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THE WORD PROBLEM FOR LATTICE-ORDER GROUPS

BY

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To. W. W. Boone on the occasion of his 60th birthday

ABSTRACT.

THEOREM. *There is a finitely generated one relator lattice-ordered group with insoluble (group) word problem.*

We prove

THEOREM. *There exists a finitely presented lattice-ordered group with insoluble word problem.*

COROLLARY. *There is a one relator finitely generated lattice-ordered group with insoluble word problem.*

The Theorem is the obvious analogue of the corresponding result for groups. However, the Corollary has no analogue for groups since any one relator finitely generated group has soluble word problem [7].

The proofs we will give are completely self-contained except for two well-known results from recursion theory (I and II below).

0. Introduction. A lattice-ordered group is a group and a lattice such that the group operation distributes over the lattice operations. An example (which we will need later) is $\mathbf{A}(\mathbf{R}) = \mathrm{Aut}(\langle \mathbf{R}, \leqslant \rangle)$, the order-preserving permutations of the real line, $\mathbf{R}$; the group operation is composition, and the lattice operations are the pointwise ones (i.e., if $f, g \in \mathbf{A}(\mathbf{R})$, then fg, $f \vee g$ and $f \wedge g$ are the order-preserving permutations of $\mathbf{R}$ defined by: $\alpha(fg) = (\alpha f)g$, $\alpha(f \vee g) = \max\{\alpha f, \alpha g\}$ and $\alpha(f \wedge g) = \min\{\alpha f, \alpha g\}$ $(\alpha \in \mathbf{R})$, respectively).

A presentation of a lattice-ordered group with generators x_i $(i \in I)$, and relations $r_j(\mathbf{x}) = e$ $(j \in J)$ can be realised by taking the quotient of the free lattice-ordered group F on $\{x_i \colon i \in I\}$ by the normal convex sublattice subgroup generated by the elements $\{r_j(\mathbf{x}) \colon j \in J\}$ of F. If I and J are both finite we will say that the lattice-ordered group is finitely presented. Note that the $r_j(\mathbf{x})$ are formed from $\mathbf{x}$ by using possibly both the lattice and group operations.

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We will obtain a certain primitive recursive set R of relations, involving only a finite set of letters correlated with elements of $\mathbf{A}(\mathbf{R})$, such that the subset S of R comprising those members of R which hold in $\mathbf{A}(\mathbf{R})$ is nonrecursive. We will then show that there is a finite set S_0 of relations which hold in $\mathbf{A}(\mathbf{R})$ and imply S. S_0 will be constructed from a finite set X_0 of letters, consisting of the letters which appear in R and new elements correlated with additional elements of $\mathbf{A}(\mathbf{R})$. A member of R will then be implied by S_0 if and only if it belongs to S. Hence the finitely presented lattice-ordered group $(X_0; S_0)$ will have insoluble word problem; i.e., the word problem for $(X_0; S_0)$ is not recursively soluble. Roughly speaking, S_0 corresponds to the characteristic function of a recursively enumerable set that is not recursive. Indeed, the representation of each recursive function by a finite set of relations is the main step in our proof—see §4.

The method we use in our proof for coding in recursive functions comes from **[9]** where a similar technique was used to give an elementary proof of the existence of a finitely presented group with insoluble word problem. The intuition gained from reading McKenzie and Thompson's paper has proved invaluable to us. We are most grateful to Professor W. W. Boone for making us aware of it.[2]

Caution. **[9]** makes essential use of an element R of order 3 (which juggles the levels). Since lattice-ordered groups have no torsion elements, we have had to develop a totally different strategy to prove our theorem. This has meant we have had to originate new results in ordered permutation groups concerning conjugating *several* elements *simultaneously* by the same conjugator (see Appendix for proofs). This new technique is necessitated by the failure of the amalgamation property [**2**, Theorem 10C] and is an essential departure from **[9]**. We do not know if the standard finitely presented groups with insoluble word problem (see, e.g.. **[0]** and **[1]**) can be made into lattice-ordered groups—they are torsion-free. Also, we do not know whether the lattice-ordered groups (as opposed to groups) given by these finite presentations have insoluble word problem—the lack of a unique normal form and the absence of the amalgamation property make this a difficult problem.

The only results on word problems for lattice-ordered groups to date have concerned free objects in certain varieties of lattice-ordered groups **[5]**, **[6]** and **[8]** and the existence of a recursively presented lattice-ordered group with insoluble word problem **[3]**.

We are most grateful to the referees, especially George McNulty, for their careful reading of the manuscripts. Their attention to detail has led to considerable amplification in many places we had left obscure. The readability of the paper (such as it is) owes a great deal to them.

1. Background. We will use ω for the set of nonnegative integers, $\mathbf{Z}$ for the set of integers, and $\mathbf{Z}^+$ for the set of positive integers. ${}^\omega\omega$ will be used for the set of functions from ω into ω. In contrast to our notation for permutations, we will write

[2]After this article was written, we learned from Ralph N. McKenzie that Richard J. Thompson had independently solved the problem but had not published it.

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elements of ${}^{\omega}\omega$ on the left; so, if $f, g \in {}^{\omega}\omega$, fg will be that member of ${}^{\omega}\omega$ defined by $fg(n) = f(g(n))$ $(n \in \omega)$.

Before embarking on the proof of the Theorem, we develop the notation and basic facts that we will need about lattice-ordered groups and **A(R)**.

LEMMA 0. *Let G be any lattice-ordered group and* $x, y, z \in G$.

(i) *If* $x \wedge y = e$, *then* $x^{-1}yx = y$.

(ii) *If* $x(x \wedge y)^{-1} \wedge y = e$ *and* $e \leqslant z \leqslant y$, *then* $(x \wedge y)^{-1}z(x \wedge y) = x^{-1}zx$.

(iii) *If* $x \wedge yx^{-1} = e$, $e \leqslant z \leqslant y$ & $yz = zy$, *then* $x \leqslant y$ & $xz = zx$.

PROOF. Since $(x \wedge y)^{-1} = x^{-1} \vee y^{-1}$, we have $x(x \wedge y)^{-1}y = y \vee x$ for all $x, y \in G$.

