

A TOPOLOGICAL GROUP HAVING NO HOMEOMORPHISMS OTHER THAN TRANSLATIONS

BY
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ABSTRACT. We give an example of a (separable metric) connected and locally connected topological group, the only autohomeomorphisms of which are group translations.

1. Introduction. The aim of this paper is to present an example of a connected and locally connected topological group G having no homeomorphisms other than group translations. It turns out that G is *uniquely homogeneous*, i.e. for any pair of points $x, y \in G$ there is a unique homeomorphism taking x onto y . Our results were motivated, on the one hand, by the result due to Barit and Renaud [2] that a uniquely homogeneous space cannot be compact, or locally compact and locally connected, and, on the other hand, by the recent result due to Dobrowolski and Toruńczyk [5] that a topologically complete ANR group is either a Lie group or a Hilbert space manifold and consequently these groups have many homeomorphisms other than those derived from their underlying algebraic structure.

Our example answers many natural questions in the negative. To my knowledge, it is the first example of a uniquely homogeneous space. In addition, it is an example of a homogeneous topological space the autohomeomorphism group of which is Abelian. We will derive many other strange properties of G . For example, if $x \in G$ then the space $G \setminus \{x\}$ is rigid, i.e. has no autohomeomorphisms other than the identity.

We use a method originally due to Kuratowski [14] which was later rediscovered or used by many authors, see e.g. [6, 7, 9, 10, 16–20]. Some specific ideas needed to apply Kuratowski's technique in the present construction are implicit in [16–18].

2. Preliminaries. *Space* means separable metric topological space. A space is called topologically complete provided that there is a complete metric on the space that generates the topology. The following classical results will be important in our construction.

2.1. LEMMA. (a) (*Laurentieff* [15]) *Let X be a space and let Y be topologically complete. If $A \subset X$ and if $f: A \rightarrow Y$ is continuous, then there is a G_δ -subset $A' \subset X$ such that f can be extended to a map $\bar{f}: A' \rightarrow Y$.*

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(b) (*Sierpiński* [21]) *If X is a continuum and if \mathcal{Q} is a partition of X in countably many closed sets, then at most one element of \mathcal{Q} is nonempty.*

(c) (*Hausdorff* [13]) *If X is topologically complete and uncountable, then X contains a Cantor set. A topologically complete space is therefore either countable or has cardinality continuum.*

A cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. c denotes 2^{\aleph_0} .

The domain and range of a function f will be denoted by $\text{dom}(f)$ and $\text{range}(f)$, respectively. Henceforth, all functions will be continuous. Observe that the cardinality of the collection of all G_δ -subsets of a given space is at most c . This implies that if X and Y are spaces, then the collection

$$\mathfrak{F} = \{f: \text{dom}(f) \text{ is a } G_\delta\text{-subset of } X \text{ and } \text{range}(f) \subset Y\}$$

has cardinality at most c .

If G is a group and if $A \subset G$ then $\langle A \rangle$ denotes the subgroup of G generated by A . A group is called *Boolean* provided that each element of the group has order at most 2. Observe that a Boolean group is Abelian. If G is an Abelian group then we let “+” denote the operation on G . Let G be a Boolean group and let $H \subset G$ be a subgroup. It is easy to verify that for each $x \in G$ the following formula holds:

$$\langle H \cup \{x\} \rangle = H \cup (x + H).$$

If A and B are sets, then $A \Delta B$ denotes their symmetric difference, i.e. $A \Delta B = (A \setminus B) \cup (B \setminus A)$. As usual, λ denotes Lebesgue measure on $[0, 1]$. Let \mathfrak{M} denote the family of all Lebesgue measurable subsets of $[0, 1]$ and let \mathfrak{N} be the ideal of null-sets. The quotient algebra $\mathfrak{M}/\mathfrak{N}$ will be denoted by \mathfrak{G} . If $A \in \mathfrak{M}$ then $[A]$ denotes the \mathfrak{N} -equivalence class of A . Metrize \mathfrak{G} by $d([A], [B]) = \lambda(A \Delta B)$. As is well known, d is a complete metric on \mathfrak{G} [12, Exercise 40.1] which, in addition, is convex [12, Exercise 40.8]. This easily implies that \mathfrak{G} is path-connected. Since obviously $\mathfrak{G} \times \mathfrak{G} \approx \mathfrak{G}$, we conclude that for any pair of points $x, y \in \mathfrak{G}$ there is a disc $D \subset \mathfrak{G}$, i.e. a topological copy of $[0, 1] \times [0, 1]$, containing both x and y . This will be of crucial importance later on. Bessaga and Pelczyński [3, VI, 7.2] have shown that \mathfrak{G} is homeomorphic to the separable Hilbert space l_2 , so using their result the above property of \mathfrak{G} becomes a triviality. If we define an operation $+: \mathfrak{G} \times \mathfrak{G} \rightarrow \mathfrak{G}$ by $[A] + [B] = [A \Delta B]$, then \mathfrak{G} with this operation becomes a Boolean topological group [12, 167]. We have given a well-known example of a topological group G satisfying the following properties:

