disk in the x-y-plane with a vertical rectangle standing on the line segment $-1 \leqslant x \leqslant 1$). Let B be the algebra of all continuous $\mathbf{C}$-valued functions f on Ω such that f restricted to the interior of D is analytic. B is a Banach algebra with respect to the sup norm over Ω. For $f \in B$, define

$$f^*(x, y, 0) = \overline{f(x, -y, 0)} \quad \text{on } D, \qquad f^*(x, 0, t) = \overline{f(x, 0, t)} \quad \text{on } R.$$

Then $f \to f^*$ is an involution on B. It is easy to see that Δ_B is identified with R (the multiplicative functionals are point evaluations), and that

$$\gamma_B(f) = \sup\{|f(w)| : w \in R\}.$$

Then $C^*(B)$ is identified with the algebra of all $\mathbf{C}$-valued continuous functions on R. It is straightforward to verify that B has a unique C^*-norm.

Now let A be the closed *-ideal of B consisting of those $f \in B$ such that $f(0, 1, 0) = f(0, -1, 0) = 0$. Now A does not have a unit, but $C^*(A) \approx C^*(B)$ which does have a unit (note that A is γ_B-dense in B, which implies the identification of $C^*(A)$ with $C^*(B)$). Since A does not have a unit but $C^*(A)$ does, A_1 does not have unique C^*-norm by Theorem 4.1. However, A is an ideal of B, and hence A does have unique C^*-norm.

5. Tensor products. Given two *-algebras A and B, $A \otimes B$ is the algebraic tensor product of A and B [**6**, §42]. In this case $A \otimes B$ is a *-algebra. A *-algebra A is a U^*-algebra if every element of A_1 is a finite linear combination of unitary elements; see [**14**]. A Banach *-algebra is a U^*-algebra. Now assume A and B are U^*-algebras. Then it is immediate that $A_1 \otimes B_1$ is a U^*-algebra since the tensor product of two unitary elements is again unitary. Now $A \otimes B$ is an ideal in $A_1 \otimes B_1$. It follows that

$A \otimes B$ is a BG^*-algebra (1.1). In particular $A \otimes B$ has a largest C^*-seminorm γ, and the γ-completion of $A \otimes B$ is $C^*(A \otimes B)$. When A and B are reduced, then $A \otimes B$ is reduced. Given a norm ν on $A \otimes B$, we denote the ν-completion of $A \otimes B$ by $A \otimes_\nu B$.

Unless explicitly noted otherwise, A and B will always be reduced U^*-algebras.

LEMMA 5.1. *Assume that π is a *-representation of $A \otimes B$. Then*

$$\|\pi(f \otimes g)\| \leqslant \gamma_A(f)\gamma_B(g), \qquad f \in A, g \in B.$$

PROOF. Since $A \otimes B$ is an ideal in $A_1 \otimes B_1$, there exists a *-representation π_1 on $A_1 \otimes B_1$ that extends π by 1.1 (4). Then for $f \in A$, $g \in B$,

$$\|\pi(f \otimes g)\| = \|\pi_1(f \otimes g)\| \leqslant \|\pi_1(f \otimes 1)\|\|\pi_1(1 \otimes g)\| \leqslant \gamma_A(f)\gamma_B(g).$$

LEMMA 5.2. *If (π, H) is a *-representation of $A \otimes B$, then π extends to a *-representation $\bar{\pi}$ of $C^*(A) \otimes C^*(B)$.*

PROOF. Consider the bilinear map ψ: $A \times B \to B(H)$ given by $\psi(f, g) = \pi(f \otimes g)$. By Lemma 5.1, $\|\psi(f, g)\| \leqslant \gamma_A(f)\gamma_B(g)$ for all $f \in A$, $g \in B$. If $f \in C^*(A)$, $g \in C^*(B)$, then choose $\{f_n\} \subset A$, $\{g_n\} \subset B$ such that $\gamma_A(f_n - f) \to 0$ and $\gamma_B(g_n - g) \to 0$. Then

$$\begin{aligned}\|\psi(f_n, g_n) - \psi(f_m, g_m)\| &= \|\pi(f_n \otimes g_n) - \pi(f_m \otimes g_m)\| \\ &\leqslant \|\pi((f_n - f_m) \otimes g_n)\| + \|\pi(f_m \otimes (g_n - g_m))\| \\ &\leqslant \gamma_A(f_n - f_m)\gamma_B(g_n) + \gamma_A(f_m)\gamma_B(g_n - g_m) \to 0 \quad \text{as } n, m \to \infty.\end{aligned}$$