(i) If $x \wedge y = e$, then $xy = y \vee x = x \vee y = yx$.

(ii) If $x(x \wedge y)^{-1} \wedge y = e$ & $e \leqslant z \leqslant y$, then $x(x \wedge y)^{-1} \wedge z = e$. By (i), $x(x \wedge y)^{-1}z = zx(x \wedge y)^{-1}$. Thus (ii) holds.

(iii) If $x \wedge yx^{-1} = e$, then $yx^{-1} \geqslant e$ (so $y \geqslant x$) and $xyx^{-1} = yx^{-1}x = y$ by (i). Since $z \geqslant e$, $x \wedge z \wedge yx^{-1} = e$; hence $yx^{-1}(x \wedge z) = (x \wedge z)yx^{-1} = y(x \wedge z)x^{-1}$ (y commutes with x & z). Thus $x \wedge z = x(x \wedge z)x^{-1} = x \wedge xzx^{-1}$. As $x, z \leqslant y$, $x \wedge (x \vee z)x^{-1} = e$. Therefore $x \vee z = (x \vee z)x^{-1}x = x(x \vee z)x^{-1}$ by (i). Consequently, $x \wedge z = x \wedge xzx^{-1}$ & $x \vee z = x \vee xzx^{-1}$. Since lattice-ordered groups are distributive lattices, $z = xzx^{-1}$, proving (iii). □

For $g \in \mathbf{A(R)}$, let $\operatorname{supp}(g) = \{\alpha \in \mathbf{R}\colon \alpha g \neq \alpha\}$. If $\operatorname{supp}(g)$ is precisely one interval of **R**, we will say that g has *one bump*; if $\operatorname{supp}(g)$ is bounded in **R**, we will say that g is *bounded*.

Observe that if $e < f, g \in \mathbf{A(R)}$, then $f \wedge g = e$ if and only if $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \varnothing$. Also, $\operatorname{supp}(h^{-1}gh) = [\operatorname{supp}(g)]h$. Hence if g is bounded and has one bump and $h > e$, then $h^{-1}gh \wedge g = e$ if and only if h maps the interval of support of g completely to the right of the interval of support of g. Further, if $f, f_1 > e$, then $f_1 \wedge ff_1^{-1} = e$ if and only if $\alpha f_1 = \alpha f$ for all $\alpha \in \operatorname{supp}(f_1)$. In this case, $\operatorname{supp}(f_1) \subseteq \operatorname{supp}(f)$ and $f = f_1 \vee ff_1^{-1}$. All of these facts are easily verified and are standard fare in the study of ordered permutation groups (see §§1.9 and 2.1 of [**2**]).

2. Interpretation in A(R).

We now construct elements of **A(R)** which will realise all the relations we require later. We present the interpretation first since it led us to the eventual finitely presented lattice-ordered group with insoluble word problem. Indeed, it is the picture that led to the construction of such a lattice-ordered group.

In **A(R)**, there are elements $a_0, b_0, c_0 > e$ each bounded and having one bump, such that:

(a) $a_0 \wedge b_0^{-n}a_0b_0^n = e = b_0 \wedge c_0^{-n}b_0c_0^n$ for all $n \in \mathbf{Z}\backslash\{0\}$,

(b) $b_0^{-m}a_0b_0^m < b_0$ and $c_0^{-m}b_0c_0^m < c_0$ for all $m \in \mathbf{Z}$,

(c) between $\operatorname{supp}(b_0^{-m}a_0b_0^m)$ and $\operatorname{supp}(b_0^{-n}a_0b_0^n)$ there is a nondegenerate interval of **R** whenever m and n are distinct integers, and

(d) similarly for $\operatorname{supp}(c_0^{-m}b_0c_0^m)$ and $\operatorname{supp}(c_0^{-n}b_0c_0^n)$.

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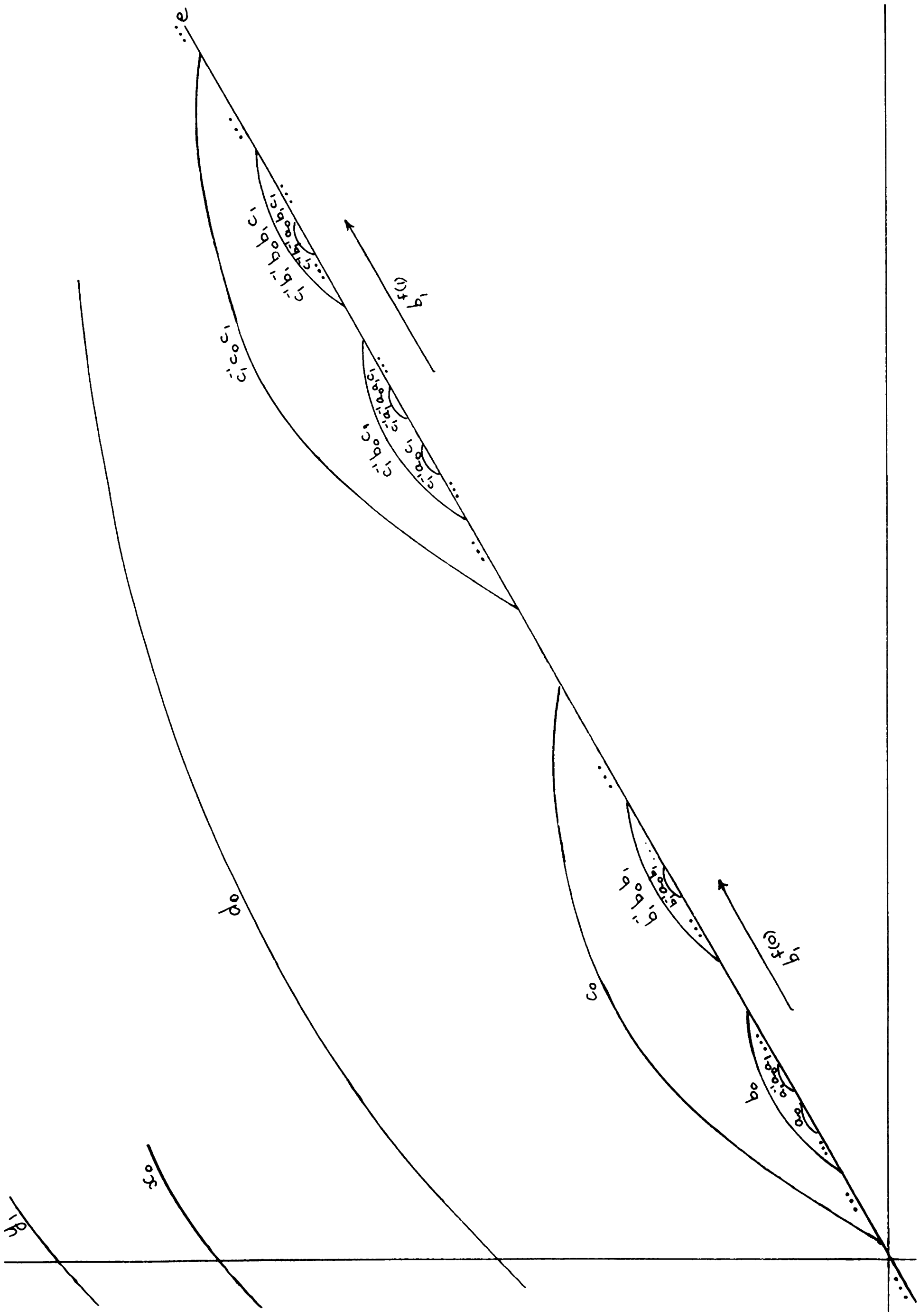