- (1) G is topologically complete.
- (2) for all $x, y \in G$ there is a disc $D \subset G$ with $x, y \in D$.
- (3) G is Boolean.

3. The construction. We will now present the main result in this paper.

3.1. THEOREM. *Let G be a topological group which is topologically complete and Boolean and which, moreover, has the property that any pair of points $x, y \in G$ is contained in a disc $D \subset G$. Then G contains a dense connected subgroup H such that*

each autohomeomorphism of H is a translation. Moreover, if G has the property that for all open $U \subset G$ and $x \in U$, there is an open $V \subset U$ with $x \in V \subset U$ such that any pair of points of V is contained in a disc $D \subset U$, then H is locally connected.

Since the proof of this result is rather lengthy, we break it up in several pieces.

3.2. *Construction of H .* Let $\mathcal{F} = \{f: \text{dom}(f) \text{ is a } G_\delta\text{-subset of } G \text{ and } \text{range}(f) \subset G\}$. Since, as was observed in §2, $|\mathcal{F}| \leq c$, we can enumerate \mathcal{F} by $\{f_\alpha: \alpha < c, \alpha \text{ even}\}$. We choose the enumeration such that each $f \in \mathcal{F}$ is listed c times. It is clear that this is possible. Let $\{K_\alpha: \alpha < c, \alpha \text{ odd}\}$ enumerate all Cantor sets in G . By transfinite induction, for every $\alpha < c$ we will construct subgroups $H_\alpha \subset G$ and subsets $V_\alpha \subset G$ such that

- (1) if $\beta < \alpha$ then $H_\beta \subset H_\alpha, V_\beta \subset V_\alpha$, and $H_\alpha \cap V_\alpha = \emptyset$,
- (2) $|H_\alpha| \leq |\alpha| \cdot \aleph_0$ and $|V_\alpha| \leq |\alpha| \cdot \aleph_0$,
- (3) if α is odd then $H_\alpha \cap K_\alpha \neq \emptyset$,
- (4) if α is even and if $|\{x \in \text{dom}(f_\alpha): f_\alpha(x) \notin \langle \bigcup_{\beta < \alpha} H_\beta \cup \{x\} \rangle}| = c$, then there is a point $x \in \text{dom}(f_\alpha) \cap (H_\alpha \setminus \bigcup_{\beta < \alpha} H_\beta)$ such that $f_\alpha(x) \in V_\alpha$.

Suppose that we have completed the construction for all $\beta < \alpha$, where $\alpha < c$.

Case 1. α is odd.

For convenience, put $H^\alpha = \bigcup_{\beta < \alpha} H_\beta$ and $V^\alpha = \bigcup_{\beta < \alpha} V_\beta$. Observe that $|H^\alpha| \leq |\alpha| \cdot \aleph_0 < c$ and, similarly, that $|V^\alpha| \leq |\alpha| \cdot \aleph_0 < c$. Since $|K_\alpha| = c$, we can pick a point

$$x \in K_\alpha \setminus (H^\alpha + V^\alpha).$$

Define $H_\alpha = \langle H^\alpha \cup \{x\} \rangle$ and $V_\alpha = V^\alpha$. An easy check shows that H_α and V_α are as required.

Case 2. α is even and if $S = \{x \in \text{dom}(f_\alpha): f_\alpha(x) \notin \langle H^\alpha \cup \{x\} \rangle\}$ then $|S| < c$.

Define $H_\alpha = H^\alpha$ and $V_\alpha = V^\alpha$.

Case 3. α is even and $|S| = c$.

By the same argument as in Case 1, we can find a point

$$x \in S \setminus ((H^\alpha + V^\alpha) \cup H^\alpha).$$

Define $H_\alpha = \langle H^\alpha \cup \{x\} \rangle$ and $V_\alpha = V^\alpha \cup \{f_\alpha(x)\}$. Since

$$H_\alpha \cap V_\alpha = (H^\alpha \cup (x + H^\alpha)) \cap (V^\alpha \cup \{f_\alpha(x)\}) = \emptyset,$$

we see that H_α and V_α are as required.

