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ANALYTICITY ON ROTATION INVARIANT FAMILIES OF CURVES

BY

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Dedicated to Professor Ivan Vidav on the occasion of his sixty-fifth birthday, January 17, 1983

ABSTRACT. Let $\mathfrak{G}$ be a rotation invariant family of smooth Jordan curves contained in Δ, the open unit disc in $\mathbf{C}$. For each $\Gamma \in \mathfrak{G}$ let D_Γ be the simply connected domain bounded by Γ. We present various conditions which imply that if f is a continuous function on $\overline{\Delta}$ such that for every $\Gamma \in \mathfrak{G}$ the function $f|\Gamma$ has a continuous extension to $\overline{D_\Gamma}$ which is analytic in D_Γ, then f is analytic in Δ.

1. Introduction. Denote by Δ the open unit disc in $\mathbf{C}$ and by G the group of conformal automorphisms of Δ. Let $\mathfrak{G}$ be a family of smooth (i.e. continuously differentiable) Jordan curves contained in Δ which is *Moebius invariant*, i.e. $\omega(\Gamma) \in \mathfrak{G}$ whenever $\Gamma \in \mathfrak{G}$ and $\omega \in G$. For each $\Gamma \in \mathfrak{G}$ denote by D_Γ the simply connected domain bounded by Γ and write $\Gamma^* \doteq \{\bar{z}: z \in \Gamma\}$. Agranovski and Valski [2] (see also [1]) proved that

$$(1)\qquad \begin{cases} \text{if } f \text{ is a continuous function on } \Delta \text{ such that for every } \Gamma \in \mathfrak{G} \text{ the} \\ \text{function } f|\Gamma \text{ has a continuous extension to } \overline{D_\Gamma}\text{, which is analytic} \\ \text{in } D_\Gamma\text{, then } f \text{ is analytic in } \Delta. \end{cases}$$

Agranovski [1] sharpened this result by proving that if f is continuous on Δ and satisfies $\int_\Gamma f(z)\,dz = 0$ for every $\Gamma \in \mathfrak{G}$ then f is analytic in Δ. This suggests that one should be able to prove (1) for families $\mathfrak{G}$ much smaller than the Moebius invariant ones.

A Moebius invariant family $\mathfrak{G}$ is always *rotation invariant* (i.e. $s\Gamma \in \mathfrak{G}$ whenever $\Gamma \in \mathfrak{G}$ and $s \in \mathbf{C}$, $|s| = 1$) and sometimes (e.g. if it consists of circles) it is also *symmetric* (i.e. $\Gamma^* \in \mathfrak{G}$ whenever $\Gamma \in \mathfrak{G}$).

In the present paper we study the rotation invariant, symmetric families $\mathfrak{G}$ which satisfy (1) and the rotation invariant families $\mathfrak{G}$ which satisfy (1) for smooth functions f. We present various examples of such minimal families.

If f is continuous then $f|\Gamma$ has a continuous extension to $\overline{D_\Gamma}$ which is analytic in D_Γ if and only if

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)\,d\zeta}{\zeta - z} = 0 \qquad \left(z \in \mathbf{C} - \overline{D_\Gamma}\right)$$

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[6]. We will often use the fact that if f is continuous on Δ and holomorphic in $\Delta - \{0\}$ then f is holomorphic in Δ.

If f is a continuous function on a circle $|z| = r$, $r > 0$, and if $n \in \mathbf{Z}$ then define

$$A_n(f, r) = r^{-n}\frac{1}{2\pi}\int_0^{2\pi} e^{-in\varphi}f(re^{i\varphi})\,d\varphi.$$

Note that if f is analytic in a neighbourhood of $|z| = r$ then $A_n(f, r)$ is the nth coefficient in the Laurent series of f.