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The picture above describes the situation—by (a) and the remarks in §1, the support of $b_0^{-n}a_0b_0^n$ is indeed to the left of the support of $b_0^{-m}a_0b_0^m$ if $m > n$. So we can satisfy (c), (a) and (b) simultaneously; similarly also (d).

Because of the bump structure, there is $a_3 \in \mathbf{A}(\mathbf{R})$ such that $a_3^{-1}a_0a_3 = b_0$ and $a_3^{-1}b_0a_3 = c_0$. Since the actual definition of a_3 (as opposed to these vital properties) is immaterial, we omit the details here—they can be found in the Appendix. Let $d_0 = a_3^{-1}c_0a_3$, $x_0 = a_3^{-1}d_0a_3$ and $y_n = a_3^{-n}x_0a_3^n$ $(n \in \mathbf{Z}^+)$. Let $W = \{y_1^{n_1}\cdots y_k^{n_k}: k \in \mathbf{Z}^+, n_1,\ldots,n_k \in \mathbf{Z}\}$ and write $x * y$ for $y^{-1}xy$ ($x * yz$ is shorthand for $x * (yz)$). By (a), (b), the remarks in §1 and the definition of a_3, $\{x_0 * w: w \in W\}$ forms a pairwise disjoint set. Let $d_1 \in \mathbf{A}(\mathbf{R})$ be the pointwise supremum of it. (Note that $x_0 \wedge d_1x_0^{-1} = e$, i.e., $a_3^{-1}d_0a_3 \wedge d_1(a_3^{-1}d_0a_3)^{-1} = e$—so $x_0 \vee d_1x_0^{-1} = d_1$.) Let $c_1 \in \mathbf{A}(\mathbf{R})$ be the pointwise supremum of the pairwise disjoint set of elements $\{d_0 * d_1^nw: n \in \mathbf{Z}, w \in W\}$, $b_1 \in \mathbf{A}(\mathbf{R})$ be the pointwise supremum of the pairwise disjoint set of elements $\{c_0 * c_1^md_1^nw: m, n \in \mathbf{Z}, w \in W\}$, and $a_1 \in \mathbf{A}(\mathbf{R})$ be the pointwise supremum of the pairwise disjoint set $\{b_0 * b_1^mc_1^nd_1^kw: m, n, k \in \mathbf{Z}, w \in W\}$. By construction a_1, b_1, c_1 and d_1 each commute with each other, $a_3^{-1}c_1a_3 = d_1$, $a_3^{-1}b_1a_3 = c_1$ and $a_3^{-1}a_1a_3 = b_1$. Also, since $a_1 \restriction \mathrm{supp}(b_0) = b_0 \restriction \mathrm{supp}(b_0)$, $a_1b_0 = b_0a_1$. Let a_2 be the pointwise supremum of the pairwise disjoint set $\{a_0 * a_1^n: n \in \mathbf{Z}^+\}$. So $a_0 \wedge a_2 = e$ and $a_1^{-1}a_2a_1 < a_2$. Let a_6 and a_7 be the pointwise suprema of the pairwise disjoint sets $\{a_0 * c_1^m: m \in \omega\}$ and $\{a_0 * c_1^{-m}: m \in \mathbf{Z}^+\}$. Note that $a_6 \wedge a_7 = e$, $c_1^{-1}(a_6 \vee a_7)c_1 = a_6 \vee a_7$, $a_0 \wedge a_6a_0^{-1} = e$, $c_1^{-1}a_6c_1 \leqslant a_6$, $c_1a_0c_1^{-1} \leqslant a_7$ and $c_1a_7c_1^{-1} \leqslant a_7$.

Because of bump structure, we can find $a_4, a_5 \in \mathbf{A}(\mathbf{R})$ such that $a_4^{-1}a_0a_4 = a_0$, $a_4^{-1}a_1a_4 = b_1$, $a_4^{-1}b_1a_4 = c_1$ and $a_4^{-1}c_1a_4 = d_1$; $a_5^{-1}a_0a_5 = a_0$, $a_5^{-1}b_0a_5 = c_0$ and $a_5^{-1}c_1a_5 = c_1$—see the Appendix for the details. Hence $a_0 * a_1^ma_5 = a_0 * b_1^m$ for all $m \in \mathbf{Z}$.