Now put $H = \bigcup_{\alpha < c} H_\alpha$. We claim that H is as required. For later use we prove a statement which is slightly more general than the statement we claimed to be true for H .

3.3. Let $A \subset H$ be countable and let $f: H \setminus A \rightarrow H$ be a map with the property that each point-inverse is at most countable. Then there is a translation $h: G \rightarrow G$ with $h|_{H \setminus A} = f$.

By Lemma 2.1(a), let S be a G_δ -subset of G such that f can be extended to a map $\bar{f}: S \rightarrow H$. Observe that $\{\alpha < c: \bar{f} = f_\alpha\}$ has cardinality c .

Case 1. For each α such that $\bar{f} = f_\alpha, |\{x \in S: f_\alpha(x) \notin \langle \bigcup_{\beta < \alpha} H_\beta \cup \{x\} \rangle}| = c$. Then by (4), there is a set $B \subset \text{dom}(\bar{f}) \cap H$ of cardinality c such that $f(B) \cap H = \emptyset$.

Take a point $x \in B \setminus A$. Then $x \in H \setminus A$ and since \bar{f} extends f , we conclude that $f(x) \notin H$, which is a contradiction.

Case 2. Not Case 1. Take $\alpha < c$ such that $\bar{f} = f_\alpha$ and

$$\left| \left\{ x \in S : f_\alpha(x) \notin \left\langle \bigcup_{\beta < \alpha} H_\beta \cup \{x\} \right\rangle \right\} \right| < c.$$

Let $H^\alpha = \bigcup_{\beta < \alpha} H_\beta$ and $T = \{x \in S : f_\alpha(x) \notin (H^\alpha + (x + H^\alpha))\}$. By assumption, $|T| < c$. Take $h \in H^\alpha$ and assume that $f_\alpha^{-1}(h)$ is uncountable. Since $f_\alpha^{-1}(h)$ is a closed subspace of the space S , which is a G_δ -subset of G , $f_\alpha^{-1}(h)$ contains a family of c pairwise disjoint Cantor sets (Lemma 2.1(c)). From (3) we conclude that $|f_\alpha^{-1}(h) \cap H| = c$, whence $|f_\alpha^{-1}(h) \cap (H \setminus A)| = c$. Since f_α extends f , this contradicts our assumptions on f . We conclude that

$$(*) \quad |f_\alpha^{-1}(H^\alpha)| \leq |H^\alpha| \cdot \aleph_0 \leq |\alpha| \cdot \aleph_0 \cdot \aleph_0 < c.$$

For each $h \in H^\alpha$ define

$$E_h = \{x \in S : f_\alpha(x) = x + h\}.$$

Observe that each E_h is closed in S and that the collection $\{E_h : h \in H^\alpha\}$ is pairwise disjoint. If $h \in H^\alpha$ define $F_h = E_h \setminus f_\alpha^{-1}(H^\alpha)$. We claim that at most one of the collection $\{F_h : h \in H^\alpha\}$ is nonempty. Suppose, to the contrary, that there exist distinct points $s, t \in H^\alpha$ such that $F_s \neq \emptyset$ and $F_t \neq \emptyset$. Pick points $x \in F_s$ and $y \in F_t$. By assumption, there is a disc $D \subset G$ containing both x and y . Since S is a G_δ , the complement $G \setminus S$ of S is the union of countably many closed subsets of G , none of which can contain a Cantor set by (3). We conclude that $G \setminus S$ is countable, Lemma 2.1(c). Therefore, $|f_\alpha^{-1}(H^\alpha) \cup (G \setminus S)| < c$ which implies that we can find an arc $J \subset D$ connecting x and y such that

$$J \cap (f_\alpha^{-1}(H^\alpha) \cup (G \setminus S) \cup T) = \emptyset,$$

whence

$$J \subset \bigcup_{h \in H^\alpha} F_h \subset \bigcup_{h \in H^\alpha} E_h.$$

Put $K = \{h \in H^\alpha : F_h \cap J \neq \emptyset\}$. By assumption, $|K| \geq 2$, whence, by Sierpiński's Lemma (Lemma 2.1(b)), $|K| > \aleph_0$. Also, $|K| \leq |H^\alpha| < c$ (so in case the Continuum Hypothesis holds, at this stage of the proof we have already derived a contradiction). We conclude that K cannot be topologically complete, Lemma 2.1(c), and that, in particular, K is not closed in G . Let $k_n \in K$ ($n = 1, 2, \dots$) be a sequence of points of K converging to a point $k \notin K$. For each n , take a point $x_n \in J \cap F_{k_n}$. By compactness of J , we may assume that $\lim_{n \rightarrow \infty} x_n = x \in J$. Then