Define $\bar{\psi}(f, g) = \lim_{n \to \infty} \psi(f_n, g_n)$. It is straightforward to verify that $\bar{\psi}$ is a well-defined bilinear map from $C^*(A) \times C^*(B)$ into $B(H)$ and that $\|\bar{\psi}(f, g)\| \leqslant \gamma_A(f)\gamma_B(g)$, $f \in C^*(A)$, $g \in C^*(B)$. Thus, there exists a unique linear map $\bar{\pi}$: $C^*(A) \otimes C^*(B) \to B(H)$ such that $\bar{\pi}(f \otimes g) = \bar{\psi}(f, g)$, $f \in C^*(A)$, $g \in C^*(B)$ [**6**, Theorem 6, p. 232]. Elementary computations verify that $\bar{\pi}$ is a *-representation of $C^*(A) \otimes C^*(B)$.

Let ν denote the largest C^*-norm on $C^*(A) \otimes C^*(B)$, and let γ denote the largest C^*-norm on $A \otimes B$. Now $\gamma \geqslant \nu | A \otimes B$. On the other hand, for $t \in A \otimes B$

$$\begin{aligned}\gamma(t) &= \sup\{\|\pi(t)\|: \text{all *-representations } \pi \text{ of } A \otimes B\} \\ &\leqslant \sup\{\|\bar{\pi}(t)\|: \text{all *-representations } \bar{\pi} \text{ of } C^*(A) \otimes C^*(B)\}\end{aligned}$$

by Lemma 5.2. Therefore $\gamma(t) \leqslant \nu(t)$. We have proved that $\gamma = \nu | A \otimes B$. Since ν is a cross-norm on $C^*(A) \otimes C^*(B)$, it follows that $A \otimes B$ is ν-dense in $C^*(A) \otimes_\nu C^*(B)$. Thus $C^*(A \otimes B)$ is isometrically *-isomorphic to $C^*(A) \otimes_\nu C^*(B)$.

PROPOSITION 5.3. *$C^*(A \otimes B)$ is isometrically *-isomorphic to $C^*(A) \otimes_\nu C^*(B)$.*

Now let $C^*(A)\hat{}$ be the usual space of equivalence classes of irreducible *-representations of $C^*(A)$. In place of the notation Π_A we use PRIM($C^*(A)$) to denote the primitive ideal space of $C^*(A)$ with the hull-kernel topology. If $C^*(A)$ is GCR, then we call A GCR. When $C^*(A)$ is GCR, then given $P \in$ PRIM($C^*(A)$) there exists an irreducible *-representation π_P of $C^*(A)$ with kernel P which is unique up to unitary equivalence [**8**, Corollaire 4.1.10]. In fact, $P \to \pi_P$ is a homeomorphism of PRIM($C^*(A)$) onto $C^*(A)\hat{}$ [**8**, Proposition 3.1.6].

LEMMA 5.4. *Assume that both A and B are GCR. Then*

$$(P, Q) \to \ker \overline{(\pi_P \otimes \pi_Q)}$$

is a homeomorphism from $\mathrm{PRIM}(C^*(A)) \times \mathrm{PRIM}(C^*(B))$ *onto*

$$\mathrm{PRIM}(C^*(A) \otimes_\nu C^*(B)) \approx \mathrm{PRIM}(C^*(A \otimes B)).$$

Here $\overline{\pi_P \otimes \pi_Q}$ *is the unique extension of* $\pi_P \otimes \pi_Q$ *from* $C^*(A) \otimes C^*(B)$ *to* $C^*(A) \otimes_\nu C^*(B)$.

PROOF. By the remarks preceding the lemma

$$P \to \pi_P \text{ is a homeomorphism of } \mathrm{PRIM}(C^*(A)) \text{ onto } C^*(A)\hat{},$$
$$Q \to \pi_Q \text{ is a homeomorphism of } \mathrm{PRIM}(C^*(B)) \text{ onto } C^*(B)\hat{},$$

and since $C^*(A) \otimes_\nu C^*(B)$ is GCR, $\mathrm{PRIM}(C^*(A) \otimes_\nu C^*(B))$, $(C^*(A) \otimes_\nu C^*(B))\hat{}$ are homeomorphic. Furthermore, by [**13**, Theorem 5.5] $(\pi, \delta) \to \overline{\pi \otimes \delta}$ is a homeomorphism of $C^*(A)\hat{} \times C^*(B)\hat{}$ onto $(C^*(A) \otimes_\nu C^*(B))\hat{}$. By Lemma 5.3, $C^*(A) \otimes_\nu C^*(B)$ can be identified with $C^*(A \otimes B)$. Combining this information, the lemma follows.