We have coded in five "levels" of $\mathbf{Z}$ via a_0, a_1, b_1, c_1 and d_1, e.g., $a_0a_1^nb_1^m$ will code in 1 at the lowest level, n at the next level, m at the third level, and 0 at the two highest levels. The purpose of a_3, a_4 and a_5 is to help us to pass from one level to another. This wll be vital in the proof of Lemma 2 when coding in the composition of two functions belonging to ${}^\omega\omega$ (see §4). Since the class of lattice-ordered groups fails to satisfy the amalgamation property, the existence of such a_3, a_4 and a_5 is by no means guaranteed. The construction of a_0, a_1, b_2, c_1 and d_1 has been carefully contrived to ensure that there are indeed elements a_3, a_4 and a_5 having the desired properties.

We now code $f \in {}^\omega\omega$ into $\mathbf{A}(\mathbf{R})$ via a_f where

$$\mathrm{supp}(a_f) = \bigcup\{\mathrm{supp}(c_0 * c_1^md_1^n): m \in \omega, n \in \mathbf{Z}\}$$

and

$$a_f \restriction \mathrm{supp}(c_0 * c_1^md_1^n) = b_1^{f(m)} \restriction \mathrm{supp}(c_0 * c_1^md_1^n) \quad \text{for all } m \in \omega \text{ and } n \in \mathbf{Z}.$$

Note that a_f commutes with a_1, b_1 and d_1, and $a_0 * c_1^ma_f = a_0 * b_1^{f(m)}c_1^m$ for all $m \in \omega$.

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Finally, for technical reasons, we need one extra element $\hat{a} \in \mathbf{A}(\mathbf{R})$ defined by $\operatorname{supp}(\hat{a}) \subseteq \bigcup \{\operatorname{supp}(c_0 * c_1^m d_1^n w)\colon m, n \in \mathbf{Z}, w \in W\}$, and $\hat{a} \restriction \operatorname{supp}(c_0 * c_1^m d_1^n w) = b_1^{m+1} \restriction \operatorname{supp}(c_0 * c_1^m d_1^n w)$ for all $m, n \in \mathbf{Z}$, $w \in W$. Note that $a_s = (\hat{a} \vee e) \wedge a_3^{-1} d_0 a_3$, s being the successor function. Also $\alpha\hat{a} \geqslant \alpha$ for all $\alpha \in \operatorname{supp}(a_6)$; hence $a_6 \wedge (\hat{a}^{-1} \vee e) = e$.

3. The basic lattice-ordered group.

In $\mathbf{A}(\mathbf{R})$ we constructed five "levels" of $\mathbf{Z}$. We now write down a finite presentation which will capture this information and will be a close approximation to the picture.

Let G be the finitely-presented lattice-ordered group with generators the formal symbols $a_0, a_1, a_2, a_3, a_4, a_5, a_6$ and a_7 and relations

$$
\begin{array}{ll}
a_0 \wedge a_2 = e, & a_1 \geqslant e, \\
a_1^{-1} a_0 a_1 \leqslant a_2, & a_1^{-1} a_2 a_1 \leqslant a_2, \\
a_i \leqslant b_i \ (i = 0, 1), & a_1 b_0 = b_0 a_1, \\
a_1 b_1 = b_1 a_1, & a_1 c_1 = c_1 a_1, \\
a_1 d_1 = d_1 a_1, & a_3^{-1} d_0 a_3 \wedge d_1 (a_3^{-1} d_0 a_3)^{-1} = e, \\
a_0 a_4 = a_4 a_0, & a_4^{-1} a_1 a_4 = b_1, \\
a_4^{-1} b_1 a_4 = c_1, & a_4^{-1} c_1 a_4 = d_1, \\
a_0 a_5 = a_5 a_0, & a_5^{-1} b_0 a_5 = c_0, \\
c_1 a_5 = a_5 c_1, & a_6 \wedge a_7 = e, \\
c_1^{-1} (a_6 \vee a_7) c_1 = a_6 \vee a_7, & a_0 \wedge a_6 a_0^{-1} = e, \\
c_1^{-1} a_6 c_1 \leqslant a_6, & c_1 a_7 c_1^{-1} \leqslant a_7, \\
c_1 a_0 c_1^{-1} \leqslant a_7, & a_1 b_0^{-1} \wedge b_0 = e
\end{array}
$$

where

$$b_i \equiv a_3^{-1} a_i a_3, \quad c_i \equiv a_3^{-2} a_i a_3^2, \quad d_i \equiv a_3^{-3} a_i a_3^3 \qquad (i = 0, 1).$$

Observe that all of the above relations hold in $\mathbf{A}(\mathbf{R})$ under the natural interpretation.

Note that we write $x \leqslant y$ as a shorthand for $x \wedge y = x$.

We will continue to use $x * y$ for $y^{-1} x y$ and $x * yz$ for $x * (yz)$. Observe that $(x * y) * z = x * yz$.

LEMMA 1. *In G, the following facts hold*:

(i) $b_1 c_0 = c_0 b_1$; $c_1 d_0 = d_1 c_1$.

(ii) $b_1 c_1 = c_1 b_1$; $c_1 d_1 = d_1 c_1$; $b_1 d_1 = d_1 b_1$.

(iii) $a_0 * a_1^n \leqslant a_2$ *for all* $n \in \mathbf{Z}^+$.

(iv) $(a_0 * a_1^n) \wedge (a_0 * a_1^m) = e$ *unless* $n = m$.

(v) $(b_0 * b_1^n) \wedge (b_0 * b_1^m) = e$ *unless* $n = m$.

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(vi) $(c_0 * c_1^n) \wedge (c_0 * c_1^m) = e$ *unless* $n = m$.

(vii) $(d_0 * d_1^n) \wedge (d_0 * d_1^m) = e$ *unless* $n = m$.

(viii) $(a_0 * a_1^m b_1^n c_1^k d_1^p) \wedge (a_0 * a_1^{m'} b_1^{n'} c_1^{k'} d_1^{p'}) = e$ *unless* ($m = m'$, $n = n'$, $k = k'$ *and* $p = p'$).

(ix) $a_0 * c_1^m \leqslant a_3^{-1} d_0 a_3$ *for all* $m \in \omega$.