LEMMA 1. *Let $0 < r_1 < r_2$ and let f be a continuous function on $\Omega = \{z\colon r_1 \leqslant |z| \leqslant r_2\}$. Suppose that for each $n \in \mathbf{Z}$ the function $r \mapsto A_n(r)$ is constant on $\{r\colon r_1 \leqslant r \leqslant r_2\}$. Then f is anlaytic in the interior of Ω.*

PROOF. By the assumption there are numbers a_n, $n \in \mathbf{Z}$, such that for each r, $r_1 \leqslant r \leqslant r_2$, $\sum a_n r^n e^{in\theta}$ is the Fourier series of the function $\theta \mapsto f(re^{i\theta})$. For $m \in \mathbf{N}$ let

$$\sigma_m(f, r, e^{i\theta}) = \frac{1}{m}\left[a_0 + \sum_{-1}^{1} a_k r^k e^{ik\theta} + \cdots + \sum_{-(m-1)}^{m-1} a_k r^k e^{ik\theta}\right]$$

be its mth Cezàro mean. By the uniform continuity of f the family $\{\theta \mapsto f(re^{i\theta});\ r_1 \leqslant r \leqslant r_2\}$ is uniformly equicontinuous. The usual proof of Fejér's theorem [5] applied to the series $\sum_{-\infty}^{\infty} a_n r^n e^{in\theta}$ shows that $\sigma_m(f, r, e^{i\theta})$ converges to $f(re^{i\theta})$ uniformly for r and θ, $r_1 \leqslant r \leqslant r_2$, $0 \leqslant \theta \leqslant 2\pi$. Consequently, on Ω, $f(z)$ is the uniform limit of the sequence

$$\frac{1}{m}\left[a_0 + \sum_{-1}^{1} a_k z^k + \cdots + \sum_{-(m-1)}^{m-1} a_k z^k\right]$$

so f is analytic in the interior of Ω. This completes the proof.

2. Analyticity on a family obtained by rotating a single curve.

LEMMA 2. *Let Γ be a smooth Jordan curve in $\mathbf{C}$ and let D be the simply connected domain bounded by Γ. Suppose that f is a continuous function on $\Omega = \{sz\colon z \in \Gamma, |s| = 1\}$ such that for every s, $|s| = 1$, the function $f|(s\Gamma)$ has a continuous extension to $s\overline{D}$ which is analytic in sD. Then for every $n \in \mathbf{Z}$ the function $z \mapsto z^n A_n(|z|)$ has a continuous extension from $\Gamma - \{0\}$ to $\overline{D}$ which is analytic in D.*

PROOF. Fix $n \in \mathbf{Z}$. Let $z \in \mathbf{C} - \overline{D}$. By the continuity of f the function $(\zeta, \varphi) \mapsto e^{-in\varphi}f(e^{i\varphi}\zeta)/(\zeta - z)$ is continuous on $\Gamma \times [0, 2\pi]$. By our assumption,

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(e^{i\varphi}\zeta)\,d\zeta}{\zeta - z} = 0 \qquad (0 \leqslant \varphi \leqslant 2\pi),$$

so by Fubini,

$$\frac{1}{2\pi i}\int_\Gamma\left[\int_0^{2\pi} e^{in\varphi}f(e^{i\varphi}\zeta)\,d\varphi\right]\frac{d\zeta}{\zeta - z} = \int_0^{2\pi} e^{-in\varphi}\left[\frac{1}{2\pi i}\int_\Gamma \frac{f(e^{i\varphi}\zeta)\,d\zeta}{\zeta - z}\right]d\varphi = 0.$$

This proves that

$$\zeta \mapsto \frac{1}{2\pi}\int_0^{2\pi} e^{in\varphi} f(e^{i\varphi}\zeta)\, d\varphi$$

has a continuous extension from Γ to $\overline{D}$ which is anlaytic in D. If $\zeta \neq 0$ then

$$\frac{1}{2\pi i}\int_0^{2\pi} e^{-in\varphi} f(e^{i\varphi}\zeta)\, d\varphi = e^{in\arg\zeta}\frac{1}{2\pi}\int_0^{2\pi} e^{-in\varphi} f(e^{i\varphi}|\zeta|)\, d\varphi = \zeta^n A_n(|\zeta|).$$

This completes the proof.

For any domain $D \subset \mathbf{C}$ we write $D^* = \{\bar{z}\colon z \in D\}$.