Now we prove the main result of this section.

THEOREM 5.5. *Assume that A and B are GCR.*

(1) *If A and B have unique C^*-norms, then $A \otimes B$ has a unique C^*-norm.*

(2) *If A and B are *-regular, then $A \otimes B$ is *-regular.*

PROOF. We prove only (2), the proof of (1) being similar. By Definition 1.2 it suffices to show that given U an open set in $\mathrm{PRIM}(C^*(A \otimes B))$ and $R_0 \in U$, there exists $h \in A \otimes B$ such that $\hat{h}(R_0) \neq 0$ while $\hat{h}(U') = \{0\}$. By Lemma 5.4 there exists an open set $V \subset \mathrm{PRIM}(C^*(A))$, an open set $W \subset \mathrm{PRIM}(C^*(B))$, $P_0 \in V$, $Q_0 \in W$ such that $\ker\overline{(\pi_{P_0} \otimes \pi_{Q_0})} = R_0$ and $\ker\overline{(\pi_P \otimes \pi_Q)} \in U$ for all $P \in V$, $Q \in W$. By the *-regularity of A and B we can choose $f \in A$, $g \in B$ such that

$$\hat{f}(P_0) \neq 0, \qquad \hat{f}(V') = \{0\}, \qquad \hat{g}(Q_0) \neq 0, \qquad \hat{g}(W') = \{0\}.$$

Then $f \otimes g$ has the properties $(f \otimes g)\hat{}(R_0) \neq 0$ and $(f \otimes g)\hat{}(U') = \{0\}$. This proves the theorem.

Before proving the converse of Theorem 5.5, we state one lemma.

LEMMA 5.6. *Let X be a linear space, and Y a linear subspace of functionals acting on X such that Y is total (i.e., if for some $x \in X$, $\alpha(x) = 0$ for all $\alpha \in Y$, then $x = 0$). If $\{x_1, \ldots, x_n\}$ is a linearly independent set in X, then there exists $\{\alpha_1, \ldots, \alpha_n\} \subset Y$ such that*

$$\alpha_j(x_k) = 1 \quad \text{if } j = k; \qquad \alpha_j(x_k) = 0 \quad \text{if } j \neq k.$$

PROOF. Routine linear algebra.

THEOREM 5.7. *Assume that A and B are GCR.*

(1) *If $A \otimes B$ has a unique C^*-norm, then both A and B have unique C^*-norms.*

(2) *If $A \otimes B$ is *-regular, then both A and B are *-regular.*

PROOF. We prove (2). Let U be an open set in $\mathrm{PRIM}(C^*(A))$ with $P \in U$. Then by Lemma 5.4 $\{\ker(\pi_R \otimes \pi_Q): P \in U, Q \in \mathrm{PRIM}(C^*(B))\}$ is an open subset of $\mathrm{PRIM}(C^*(A \otimes B))$. Fix $S \in \mathrm{PRIM}(C^*(B))$. Since $A \otimes B$ is *-regular, there exists

$$h = \sum_{k=1}^{n} f_k \otimes g_k \in A \otimes B,$$

such that $\hat{h}(\ker(\pi_R \otimes \pi_Q)) = 0$ for all $R \notin U$ and all $Q \in \mathrm{PRIM}(C^*(B))$ while $\hat{h}(\ker(\pi_P \otimes \pi_S)) \neq 0$. Choose m such that $(f_m \otimes g_m)\hat{}\,(\ker(\pi_P \otimes \pi_S)) \neq 0$. Then $\hat{f}_m(P) \neq 0$. We now show that $\hat{f}_m(R) = 0$ for all $R \notin U$, and this will complete the proof that A is *-regular. We have

$$\sum_{k=1}^{n} (\pi_R(f_k)\xi_1, \eta_1)(\pi_Q(g_k)\xi_2, \eta_2) = 0, \tag{3}$$

whenever $R \notin U$, $Q \in \mathrm{PRIM}(C^*(B))$, $\xi_1, \eta_1 \in H_{\pi_R}$ and $\xi_2, \eta_2 \in H_{\pi_Q}$. Let Y be the span of the following set of coordinate functionals on $C^*(B)$:

$$\left\{g \to (\pi_Q(g)\xi, \eta): Q \in \mathrm{PRIM}(C^*(B)), \xi, \eta \in H_{\pi_Q}\right\}.$$

The space Y is total in the dual of $C^*(B)$. Also, if $\alpha \in Y$, then by (3)

$$\sum_{k=1}^{n} (\pi_R(f_k)\xi_1, \eta_1)\alpha(g_k) = 0,$$

for all $R \notin U$ and all $\xi_1, \eta_1 \in H_{\pi_R}$. We may assume that $\{g_1, \ldots, g_n\}$ is a linearly independent set. Then applying Lemma 5.6, we have in particular that $\pi_R(f_m) = 0$ for all $R \notin U$. Thus $\hat{f}_m(R) = 0$ for all $R \notin U$, while as previously noted, $\hat{f}_m(P) \neq 0$.