(x) $a_0 * c_1^m \leqslant a_6$ *if and only if* $m \in \omega$.

(xi) $a_0 * a_1^m a_5 = a_0 * b_1^m$ *for all* $m \in \mathbf{Z}$.

(xii) $a_1 \leqslant b_1 \leqslant c_1 \leqslant d_1$.

PROOF. (i) and (ii) follow immediately by conjugating the relations $a_1 b_0 = b_0 a_1$, $a_1 b_1 = b_1 a_1$ and $a_1 c_1 = c_1 a_1$ of G by a_3 or a_3^2.

(iii) follows from the relations $a_1^{-1} a_0 a_1 \leqslant a_2$ and $a_1^{-1} a_2 a_1 \leqslant a_2$ by induction on n.

(iv) follows from (iii) and the relation $a_0 \wedge a_2 = e$.

(v), (vi) and (vii) now follow by conjugating (iv) by a_3, a_3^2 and a_3^3 respectively.

(viii) Since a_1 commutes with b_0, $a_0 \leqslant b_0$ implies $a_0 * a_1^n \leqslant b_0$ for all $n \in \mathbf{Z}$. Conjugating by a_3 and a_3^2 gives $b_0 * b_1^n \leqslant c_0$ and $c_0 * c_1^n \leqslant d_0$, respectively, for all $n \in \mathbf{Z}$. Hence $a_0 * a_1^m b_1^n c_1^k \leqslant d_0$ and $a_0 * a_1^{m'} b_1^{n'} c_1^{k'} \leqslant d_0$. (viii) now follows by successively using (vii), (vi), (v) and (iv).

(ix) Note that $a_0 * c_1^m \leqslant d_0$ for all $m \in \omega$ (as in (viii)). Since $a_0 \leqslant b_0$,

$$d_0 = a_0 * a_3^3 \leqslant b_0 * a_3^3 = a_3^{-1} d_0 a_3.$$

Hence $a_0 * c_1^m \leqslant a_3^{-1} d_0 a_3$ for all $m \in \omega$.

(x) is similar to (iii) using the relations involving a_6 and a_7.

(xi) Since $a_1 b_0^{-1} \wedge b_0 = e$, $a_1 \wedge b_0 = b_0$ by Lemma 0(iii). But $a_0 \leqslant b_0$; so $a_0 * a_1^m = a_0 * b_0^m$ for all $m \in \mathbf{Z}$ by Lemma 0(ii). Thus $a_0 * a_1^m a_5 = a_0 * b_0^m a_5 = a_0 * c_0^m$. But $b_1 c_0^{-1} \wedge c_0 = e$ (conjugate $a_1 b_0^{-1} \wedge b_0 = e$ by a_3); so $a_0 * c_0^m = a_0 * b_1^m$ by Lemma 0(ii) again. Therefore $a_0 * a_1^m a_5 = a_0 * b_1^m$ for all $m \in \mathbf{Z}$.

(xii) Since $a_1 \leqslant b_1 = a_3^{-1} a_1 a_3$, $b_1 \leqslant c_1$ & $c_1 \leqslant d_1$ (conjugate by a_3 & a_3^2).

4. Coding recursive functions into lattice-ordered groups.

We write $(\mathbf{a}; \mathbf{r}(\mathbf{a}) = e)$ for the G described above.

We observe that G contains the five "levels" of $\mathbf{Z}$, as described above.

Our main goal in this section will be to code recursive functions (from ω into ω) into finitely presented lattice-ordered groups.

A function $f \in {}^\omega\omega$ is said to be *representable* if there are a finite number of generators $\mathbf{x}(f)$—including $\mathbf{a}$ and a_f — and a finite number of words $\mathbf{s}(f)$ in these generators such that the relations $\mathbf{s}(f) = e$ hold in $\mathbf{A}(\mathbf{R})$ in the natural interpretation and, in $G(f) = (\mathbf{x}(f); \mathbf{r}(\mathbf{a}) = e, \mathbf{s}(f) = e)$, a_f commutes with a_1, b_1 and d_1, and for all $m \in \omega$

$$(*)_m \qquad a_0 * c_1^m a_f = a_0 * b_1^{f(m)} c_1^m.$$

By Lemma 1(viii), a_f is well defined ($a_0 * c_1^m$ & $a_0 * f_1^{f(m)} c_1^m$ are disjoint from $a_0 * c_1^n$ & $a_0 * b_1^{f(n)} c_1^n$ for all $n \neq m$).

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Note that in $\mathbf{A}(\mathbf{R})$, a_f has all these properties. Indeed, all the relations that we will write down hold in $\mathbf{A}(\mathbf{R})$ in the natural interpretation as can be easily checked.

All generators of the finitely presented lattice-ordered groups we will construct will be of the form $\mathbf{a}$, $\hat{a}$ or a_g for some functions $g \in {}^\omega\omega$.

Our main goal will be to prove that every recursive $f \in {}^\omega\omega$ is representable.

To achieve this, it is easiest to take a classification of recursive functions of one variable given by Julia Robinson:

A function $f \in {}^\omega\omega$ is obtained from $g, h, u, v \in {}^\omega\omega$ by *general recursion* if (i) $fg = u$, (ii) $fh = vf$, and (iii) each $n \in \omega$ belongs to the range of one of the functions $h^k g$ $(k \in \omega)$.

The first result from recursion theory that we will need is:

I. (Julia Robinson [**10**]) *The class of recursive functions is the smallest class of numerical functions which is closed under composition and general recursion, and contains the zero function θ ($\theta(n) = 0$) and the successor function s ($s(n) = n + 1$).*

LEMMA 2. *Every recursive function is representable.*

PROOF. Firstly, $\theta(n) \equiv 0$ is clearly representable—adjoin to G the generator a_θ and the relation $a_\theta = e$.