THEOREM 1. *Let Γ be a smooth Jordan curve in $\mathbf{C}$. Denote by D the simply connected domain bounded by Γ and assume that $0 \notin \overline{D}$. If f is a continuous function on $\Omega = \{sz\colon z \in \Gamma, |s| = 1\}$ such that for every s, $|s| = 1$:*

(i) *the function $f|(s\Gamma)$ has a continuous extension to $s\overline{D}$ which is analytic in sD and*

(ii) *the function $f|(s\Gamma^*)$ has a continuous extension to $s\overline{D}^*$ which is analytic in sD^*, then f is analytic in the interior of Ω. If f is smooth (i.e. of class C^1) on Ω then* (i) *alone implies that f is analytic in the interior of Ω.*

PROOF. Let f be continuous on Ω and suppose that (i) holds. Since $0 \notin \overline{D}$ it follows by Lemma 2 that for each $n \in \mathbf{Z}$ the function $z \mapsto A_n(|z|)$ has a continuous extension F_n from Γ to $\overline{D}$ which is analytic in D.

If (ii) also holds, then by Lemma 2 for each $n \in \mathbf{Z}$, the function $z \mapsto A_n(|z|)$ has a continuous extension G_n from Γ^* to $\overline{D}^*$ which is analytic in D. Fix $n \in \mathbf{Z}$. Note that the function $z \mapsto G_n(\bar{z})$ is continuous on $\overline{D}$ and antianalytic in D. For every $z \in \Gamma$ we have $F_n(z) = A_n(|z|) = G_n(\bar{z})$ which implies that the functions $z \mapsto F_n(z)$, $z \mapsto G_n(\bar{z})$, continuous on $\overline{D}$ and harmonic in D, coincide in D. It follows that F_n is a constant, i.e. $z \mapsto A_n(|z|)$ is constant on Γ. By Lemma 1 it follows that f is analytic in the interior of Ω.

Suppose now that f is smooth on Ω and that (i) holds. It is easy to see that for each $n \in \mathbf{Z}$ the function $z \mapsto A_n(|z|)$ is smooth on Γ. Thus the proof will be complete once we have shown that if $z \mapsto \Phi(z)$ is a smooth function on Γ that depends only on $|z|$ and if Φ has a continuous extension $\tilde{\Phi}$ to $\overline{D}$ which is analytic in D, then Φ is a constant. We follow Agranovski and Valski **[2]** and use an argument used by Browder and Wermer **[3]**. Suppose that Φ is not a constant. Then $\tilde{\Phi}(D)$ is open, and so by the smoothness of Φ, $\tilde{\Phi}(D) \not\subset \Phi(\Gamma)$. Let $c \in \tilde{\Gamma}(D) - \Phi(\Gamma)$. Then the function $z \mapsto \tilde{\Phi}(z) - c$ is continuous on $\overline{D}$, analytic in D, has a zero in D, and has no zero on Γ. On Γ it depends only on $|z|$ so the variation of its argument along Γ around D is zero, a contradiction. This completes the proof.

REMARK 1. Let Γ and D be as in Theorem 1. Suppose that f is continuous on Ω. It is an open question whether (i) alone implies that f is analytic in the interior of Ω. Standard arguments, e.g. convolving f with an approximate identity of a group to get a smooth function **[2, 9]**, do not apply since the rotation group is too small and the process smoothens the function only in the direction perpendicular to the radius. From the proof of Theorem 1 it follows that we could answer the question if we knew that if $z \mapsto F(z)$ is a continuous function on $\overline{D}$, analytic in D and such that its boundary values depend only on $|z|$, then F is a constant. We are not able to prove this. For certain domains D this problem is related to Question 1 in [**3**, p. 129].

The following corollary gives a characterization of analytic functions in an annulus in terms of the behavior on certain circles (compare with [**11**, p. 169; **7**, Theorem 12.3.11]).

COROLLARY 1. *Let* 0 *be in the exterior of a circle* $\Gamma \subset \mathbf{C}$ *and let* Ω *be the annulus obtained by rotating* Γ *around the origin. If* f *is a continuous function on* Ω *such that for every* s, $|s|=1$, *the function* $f|(s\Gamma)$ *has a continuous extension to the closed disc bounded by* $s\Gamma$ *which is analytic in its interior, then* f *is analytic in the interior of* Ω.