Now we give some applications of the main result, the most important of these being a theorem concerning locally compact groups G_1 and G_2 that are GCR: if $L^1(G_1)$ and $L^1(G_2)$ have unique C^*-norms (are *-regular), then $L^1(G_1 \times G_2)$ has a unique C^*-norm (is *-regular).

We denote the projective tensor product of A and B by $A \otimes_p B$; see [**6**, pp. 233–234].

THEOREM 5.8. *Assume A and B are U^*-algebras and that both A and B are GCR.*

(1) *If A and B have unique C^*-norms, then $A \otimes_P B$ has a unique C^*-norm.*

(2) *If A and B are *-regular, then $A \otimes_p B$ is *-regular.*

Theorem 5.8 follows immediately by applying Proposition 2.5 and Theorem 5.5.

Let G_1 and G_2 be locally compact groups. It is well known that $L^1(G_1 \times G_2)$ is *-isomorphic to $L^1(G_1) \otimes_p L^1(G_2)$; see [**6**, Example 14, p. 234]. Thus the following application is a corollary of Theorem 5.8.

THEOREM 5.9. *Let G_1 and G_2 be locally compact groups which are GCR.*

(1) *If $L^1(G_1)$ and $L^1(G_2)$ have unique C^*-norms, then $L^1(G_1 \times G_2)$ has a unique C^*-norm.*

(2) *If $L^1(G_1)$ and $L^1(G_2)$ are *-regular, then $L^1(G_1 \times G_2)$ is *-regular.*

EXAMPLE 5.10. Let Ω be a locally compact Hausdorff space, and let B be a Banach *-algebra. The algebra $C_0(\Omega, B)$ consisting of all continuous B-valued functions on Ω

that vanish at infinity is a Banach *-algebra with respect to the norm

$$\|f\| = \sup\{\|f(\omega)\|_B : \omega \in \Omega\},$$

and the involution $f \to f^*$ where

$$f^*(\omega) = (f(\omega))^*, \qquad \omega \in \Omega, f \in C_0(\Omega, B).$$

If $g \in C_0(\Omega)$ and $b \in B$, we denote by bg the function in $C_0(\Omega, B)$ given by

$$(bg)(\omega) = b(g(\omega)), \qquad \omega \in \Omega.$$

We prove that $\operatorname{span}\{bg: b \in B, g \in C_0(\Omega)\}$ is dense in $C_0(\Omega, B)$. Fix $f \in C_0(\Omega, B)$ and $\varepsilon > 0$. Let

$$K = \{\omega \in \Omega : \|f(\omega)\|_B \geqslant \varepsilon/4\}.$$

K is compact. For $\omega \in K$ let

$$U_\omega = \{\delta \in \Omega : \|f(\delta) - f(\omega)\|_B < \varepsilon/2\}.$$

Let $\{V_1 = U_{\omega_1}, V_2 = U_{\omega_2}, \ldots, V_n = U_{\omega_n}\}$ be a finite cover for K. By [**19**, Theorem 2.13] there exists $\{g_1, \ldots, g_n\} \subset C_0(\Omega)$ with the properties

(i) $g_1(\omega) + \cdots + g_n(\omega) = 1$ for all $\omega \in K$;

(ii) $0 \leqslant g_j(\omega) \leqslant 1$, $\omega \in \Omega$, $1 \leqslant j \leqslant n$;

(iii) support of $g_j \subset V_j$, $1 \leqslant j \leqslant n$.