Secondly, for s the successor function, adjoin to G the generators $\hat{a}$ and a_s, and the relations: $\hat{a}$ commutes with a_1, b_1 and d_1;

$$c_1\hat{a} = \hat{a}b_1c_1; \quad a_0 * \hat{a} = a_0 * b_1;$$

$$(\hat{a} \vee e)\left[(\hat{a} \vee e) \wedge a_3^{-1}d_0a_3\right]^{-1} \wedge a_3^{-1}d_0a_3 = e;$$

$$a_s = (\hat{a} \vee e) \wedge a_3^{-1}d_0a_3 \quad \text{and} \quad a_6 \wedge \left(\hat{a}^{-1} \vee e\right) = e.$$

Since $\hat{a}$ commutes with a_1, b_1 and d_1, so does $\hat{a} \vee e$. Moreover, so does $a_3^{-1}d_0a_3$ by Lemmas 0(iii) and 1(xii) (with $x = a_3^{-1}d_0a_3$, $y = d_1$ and $z = a_1$, b_1 and d_1). Thus a_s commutes with a_1, b_1 and d_1. We now prove $a_0 * c_1^m\hat{a} = a_0 * b_1^{s(m)}c_1^m$ for all $m \in \omega$ by induction.

For $m = 0$, $a_0 * c_1^0\hat{a} = a_0 * \hat{a} = a_0 * b_1 = a_0 * b_1^{s(0)}c_1^0$.
Assume $a_0 * c_1^m\hat{a} = a_0 * b_1^{s(m)}c_1^m$. Then

$$\begin{aligned} a_0 * c_1^{m+1}\hat{a} &= a_0 * c_1^m c_1\hat{a} = a_0 * c_1^m\hat{a}b_1c_1 \\ &= a_0 * b_1^{s(m)}c_1^m b_1c_1 = a_0 * b_1^{s(m+1)}c_1^{m+1} \end{aligned}$$

as required.

Now $e = a_3^{-1}d_0a_3 \wedge d_1(a_3^{-1}d_0a_3)^{-1}$, so $d_1 = a_3^{-1}d_0a_3 \vee d_1(a_3^{-1}d_0a_3)^{-1}$. Since $(\hat{a} \vee e)[(\hat{a} \vee e) \wedge a_3^{-1}d_0a_3]^{-1} \wedge a_3^{-1}d_0a_3 = e$ and $a_0 * c_1^m \leqslant a_3^{-1}d_0a_3$ (by Lemma 1(ix)), $a_0 * c_1^m[(\hat{a} \vee e) \wedge a_3^{-1}d_0a_3] = a_0 * c_1^m(\hat{a} \vee e)$ for all $m \in \omega$ by Lemma 0(ii). Hence, by the above and the definition of a_s $(= (\hat{a} \vee e) \wedge a_3^{-1}d_0a_3)$,

$$a_0 * c_1^m a_s = a_0 * c_1^m(\hat{a} \vee e) \quad \text{for all } m \in \omega.$$

But $a_6 \wedge (\hat{a}^{-1} \vee e) = e$, so $a_0 * c_1^m \wedge (\hat{a}^{-1} \vee e) = e$ $(m \in \omega)$ by Lemma 1(x). By Lemma 0(i),

$$a_0 * c_1^m(e \vee \hat{a})\hat{a}^{-1} = a_0 * c_1^m\left(\hat{a}^{-1} \vee e\right) = a_0 * c_1^m$$

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and hence

$$a_0 * c_1^m(e \vee \hat{a}) = a_0 * c_1^m \hat{a} = a_0 * b_1^{s(m)} c_1^m$$

for all $m \in \omega$. Therefore $a_0 * c_1^m a_s = a_0 * b_1^{s(m)} c_1^m$ for all $m \in \omega$. Thus s is representable.

We next show that if f and g are representable, then so is $h = gf$. Let $G(h)$ have generators $\mathbf{x}(f) \cup \mathbf{x}(g) \cup \{a_h\}$, and relations: $\mathbf{s}(f) = e$, $\mathbf{s}(g) = e$, a_h commutes with a_1, b_1 and d_1, and $a_h(a_g \dagger a_f)^{-1} \wedge a_6 = e$, where $x \dagger y$ is shorthand for $x * a_4^{-1} y^{-1} a_5$. Since $a_0 * c_1^m \leqslant a_6$ for all $m \in \omega$ (Lemma 1(x)) and $a_h(a_g \dagger a_f)^{-1} \wedge a_6 = e$,

$$a_0 * c_1^m a_h = a_0 c_1^m (a_g \dagger a_f)$$

by Lemma 0(i). So, in $G(h)$, using Lemma 1(xi),

$$\begin{aligned} a_0 * c_1^m a_h &= a_0 * c_1^m \left(a_5^{-1} a_f a_4 a_g a_4^{-1} a_f^{-1} a_5 \right) \\ &= a_0 * b_1^{f(m)} c_1^m a_4 a_g a_4^{-1} a_f^{-1} a_5 \\ &= a_0 * c_1^{f(m)} d_1^m a_g a_4^{-1} a_f^{-1} a_5 \\ &= a_0 * b_1^{gf(m)} c_1^{f(m)} d_1^m a_4^{-1} a_f^{-1} a_5 \\ &= a_0 * a_1^{h(m)} b_1^{f(m)} c_1^m a_f^{-1} a_5 \\ &= a_0 * a_1^{h(m)} c_1^m a_5 = a_0 * b_1^{h(m)} c_1^m. \end{aligned}$$

Hence the composition of representable functions is representable.

Lastly, we must show that if g, h, u and v are representable and f is obtained from them by general recursion, then f is representable. So assume the hypothesis and let $G(f)$ have a generators a_f together with those used to show the representability of g, h, u and v, and relations those required for g, h, u and v together with: a_f commutes with a_1, b_1 and d_1;

$$a_u \left(a_f \dagger a_g \right)^{-1} \wedge a_6 = e \quad \text{and} \quad \left(a_f \dagger a_h \right) \left(a_v \dagger a_f \right)^{-1} \wedge a_6 = e.$$

Now each $m \in \omega$ belongs to the range of some $h^k g$. We prove, by induction on k, that for each $m \in$ range$(h^k g)$, $a_0 * c_1^m a_f = a_0 * b_1^{f(m)} c_1^m$.

For $k = 0$, let $m = g(n)$, say.