EXAMPLE 1. The function

$$z \mapsto f(z) = \begin{cases} z^3/\bar{z}, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

is smooth on $\mathbf{C}$. Let $a, b \in \mathbf{C}$, $b \neq 0$. If $|s|=1$ and $a + sb \neq 0$ then $f(a+sb) = s(a+sb)^3/(\bar{b} + s\bar{a})$ which shows that if $|b| \geqslant |a|$ then $s \mapsto f(a+sb)$ has a continuous extension from $\partial\Delta$ to $\bar{\Delta}$ which is analytic in Δ. Consequently, for any open disc $D \subset \mathbf{C}$ such that $0 \in \bar{D}$, the function $f|\partial D$ has a continuous extension to $\bar{D}$ which is analytic in D. This shows that in Theorem 1 one cannot drop the assumption that $0 \notin \bar{D}$.

REMARK 2. Let $r > 0$ and let $\mathfrak{G}$ be the family of all circles in $\mathbf{C}$ of radius r. Let f be continuous on $\mathbf{C}$ and let $n \in \mathbf{N}$. If

$$\int_\Gamma f(z)\, dz = \int_\Gamma z^n f(z)\, dz = 0$$

for every $\Gamma \in \mathfrak{G}$, then by a result of Zalcman [**10**, **11**] f is entire. So, vanishing of only two negative Fourier coefficients implies the analyticity of f. One cannot relax the assumptions in Corollary 1 in this direction—here one needs the vanishing of all negative Fourier coefficients. To see this, let $n \in \mathbf{N}$ and define

$$f\left(e^{i\tau}(2 + e^{i\theta})\right) = \cos n\theta \qquad (0 \leqslant \tau \leqslant 2\pi, 0 \leqslant \theta \leqslant 2\pi).$$

Then f is well defined in $\{z: 1 \leqslant |z| \leqslant 3\}$. We have

$$\begin{aligned} \int_0^{2\pi} e^{im\varphi} f(2e^{i\tau} + e^{i\varphi})\, d\varphi &= \int_0^{2\pi} e^{im\varphi} \cos n(\varphi - \tau)\, d\varphi \\ &= 2^{-1} \int_0^{2\pi} e^{im\varphi} \left[e^{-in\tau} e^{in\varphi} + e^{in\tau} e^{-in\varphi}\right] d\varphi = 0 \end{aligned}$$

whenever $0 \leqslant \tau \leqslant 2\pi$ and $m \in \mathbf{N}$, $m \neq n$. So, all negative Fourier coefficients vanish, except one, and yet f is not analytic in $\{z: 1 < |z| < 3\}$.

3. Analyticity on general families. The analogue of Corollary 1 in the case when Ω is a disc does not hold: If $\mathfrak{G}$ is the family of circles obtained by rotating the circle $|z - 1/2| = 1/2$ around the origin, then Example 1 shows that if

$$\text{(2)} \quad \begin{cases} f \text{ is a continuous function on } \bar{\Delta} \text{ such that for every } \Gamma \in \mathfrak{G} \text{ the function} \\ f|\Gamma \text{ has a continuous extension to the closed disc bounded by } \Gamma \text{ which} \\ \text{is analytic in its interior} \end{cases}$$

then it does not necessarily follow that f is analytic in Δ. Consequently, for no family $\mathfrak{G}$ obtained by rotating only one circle, (2) implies the analyticity of f in Δ (see the discussion following Remark 3). It is interesting that two (suitable chosen) circles suffice: If $\Gamma_0 \subset \overline{\Delta}$ is any circle having 0 in its exterior and if $\mathfrak{G}$ is the family of all circles obtained by rotating the circles Γ_0 and $|z - 1/2| = 1/2$ around the origin, then (2) implies that f is analytic in Δ. This follows from Theorem 1, Lemma 2, and the following

LEMMA 3. *Let Γ, D, Ω and f be as in Lemma 2. Suppose that f is analytic in an open annulus contained in Ω. Then f is analytic in the interior of Ω.*