Thus by (i)

$$f = g_1 f + g_2 f + \cdots + g_n f \quad \text{on } K.$$

Let $h = f(\omega_1)g_1 + f(\omega_2)g_2 + \cdots + f(\omega_n)g_n$. Fix $\omega \in K$. Let $M = \{j : \omega \in V_j\}$. Then

$$\|f(\omega) - h(\omega)\|_B \leqslant \sum_{j \in M} g_j(\omega)\|f(\omega) - f(\omega_j)\|_B < \frac{\varepsilon}{2}.$$

Let $J = \{\omega : \|f(\omega)\|_B \geqslant \varepsilon/2\} \subset K$. Choose $g \in C_0(\Omega)$ such that $g(\omega) = 1$ for all $\omega \in J$, $g(\omega) = 0$ for all $w \notin K$, and $0 \leqslant g(\omega) \leqslant 1$ for all ω. It is straightforward to check that $\|f - gf\| \leqslant \varepsilon/2$. Also,

$$\|g(\omega)f(\omega) - g(\omega)h(\omega)\|_B < \frac{\varepsilon}{2} \quad \text{for } \omega \in K,$$

$$g(\omega)f(\omega) - g(\omega)h(\omega) = 0 \quad \text{for } \omega \notin K.$$

Thus, $\|g - gh\| < \varepsilon$. Since $gh \in \operatorname{span}\{bk : b \in B, k \in C_0(\Omega)\}$, we are done.

PROPOSITION 5.11. *Assume that B is a Banach *-algebra which is GCR.*

(1) *If B has a unique C^*-norm, then $C_0(\Omega, B)$ has a unique C^*-norm.*

(2) *If B is *-regular, then $C_0(\Omega, B)$ is *-regular.*

PROOF. We prove (1); the proof of (2) is similar. The map $\varphi: C_0(\Omega) \times B \to C_0(\Omega, B)$ defined by $\varphi(f, b) = bf$ is bilinear. Let $\bar{\varphi}: C_0(\Omega) \otimes B \to C_0(\Omega, B)$ be the corresponding linear map determined by φ. It is straightforward to check that $\bar{\varphi}$ is an algebra *-isomorphism. Assume that B has a unique C^*-norm, so that by Theorem 5.5, $C_0(\Omega) \otimes B$ has a unique C^*-norm. Let γ be the largest C^*-norm on $C_0(\Omega, B)$. Then $\bar{\varphi}(C_0(\Omega) \otimes B)$ has a unique C^*-norm and is norm-dense, hence γ-dense, in $C_0(\Omega, B)$. Therefore $C_0(\Omega, B)$ has a unique C^*-norm by Proposition 2.5.

6. Some cases where uniqueness of C^*-norm and *-regularity are equivalent. In this section we consider some special cases where A is a *-algebra for which the properties of uniqueness of C^*-norm and *-regularity are equivalent. Throughout, A is as usual a reduced BG^*-algebra.

First assume that G is a topological group that acts on Π_A, and denote the action of $x \in G$ on $P \in \Pi_A$ by $x \cdot P$. We write the group operation of G as multiplication and the unit of G as e. We make the natural assumptions that

$$(xy)\cdot P = x \cdot (y \cdot P), \qquad x, y \in G, P \in \Pi_A;$$
$$e \cdot P = P \quad \text{for all } P \in \Pi_A.$$

Furthermore, we assume that

(#) for each fixed $P \in \Pi_A$ the collection of sets $\{U \cdot P\}$, where U runs through the set of open neighborhoods of e in G, is a collection of open neighborhoods of P that forms a neighborhood base at P.

For $f \in A$, $x \in G$, denote by $\hat{f}_x$ the vector-valued function on Π_A given by $\hat{f}_x(P) = \hat{f}(x \cdot P)$.

THEOREM 6.1. *Let A and G be as above and assume (#). Also assume that if $f \in A$ and $x \in G$, there exists $g \in A$ such that $\hat{g} = \hat{f}_x$. We denote this element g by f_x. Then A has a unique C^*-norm if and only if A is *-regular.*

PROOF. Assume that A has a unique C^*-norm. Let V be an open set in Π_A and $P \in V$. By (#) there exists an open neighborhood U of e such that $U \cdot P \subset V$. Since G is a topological group, we can choose an open neighborhood W of e such that $W^2 \subset U$ and $W = W^{-1}$. Now A has a unique C^*-norm so that by Proposition 1.3 there exists $f \in A$ and $w \in W$ such that $\hat{f}((W \cdot P)') = \{0\}$ while $\hat{f}(w \cdot P) \neq 0$. If $Q \in V'$, then $w \cdot Q \notin W \cdot P$ (for otherwise, $Q \in (w^{-1}W) \cdot P \subset W^2 \cdot P \subset V$). Thus $f_w \in A$ by assumption, $\hat{f}_w(P) = \hat{f}(w \cdot P) \neq 0$, and $\hat{f}_w(Q) = 0$ for all $Q \in V'$. This proves that A is *-regular.