Thus $a_0 * c_1^n a_u = a_0 * b_1^{u(n)} c_1^n = a_0 * b_1^{f(m)} c_1^n$ since u is representable and $u = fg$. Since $a_u(a_f \dagger a_g)^{-1} \wedge a_6 = e$, it follows that

$$a_0 * c_1^n a_u = a_0 * c_1^n \left(a_f * a_4^{-1} a_g^{-1} a_5 \right).$$

Hence

$$a_0 * b_1^{f(m)} c_1^n a_5^{-1} a_g a_4 = a_0 * c_1^n a_5^{-1} a_g a_4 a_f.$$

So

$$a_0 * a_1^{f(m)} c_1^n a_g a_4 = a_0 * b_1^{g(n)} c_1^n a_4 a_f.$$

Therefore

$$a_0 * b_1^{f(m)} c_1^m d_1^n = a_0 * c_1^m d_1^n a_f.$$

Since $d_1 a_f = a_f d_1$, $a_0 * b_1^{f(m)} c_1^m = a_0 * c_1^m a_f$.

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Now assume that if $n \in \text{range}(h^k g)$, then $a_0 * c_1^n a_f = a_0 * b_1^{f(n)} c_1^n$ and suppose $m \in \text{range}(h^{k+1} g)$. So $m = h(n)$ for some $n \in \text{range}(h^k g)$.

But

$$a_0 * c_1^n (a_f \dagger a_h) = a_0 * c_1^n (a_v \dagger a_f)$$

so

$$a_0 * c_1^n a_5^{-1} a_h a_4 a_f a_4^{-1} a_h^{-1} a_5 = a_0 * c_1^n a_5^{-1} a_f a_4 a_v a_4^{-1} a_f^{-1} a_5.$$

By the hypotheses,

$$a_0 * c_1^{h(n)} d_1^n a_f a_4^{-1} a_h^{-1} = a_0 * c_1^{f(n)} d_1^n a_v a_4^{-1} a_f^{-1}.$$

Thus

$$a_0 * c_1^m a_f d_1^n a_4^{-1} a_h^{-1} = a_0 * b_1^{vf(n)} c_1^{f(n)} d_1^n a_4^{-1} a_f^{-1}.$$

Hence

$$a_0 * c_1^m a_f d_1^n = a_0 * b_1^{vf(n)} c_1^{f(n)} d_1^n a_4^{-1} a_f^{-1} a_h a_4.$$

Since $vf(n) = fh(n) = f(m)$ and a_1 commutes with b_1, c_1 and a_f,

$$a_0 * c_1^m a_f = a_0 * b_1^{f(n)} c_1^n a_f^{-1} a_1^{f(m)} a_h a_4 d_1^{-n}.$$

Therefore

$$a_0 * c_1^m a_f = a_0 * c_1^n a_h a_1^{f(m)} a_4 d_1^{-n}.$$

So

$$a_0 * c_1^m a_f = a_0 * b_1^{h(n)} c_1^n a_1^{f(m)} a_4 d_1^{-n}.$$

As $h(n) = m$ and b_1, c_1 and d_1 commute,

$$a_0 * c_1^m a_f = a_0 * b_1^{f(m)} c_1^m.$$

By I, every recursive function is representable, and the proof of the lemma complete.

5. The proofs of the Theorem and Corollary. We now apply the results of §§2 and 4 to prove the Theorem. We need the following fact:

II. *There is a recursive function whose range is not a recursive set.*

Let f be a recursive function whose range X is not recursive. Let h be the characteristic function of $\omega \setminus X$; i.e.,

$$h(m) = \begin{cases} 0 & \text{if } m \in X, \\ 1 & \text{if } m \notin X. \end{cases}$$

Note that h is not recursive and so is not necessarily representable.

By Lemma 2, f is representable. Adjoin to $G(f)$ a new generator a_h and relations: a_h commutes with a_1, b_1 and d_1, and $(a_h \dagger a_f) \wedge a_6 = e$. Let H_X be the resulting finitely presented lattice-ordered group. The set of generators corresponds to the set X_0 mentioned in the introduction; the relations to S_0 mentioned there. Now since $a_0 * c_1^n \leqslant a_6$ for all $n \in \omega$,

$$a_0 * c_1^n (a_h \dagger a_f) = a_0 * c_1^n$$

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by Lemma 0(i). So

$$a_0 * c_1^n a_5^{-1} a_f a_4 a_h = a_0 * c_1^n a_5^{-1} a_f a_4.$$

Thus

$$a_0 * c_1^{f(n)} a_h = a_0 * c_1^{f(n)}.$$

This set of relations corresponds to the nonrecursive set S of the introduction, whereas R corresponds to the set $a_0 * c_1^m a_h = a_0 * c_1^m$ $(m \in \omega)$.

However (by §2), in $\mathbf{A}(\mathbf{R})$, if $m \notin X$,

$$a_0 * c_1^m a_h = a_0 * b_1 c_1^m \neq a_0 * c_1^m.$$

Since all the defining relations of H_X hold in $\mathbf{A}(\mathbf{R})$ in the given interpretation, in H_X,

$$a_0 * c_1^m a_h = a_0 * c_1^m \quad \text{if and only if } m \in X.$$

Consequently, we have achieved our goal and the finitely presented lattice-ordered group H_X has insoluble word problem. Q.E.D.

Note that the word problem for H_X has Turing degree at least that of X.

We now prove the Corollary.

If L is any lattice-ordered group and $x \in L$, then $|x| = x \vee x^{-1} \geqslant e$; indeed $|x| = e$ if and only if $x = e$ [**2**, Lemma 1.11.4]. Hence $w_1, \ldots, w_n$ are simultaneously equal to e if and only if $|w_1| \vee \cdots \vee |w_n| = e$. So any finite set of relations in any lattice-ordered group is equivalent to a single relation. The Corollary therefore follows immediately from the Theorem.

Appendix. In §2 we asserted the existence of certain elements of $\mathbf{A}(\mathbf{R})$. We did not wish to elaborate on the reasons for their existence earlier as we felt that a better understanding of our ideas would result if we postponed these technicalities. However, we must now prove the existence of $a_3, a_4, a_5 \in \mathbf{A}(\mathbf{R})$. The proof is essentially in [**2**, §2.2]; its root is [**4**].