PROOF. By the assumption there are a sequence a_n, $n \in \mathbf{Z}$, and an open annulus $\Sigma \subset \Omega$ such that $A_n(|z|) = a_n$ $(z \in \Sigma,\ n \in \mathbf{Z})$. It follows that there is a relatively open set $U \subset \Gamma$ such that

$$A_n(|z|) = a_n \qquad (z \in U, n \in \mathbf{Z}). \tag{3}$$

By Lemma 2, for each $n \in \mathbf{Z}$ the function $z \mapsto z^n A_n(|z|)$ has a continuous extension from $\Gamma - \{0\}$ to $\overline{D}$ which is analytic in D. Let $n \geqslant 0$. By (3) we have $z^n A_n(|z|) = a_n z^n$ $(z \in U)$. Since $z \mapsto a_n z^n$ also has a continuous extension from $\Gamma - \{0\}$ to $\overline{D}$, which is analytic in D, it follows that $z^n A_n(|z|) = a_n z^n$ $(z \in \Gamma - \{0\})$ and consequently, $A_n(|z|) = a_n$ $(z \in \Gamma - \{0\})$. Let $n < 0$. Then $z \mapsto z^{-n}$ has a continuous extension from $\Gamma - \{0\}$ to $\overline{D}$ which is analytic in D so the same holds for the function $z \mapsto z^{-n}[z^n A_n(|z|)] = A_n(|z|)$. By (3) it follows that $A_n(|z|) = a_n$ $(z \in \Gamma - \{0\})$. Thus we have proved that $A_n(|z|) = a_n$ $(z \in \Gamma - \{0\}, n \in \mathbf{Z})$. Now the assertion follows from Lemma 1. This completes the proof.

THEOREM 2. *Let $\mathfrak{G}$ be a rotation invariant, symmetric family of smooth Jordan curves contained in Δ. For each $\Gamma \in \mathfrak{G}$ let D_Γ be the simply connected domain bounded by Γ, and let B_Γ be the interior of the set $\{sz\colon z \in \Gamma, |s| = 1\}$. Let $B = \bigcup_{\Gamma \in \mathfrak{G}} B_\Gamma$. Suppose that*

(a)(i) *every continuous function on Δ, which is analytic in B, is analytic in Δ,*

(ii) *every connected component of B contains a domain D_Γ, $\Gamma \in \mathfrak{G}$, such that $0 \notin \overline{D}_\Gamma$.*

Then

(b) *if f is a continuous function on Δ such that for every $\Gamma \in \mathfrak{G}$ the function $f|\Gamma$ has a continuous extension to $\overline{D}_\Gamma$ which is analytic in D_Γ, then f is analytic in Δ.*

If we assume (a)(i) *and if every $\Gamma \in \mathfrak{G}$ is a circle then* (a)(ii) *and* (b) *are equivalent. If we assume that* 0 *is in the exterior of every $\Gamma \in \mathfrak{G}$ then* (a)(i) *and* (b) *are equivalent.*

PROOF. Note first that a connected component of B is either a disc $\{z\colon |z| < r\}$ or an annulus $\{z\colon r_1 < |z| < r_2\}$.

The last statement is an easy consequence of Theorem 1.

Suppose that (a) holds. Assume that f is a continuous function on Δ such that for every $\Gamma \in \mathfrak{G}$ the function $f|\Gamma$ has a continuous extension to $\overline{D}_\Gamma$ which is analytic in D_Γ. Let B_0 be a connected component of B. By (a)(ii) there is a $\Gamma_0 \in \mathfrak{G}$ such that $D_{\Gamma_0} \subset B_0$ and $0 \notin \overline{D}_{\Gamma_0}$. By Theorem 1, f is analytic in B_{Γ_0}. Let K be a closed annulus contained in B_0 such that $K \cap B_{\Gamma_0} \neq \varnothing$. There are $\Gamma_1, \Gamma_2, \ldots, \Gamma_n \in \mathfrak{G}$ such that

$K \subset \bigcup_{i=1}^{n} B_{\Gamma_i} \subset B_0$ and such that $B_{\Gamma_i} \cap B_{\Gamma_{i+1}} \neq \varnothing$ $(1 \leqslant i \leqslant n-1)$. Further, there is some j, $1 \leqslant j \leqslant n$, such that $B_{\Gamma_j} \cap B_{\Gamma_0} \neq \varnothing$. By Lemma 3, f is analytic in B_{Γ_j}. Using Lemma 3 step by step we prove that f is analytic in $\bigcup_{i=1}^{n} B_{\Gamma_i}$. As K was arbitrary it follows that f is analytic in B_0 and as B_0 was an arbitrary connected component of B it follows by (a)(i) that f is analytic in Δ. This proves that (a) implies (b).