COROLLARY 6.2. *Let G be a locally compact abelian group. Assume that A is a dense *-subalgebra of $L^1(G)$ such that*

(i) *A is a BG^*-algebra;*

(ii) *Δ_A is homeomorphic to $\hat{G}$; and*

(iii) *if $f \in A$, $\chi \in \hat{G}$, then $\chi f \in A$.*

*Then A has a unique C^*-norm if and only if A is *-regular.*

PROOF. The topological group $\hat{G}$ acts on $\Pi_A \approx \Delta_A \approx \hat{G}$ by multiplication. Hypothesis (iii) implies that for $f \in A$ and $\chi \in \hat{G}$, $f_\chi \in A$. Clearly (#) is satisfied in this situation. Thus the result follows from the theorem.

THEOREM 6.3. *Let A be a commutative Banach *-algebra with Φ_A homeomorphic to a subset of the real line (or of the unit circle). Also assume that $\Delta_A = \Phi_A$ (A is symmetric). Then A has a unique C^*-norm if and only if A is *-regular.*

PROOF. Assume that A has a unique C^*-norm. Since A is symmetric, by Corollary 4.2 we may adjoin an identity to A if necessary and the algebra A_1 will have a unique C^*-norm. We give the proof in the case where A has a unit and Φ_A is homeomorphic

to a compact subset of the real line. We identify Φ_A completely with this compact subset of $\mathbf{R}$ in what follows.

For $\alpha \in \mathbf{R}$, let $K_\alpha = \Phi_A \cap (-\infty, \alpha]$ and $J_\alpha = \Phi_A \cap [\alpha, +\infty)$. Fix $\alpha, \beta \in \Phi_A$ with $\alpha < \beta$. We prove that there exists $f \in A$ such that $\hat{f}(K_\alpha) = \{0\}$ and $\hat{f}(\beta) = 1$. First suppose there exists $\delta \notin \Phi_A$ such that $\alpha < \delta < \beta$. Then J_δ is open and compact in Φ_A so by the Shilov Idempotent Theorem there exists $e \in A$ such that $\hat{e}(K_\delta) = \{0\}$ and $\hat{e}(J_\delta) = \{1\}$.

Now assume $(\alpha, \beta) \subset \Phi_A$. Let

$$\Gamma = \{\varphi \in \Phi_A : \hat{f}(\varphi) = 0 \text{ for all } f \in A \text{ such that } \hat{f}(K_\alpha) = \{0\}\}.$$

Suppose $\beta \in \Gamma$. Since there exists $g \in A$ such that $\hat{g}(\varphi) = 0$ for all $\varphi \in \Phi_A \backslash (\alpha, \beta)$ and $g \neq 0$, there exists $\tau \notin \Gamma$ with $\alpha < \tau < \beta$. Let

$$I = \{f \in A : \hat{f}(\Gamma) = \{0\}\}.$$

Now $\Phi_{A/I}$ is homeomorphic to Γ. Thus there exists an idempotent $h + I$ in A/I such that $(h + I)\hat{}(\varphi) = 0$ for all $\varphi \in \Gamma$, $\varphi < \tau$ and $(h + I)\hat{}(\varphi) = 1$ for $\varphi \in \Gamma$, $\tau < \varphi$. But then $\hat{h}(K_\alpha) = \{0\}$ while $\hat{h}(\beta) = 1$, a contradiction. Thus there exists $f \in A$ such that $\hat{f}(K_\alpha) = \{0\}$ and $\hat{f}(\beta) = 1$.

A similar proof shows that if $\alpha, \beta \in \Phi_A$, $\beta < \alpha$, then there exists $f \in A$ such that $\hat{f}(J_\alpha) = \{0\}$ and $\hat{f}(\beta) = 1$.

The result follows.

7. Some examples from general Banach algebra theory. In this section we discuss several general types of Banach *-algebras A which are either *-regular or have unique C^*-norm. In each case the result depends on the existence of "sufficiently many" selfadjoint idempotents in A.