Choose $\alpha_0 \in \operatorname{supp}(a_0)$. Let $\beta_0 = \inf(\operatorname{supp}(a_0))$ and $\gamma_0 = \inf(\operatorname{supp}(b_0))$. Observe that $\sup(\operatorname{supp}(a_0)) < \beta_0 b_0$ and $\sup(\operatorname{supp}(b_0)) < \gamma_0 c_0$ by §2(c) and (d) respectively. Let f_0 be an order-preserving one-to-one map of $[\alpha_0, \alpha_0 a_0]$ onto $[\alpha_0, \alpha_0 b_0]$. Let $f_n = a_0^{-n} f_0 b_0^n$, an order-preserving one-to-one map of $[\alpha_0 a_0^n, \alpha_0 a_0^{n+1}]$ onto $[\alpha_0 b_0^n, \alpha_0 b_0^{n+1}]$ $(n \in \mathbf{Z})$. Define f: $\operatorname{supp}(a_0) \to \operatorname{supp}(b_0)$ by $f \upharpoonright [\alpha_0 a_0^n, \alpha_0 a_0^{n+1}] = f_n$ $(n \in \mathbf{Z})$; i.e. $f = \bigcup \{f_n \colon n \in \mathbf{Z}\}$. We can extend f to an order-preserving one-to-one map g_0 of $[\beta_0, \beta_0 b_0]$ onto $[\gamma_0, \gamma_0 c_0]$. Let $g_n = b_0^{-n} g_0 c_0^n$: $[\beta_0 b_0^n, \beta_0 b_0^{n+1}] \to [\gamma_0 c_0^n, \gamma_0 c_0^{n+1}]$ $(n \in \mathbf{Z})$ and $g = \bigcup \{g_n \colon n \in \mathbf{Z}\}$. So g: $\operatorname{supp}(b_0) \to \operatorname{supp}(c_0)$. We can extend g to an element of $\mathbf{A}(\mathbf{R})$, say a_3. Since a_3 extends both f and g, it follows by an easy computation that $a_3^{-1} a_0 a_3 = b_0$ and $a_3^{-1} b_0 a_3 = c_0$. ($\operatorname{supp}(a_3^{-1} a_0 a_3) = \operatorname{supp}(a_0) a_3 = \operatorname{supp}(a_0) f = \operatorname{supp}(b_0)$, and if $\sigma \in [\alpha_0 b_0^n, \alpha_0 b_0^{n+1}]$, $\sigma a_3^{-1} a_0 a_3 = \sigma f_n^{-1} a_0 f_{n+1} = \sigma b_0$. The other equality is proved similarly.) This shows that a_3 exists.

The constructions of a_4 and a_5 are similar. Let $\delta_0 = \inf(\operatorname{supp}(c_0))$. Note that $\sup(\operatorname{supp}(c_0)) < \delta_0 d_0$. Let f_0 be a one-to-one order-preserving map of $[\beta_0, \beta_0 b_0]$ onto $[\beta_0, \beta_0 c_0]$ such that $f_0 \upharpoonright \operatorname{supp}(a_0) = e$. Let $f_n = b_0^{-n} f_0 c_0^n$ $(n \in \mathbf{Z})$, and f: $\operatorname{supp}(b_0) \to \operatorname{supp}(c_0)$ be given by $f = \bigcup \{f_n \colon n \in \mathbf{Z}\}$. Let g_0: $[\gamma_0, \gamma_0 c_0] \to [\delta_0, \delta_0 d_0]$ be a

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one-to-one order-preserving onto map extending f. Let $g_n = c_0^{-n} g_0 d_0^n$ ($n \in \mathbf{Z}$) and g: $\operatorname{supp}(c_0) \to \operatorname{supp}(d_0)$ be defined by $g = \bigcup\{g_n : n \in \mathbf{Z}\}$. We continue this extension process to

$$\operatorname{supp}(d_0) \to \operatorname{supp}(x_0), \operatorname{supp}(x_0) \to \operatorname{supp}(y_1), \ldots, \operatorname{supp}(y_n) \to \operatorname{supp}(y_{n+1}), \ldots$$

to obtain h, an order-preserving permutation of the interval $\bigcup\{\operatorname{supp}(y_n) : n \in \mathbf{Z}^+\}$. By construction, $h^{-1}a_0h = a_0$, $h^{-1}a_1h = b_1$, $h^{-1}b_1h = c_1$ and $h^{-1}c_1h = d_1$. Extend h to an element of $\mathbf{A}(\mathbf{R})$, call it a_4. (Of course, if $\mathbf{R} = \bigcup\{\operatorname{supp}(y_n) : n \in \mathbf{Z}^+\}$, this step is unnecessary.) Then $a_4 \in \mathbf{A}(\mathbf{R})$ has all the desired properties.

For a_5, let f be defined as for a_4. Let g_0: $[\delta_0, \delta_0 d_0] \to [\delta_0, \delta_0 d_0]$ be an order-preserving one-to-one onto map extending f. Let $g_n = d_0^{-n} g_0 d_0^n$ ($n \in \mathbf{Z}$) and g be the order-preserving permutation of $\operatorname{supp}(d_0)$ given by $\bigcup\{g_n : n \in \mathbf{Z}\}$. Then $g^{-1}d_0g = d_0$. Since g extends f, $g^{-1}a_0g = a_0$ and $g^{-1}b_0g = c_0$. Let $h_{n,w}$ be that order-preserving permutation of $\operatorname{supp}(d_0 * d_1^n w)$ defined by $g * d_1^n w$ ($n \in \mathbf{Z}$, $w \in W$). Then $h = \bigcup\{h_{n,w} : n \in \mathbf{Z},\ w \in W\}$ has the properties $h^{-1}a_0h = a_0$, $h^{-1}b_0h = c_0$ and $h^{-1}c_1h = c_1$. Any extension of h to an element of $\mathbf{A}(\mathbf{R})$ can be taken as a_5.

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