Assume that every $\Gamma \in \mathfrak{G}$ is a circle and that (a)(i) holds. Suppose that (b) holds. We have to prove that this implies (a)(ii). Assume, contrarily to (a)(ii), that there is a connected component B_0 of B such that $0 \in \overline{D}_\Gamma$ whenever $\Gamma \in \mathfrak{G}$ satisfies $B_\Gamma \subset B_0$. If $B_0 = \{z: |z| < r\}$ then define

$$f(z) = \begin{cases} 0, & z = 0, \\ z^2/\bar{z}, & 0 < |z| \leqslant r, \\ z^3/r^2, & r \leqslant |z| < 1, \end{cases}$$

and if $B_0 = \{z: r_1 < |z| r_2\}$ then define

$$f(z) = \begin{cases} z^3/r_1^2, & |z| \leqslant r_1, \\ z^2/\bar{z}, & r_1 \leqslant |z| \leqslant r_2, \\ z^3/r_2^2, & r_2 \leqslant |z| \leqslant 1. \end{cases}$$

Then f is continuous on Δ and (see Example 1) for every $\Gamma \in \mathfrak{G}$ the function $f|\Gamma$ has a continuous extension to $\overline{D}_\Gamma$ which is analytic in D_Γ yet f is not analytic in Δ, a contradiction. This complete the proof.

Remark 3. It is not known whether in Theorem 2 one can drop the assumption that $\mathfrak{G}$ is symmetric. In our proof we used the symmetry when we applied Theorem 1. So this question is related to Remark 1. However, for smooth functions one can prove a theorem, analogous to Theorem 1, without assuming that $\mathfrak{G}$ is symmetric. In the proof, analogous to the one above, one uses the second half of Theorem 1.

Now we present two examples of minimal rotation invariant families that satisfy (b) in Theorem 2. Note that if $\mathfrak{G}$ satisfies (b) then $\bigcup_{\Gamma \in \mathfrak{G}} \Gamma$ must be dense in Δ. Were this not so, there would be a continuous function f on Δ, vanishing on an open subset of Δ containing $\bigcup_{\Gamma \in \mathfrak{G}} \Gamma$ and not vanishing identically. By (b), f would be analytic in Δ, a contradiction.

Example 2. Let r_n, $n \in \mathbf{Z}$, be a sequence satisfying $0 < r_n < r_{n+1} < 1$ $(n \in \mathbf{Z})$, $\lim_{n \to \infty} r_n = 1$, $\lim_{n \to -\infty} r_n = 0$. For each $n \in \mathbf{Z}$ let Γ_n be the circle of radius $(r_{n+1} - r_n)/2$ with center $(r_{n+1} + r_n)/2$. By Morera's theorem, $\mathfrak{G} = \{s\Gamma_n: |s| = 1, n \in \mathbf{Z}\}$ satisfies (a)(i) so, by Theorem 2, $\mathfrak{G}$ is a minimal rotation invariant family satisfying (b).

Example 3. Let r_n, $n \geqslant 2$, be a strictly increasing sequence, $r_2 = 1/2$, $\lim_{n \to \infty} r_n = 1$. Let $\Gamma_1 = \{z: |z - 1/4| = 1/4\}$, $\Gamma_2 = \{z: |z - 1/4| = 1/8\}$, and for each $n > 2$, let Γ_n be the circle of radius $(r_n - r_{n-1})/2$ with center $(r_n + r_{n-1})/2$. Again, by Theorem 2, $\mathfrak{G} = \{s\Gamma_n: |s| = 1, n \in \mathbf{N}\}$ is a minimal rotation invariant family satisfying (b).

Our next example shows that in general (a) and (b) in Theorem 2 are not equivalent.