The first case we consider is where there exists $S \subset A$ such that if $g \in S$, then $g = g^*$ and $\sigma_A(g)$ is at most countable, and S is γ-dense in $\{f \in A : f = f^*\}$. We may assume that A has a unit 1. If $g \in S$, then the closed *-subalgebra $\langle g \rangle$ of A generated g and 1 is regular since $\Phi_{\langle g \rangle}$ is homeomorphic to $\sigma_A(g)$ which is at most countable. If τ is a C^*-norm on A, then $\tau(g) = \gamma_A(g)$ for all $g \in S$ by Theorem 1.5. If $f = f^* \in A$, then there exists $\{g_n\} \subset S$ such that $\gamma_A(g_n - f) \to 0$ and $\tau(g_n - f) \to 0$. Thus

$$\gamma_A(f) \leftarrow \gamma_A(g_n) = \tau(g_n) \to \tau(f).$$

Thus $\gamma_A(f) = \tau(f)$ when $f = f^* \in A$, and hence for all $f \in A$. This proves that A has a unique C^*-norm. Now if I is a γ-closed ideal of A, then A/I satisfies the same hypotheses that A did. Therefore A/I has unique C^*-norm, and A is *-regular by Theorem 2.3. We state this result as a theorem.

THEOREM 7.1. *Let A be an A^*-algebra. Assume that if $f = f^* \in A$, then there exists $\{g_n\} \subset A$, $g_n = g_n^*$, such that*

(1) $\gamma_A(g_n - f) \to 0$; *and*

(2) $\sigma_A(g_n)$ *is at most countable.*

*Then A is *-regular.*

The next example concerns the situation where $C^*(A)$ has the property that for every maximal commutative *-subalgebra C of $C^*(A)$, Φ_C is totally disconnected. Before stating and proving the result we develop some preliminary information concerning this situation.

LEMMA 7.2. *Let B be a C^*-algebra with the property that every maximal commutative *-subalgebra of B has totally disconnected carrier space. If Γ is a closed set in Π_B and $P \notin \Gamma$, then there exists $e^* = e^2 = e \in B$ such that $\hat{e}(\Gamma) = \{0\}$ and $\hat{e}(P) \neq 0$.*

PROOF. Choose $f \in B$, $f \geqslant 0$, such that $\hat{f}(\Gamma) = \{0\}$ and $\|\hat{f}(P)\| = 1$. Let C be a maximal commutative *-subalgebra of B containing f. Let

$$K = \{\omega \in \Phi_C : \hat{f}(\omega) \geqslant \tfrac{1}{2}\} \quad \text{and} \quad U = \{\omega \in \Phi_C : \hat{f}(\omega) > \tfrac{1}{4}\}.$$

Since the open-closed sets form a basis for the topology of Φ_C (Φ_C being totally disconnected), a standard topological argument using the compactness of K yields the existence of an open-closed set Ω such that $K \subset \Omega \subset U$. Let $e \in C$ be the function with $\hat{e}(\omega) = 1$, $\omega \in \Omega$, $\hat{e}(\omega) = 0$, $\omega \notin \Omega$. Then $f \geqslant fe \geqslant \frac{1}{4}e$ and $\|f - fe\| \leqslant \frac{1}{2}$.

If $Q \in \Gamma$, let π_Q be a *-representation of B with kernel Q. Since $f \geqslant \frac{1}{4}e$, we have for $Q \in \Gamma$, $0 = \pi_Q(f) \geqslant \frac{1}{4}\pi_Q(e)$, so that $e \in Q$. Thus $\hat{e}(\Gamma) = \{0\}$. On the other hand, since $\|f - fe\| \leqslant \frac{1}{2}$ and $\|\hat{f}(P)\| = 1$, it follows that $\hat{e}(P) \neq 0$.

We state the next piece of information we need as a lemma. The result is well known; see for example the proof of [**2**, Proposition 3].

LEMMA 7.3. *Let A be a symmetric A^*-algebra. Assume $e = e^2 = e^* \in C^*(A)$. Then there exists $f = f^2 = f^* \in A$ such that $\gamma(e - f) < 1$.*

We shall apply Lemma 7.3 in conjunction with the following.

NOTE. Assume that e and f are selfadjoint idempotents in a C^*-algebra B and that $\gamma(e - f) < 1$. Now for all $P \in \Pi_B$, $\|\hat{e}(P)\| = 0$ or 1 and $\|\hat{f}(P)\| = 0$ or 1. Since $1 > \gamma(e - f) \geqslant \|\hat{e}(P) - \hat{f}(P)\|$, it is clear that $\|\hat{e}(P)\| = \|\hat{f}(P)\|$ for all $P \in \Pi_B$.

Now we prove the theorem.

THEOREM 7.4. *Assume that A is a symmetric A^*-algebra with the property that every maximal commutative *-subalgebra of $C^*(A)$ has totally disconnected carrier space. Then A is *-regular.*

PROOF. Assume Γ is a closed subset of Π_A and $P \notin \Gamma$. By Lemma 7.2 there exists a selfadjoint idempotent $e \in C^*(A)$ such that $\hat{e}(\Gamma) = \{0\}$ and $\|\hat{e}(P)\| = 1$. Then by Lemma 7.3 there exists $f = f^2 = f^* \in A$ with $\gamma(e - f) < 1$. Finally, the note stated above implies that $\hat{f}(\Gamma) = \{0\}$ and $\hat{f}(P) \neq 0$.

A symmetric Banach *-algebra A has the important property that

$$\sigma_A(f) = \sigma_{C^*(A)}(f) \quad \text{for all } f \in A.$$

We prove this with the assumption that A has a unit 1. It suffices to show that if $f \in A$ and $f^{-1} \in C^*(A)$ then $f^{-1} \in A$. By [**15**, Theorem, p. 523] the set of invertible elements in A is an open set with respect to γ_A. Now $(\frac{1}{n}1 + f^*f)^{-1} \in A$ for $n \geqslant 1$, and

$$\gamma_A\left((\tfrac{1}{n}1 + f^*f)^{-1} f^*f - 1\right) \to 0 \quad \text{as } n \to \infty.$$

This implies that $(\frac{1}{n}1 + f^*f)^{-1}f^*f$ has an inverse in A for n sufficiently large. But then f has a left inverse $g \in A$. Therefore $f^{-1} = g \in A$.

COROLLARY 7.5. *Let B be a C^*-algebra with the property that for every maximal commutative $*$-subalgebra C of B, Φ_C is totaly disconnected. Let A be a $*$-subalgebra of B such that A is dense in B and such that A is a symmetric Banach $*$-algebra. Then the following are equivalent*:

(1) *A is $*$-regular*;

(2) *A has a unique C^*-norm*;

(3) $\gamma_A(f) = \|f\|_B$ *all* $f = f^* \in A$;

(4) $\sigma_A(f) = \sigma_B(f)$ *all* $f = f^* \in A$.

PROOF. That (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear.

When (3) holds $C^*(A)$ can be identified with B. As noted above, because A is symmetric, $\sigma_A(f) = \sigma_{C^*(A)}(f)$ for all $f = f^* \in A$, so (4) holds.

Now assume (4). As just noted, $\sigma_A(f) = \sigma_{C^*(A)}(f)$ for all $f = f^* \in A$, and therefore

$$\gamma_A(f) = \sup\{|\lambda| : \lambda \in \sigma_{C^*(A)}(f)\} \leqslant \sup\{|\lambda| : \lambda \in \sigma_B(f)\} = \|f\|_B.$$

Thus again, $C^*(A)$ can be identified with B. Then A is $*$-regular by Theorem 7.4.

Let H be a Hilbert space and denote by $B(H)$ the algebra of all bounded linear operators on H. We denote the usual operator norm of $T \in B(H)$ by $\|T\|_{\mathrm{op}}$. Let $F(H)$ be the ideal in $B(H)$ consisting of those $T \in B(H)$ with finite-dimensional range. A subalgebra $A \subset B(H)$ is finite dimensionally spanned (FDS) if

$$\operatorname{span}\{T\varphi : T \in A \cap F(H), \varphi \in H\} \text{ is dense in } H.$$

THEOREM 7.7. *Assume that A is a $*$-subalgebra of $B(H)$ and A is FDS. If τ is a C^*-norm on A, then*

$$\tau(T) \geqslant \|T\|_{\mathrm{op}}, \qquad T \in A.$$

Thus if A has a faithful FDS representation π such that the extension of π to $C^*(A)$ is also faithful, then A has a unique C^*-norm.

PROOF. It is sufficient to show that for $T = T^* \in A$

$$\tau(T) \geqslant \|S\psi\|^{-1}(T(S\psi), S\psi)$$

whenever $\psi \in H$ and $S = S^* \in A \cap F(H)$. Now for S as above, SAS is a finite-dimensional $*$-subalgebra of $B(H)$, and such an algebra must have a unit E. In fact, E is the range projection of S. It suffices then to show that

$$\tau(T) \geqslant \|ETE\|_{\mathrm{op}}.$$

Since EAE has a unique C^*-norm, we have

$$\tau(T) \geqslant \tau(ETE) = \|ETE\|_{\mathrm{op}}.$$

Now let π be as in the statement of the theorem, and let π also denote its extension to $C^*(A)$. Since this extension is faithful, $\gamma(f) = \|\pi(f)\|_{\mathrm{op}}$, $f \in C^*(A)$. If τ is any C^*-norm on A, then by the previous argument we have for all $f \in A$,

$$\tau(f) \geqslant \|\pi(f)\|_{\mathrm{op}} = \gamma_A(f).$$

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