EXAMPLE 4. Let D be the convex hull of

$$\{z\colon |4z - 3e^{i\pi/4}| < 1\} \cup \{z\colon |4z - 3e^{-i\pi/4}| < 1\} \cup \{re^{i\varphi}\colon 0 \leqslant r < 1, \pi/4 \leqslant |\varphi| \leqslant \pi\}.$$

Then $\Gamma = \partial D$ is smooth and satisfies $\Gamma = \Gamma^*$. Let $\Omega = \{sz\colon z \in \Gamma, |s| = 1\}$ and let f be a continuous function on Ω such that for every s, $|s| = 1$, the function $f|(s\Gamma)$ has a continuous extension to $s\overline{D}$ which is analytic in sD. Fix s, $|s| = 1$. We show that f is analytic in $(sD) \cap \operatorname{Int}\Omega$ by showing that, in $(sD) \cap \operatorname{Int}\Omega$, f coincides with the continuous extension Φ of $f|(s\Gamma)$ to $s\overline{D}$ which is analytic in sD. Fix a point $z \in (sD) \cap \operatorname{Int}\Omega$. By the definition of Γ there is an s_1, $|s_1| = 1$, such that $z \in s_1\Gamma$ and such that $(s_1\Gamma) \cap (s\Gamma)$ contains an arc. Consequently, if Φ_1 is the continuous extension of $f|(s_1\Gamma)$ to $s_1\overline{D}$ which is analytic in s_1D then $\Phi_1 = \Phi$ on $(s\overline{D}) \cap (s_1\overline{D})$ so $f(z) = \Phi_1(z) = \Phi(z)$. This proves that for any s, $|s| = 1$, f is analytic in $(sD) \cap \operatorname{Int}\Omega$, so f is analytic in $\operatorname{Int}\Omega$. This shows that in Theorem 2, (b) does not imply (a)(ii).

Example 4 also shows that, for some curves, Γ one can drop the assumption that $0 \notin \overline{D}$ in Theorem 1. By Example 1 one cannot do this for circles so it is natural question whether the circles are the only curves Γ for which one cannot drop the assumption that $0 \notin \overline{D}$ in Theorem 1. There are other curves having this property:

EXAMPLE 5. Let Γ be a smooth curve which can be parametrized as $z = r(\varphi)e^{i\varphi}$ $(-\pi \leqslant \varphi \leqslant \pi)$ where $\varphi \mapsto r(\varphi)$ is strictly increasing and positive on $[0, \pi]$ and satisfies $r(\varphi) = r(-\varphi)$ $(0 \leqslant \varphi \leqslant \pi)$. Clearly $\Gamma = \Gamma^*$ and if $z \in \Gamma$ then $\bar{z} \in \Gamma$ and there are no other points $w \in \Gamma$ satisfying $|w| = |z|$. Denote by D the simply connected domain bounded by Γ and let $\psi\colon \overline{D} \to \overline{\Delta}$ be a continuous map which maps D bianalytically onto Δ and which satisfies $\psi(0) = 0$, $\psi(r(0)) = 1$. Then by symmetry, $\psi(\bar{z}) = \overline{\psi(z)}$ $(z \in \Gamma)$. Define the function Φ on Γ by $\Phi(z) = \psi(z) + 1/\psi(z)$. Since

$$\begin{aligned}\Phi(\bar{z}) &= \psi(\bar{z}) + 1/\psi(\bar{z}) = \overline{\psi(z)} + 1/\overline{\psi(z)} \\ &= 1/\psi(z) + \psi(z) = \Phi(z) \qquad (z \in \Gamma),\end{aligned}$$

it follows by the properties of Γ that one can define a continuous function F on $\Omega = \{sz\colon z \in \Gamma, |s| = 1\}$ by $F(|z|) = \Phi(z)$ $(z \in \Gamma)$. Define f on Ω by $f(z) = zF(|z|)$ $(z \in \Omega)$. Fix s, $|s| = 1$. We have $f(sz) = szF(|z|) = sz[\psi(z) + 1/\psi(z)]$ $(z \in \Gamma)$. Since ψ has a single zero at 0 the function $z \mapsto sz[\psi(z) + 1/\psi(z)]$ has a continuous extension from Γ to $\overline{D}$ which is holomorphic in D. Consequently, f is a continuous function on Ω such that for every s, $|s| = 1$, the function $f|(s\Gamma)$ has a continuous extension to $s\overline{D}$ which is analytic in sD, yet f is not analytic in the interior of Ω.

Examples 1, 4 and 5 indicate that there may be no simple characterization of the curves Γ for which one can drop the assumption that $0 \notin \overline{D}$ in Theorem 1 and therefore that there may be no simple characterization of rotation invariant, symmetric families $\mathfrak{G}$ which satisfy (b) in Theorem 2.

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