$$f_\alpha(x) = \lim_{n \rightarrow \infty} f_\alpha(x_n) = \lim_{n \rightarrow \infty} (x_n + k_n) = x + k,$$

since $\lim_{n \rightarrow \infty} x_n = x$, $\lim_{n \rightarrow \infty} k_n = k$ and “+” is continuous. Since $x \in J$, there is an $h \in K$ such that $x \in F_h \subset E_h$. Consequently, $f_\alpha(x) = x + h$, which implies that $k = h \in K$, which is a contradiction.

We conclude that at most one of the collection $\{F_h: h \in H^\alpha\}$ is nonempty. If each F_h is empty, then $f_\alpha(H \setminus T) \subset H^\alpha$ and consequently, $|H| < c$, by (*), which obviously contradicts (3). Therefore, there is a unique point $h \in H^\alpha$ such that $F_h \neq \emptyset$. Suppose that $E_h \neq S$. Then $S \setminus E_h$ is nonempty and open in S . Since $|G \setminus S| \leq \aleph_0$ and since each nonempty open subspace of G has cardinality c (G is path-connected), this implies that $|S \setminus E_h| = c$, which contradicts (*). Consequently, $E_h = S$ which completes the proof of 3.3.

3.4. *H is connected. Moreover, if G has the property that for all open $U \subset G$ and $x \in U$, there is an open $V \subset U$ with $x \in V \subset U$ such that any pair of points of V is contained in a disc $D \subset U$, then H is locally connected.*

Suppose, to the contrary, that H is not connected, i.e. $H = U \cup V$, where U and V are disjoint nonempty open subsets of H . Since H is dense in G , we can find disjoint open sets $U', V' \subset G$ with $U' \cap G = U$ and $V' \cap H = V$. Then $K = G \setminus (U' \cup V')$ is closed and therefore has to be countable since it misses H (H intersects every Cantor set in G). Take $x \in U'$ and $y \in V'$. By assumption, there is a disc $D \subset G$ containing both x and y . Then $K \cap D$ separates D and since no countable set separates D , this is a contradiction.

To prove that H is locally connected under the given hypotheses, first observe that G is locally connected. We will show that if $W \subset G$ is open and connected, then $W \cap H$ is connected, which proves, among others, that H is locally connected. Let $W \subset G$ be open and connected, and suppose that $W \cap H$ is not connected, i.e. $W \cap H = W_0 \cup W_1$, where W_0 and W_1 are disjoint nonempty open subsets of H . As above, find disjoint open subsets $W'_0, W'_1 \subset W$ such that $W'_i \cap H = W_i$ ($i = 0, 1$). Put $L = W \setminus (W'_0 \cup W'_1)$. As above, L is countable. Since $H \cap W$ is dense in W , and since W is connected, L is nonempty. In fact, there is even a point $x \in L$ which is a limit point both from W'_0 and W'_1 . By assumption, there is a neighborhood V of x which is contained in W such that any pair of points of V is contained in a disc $D \subset W$. Since V intersects both W'_0 and W'_1 , we may, therefore, conclude that there exists a disc $D \subset W$ such that D intersects both W'_0 and W'_1 . Since L is countable, this is a contradiction.

This completes the proof of the theorem.

3.5. **REMARK.** From now on, H will denote the subgroup of \mathcal{G} that we get from the proof of Theorem 3.1. Since $\mathcal{G} \approx I_2$, or by a direct argument, \mathcal{G} satisfies all conditions required in Theorem 3.1. We therefore conclude that H is a Boolean group which is connected and locally connected and which has the property that all autohomeomorphisms are translations.

3.6. **REMARK.** In fact, the assumption on G that any pair of points is contained in a disc $D \subset G$ can be weakened considerably. The reader can easily check that the only thing we need for the proof, is the following:

for all $x, y \in G$ there is a family \mathcal{K} of c subcontinua of G each containing x and y such that if K_0 and $K_1 \in \mathcal{K}$ are distinct, then

$$(K_0 \cap K_1) \setminus \{x, y\} = \emptyset.$$

4. A result under the Continuum Hypothesis. We have seen that there are topological groups having no homeomorphisms other than translations. Under the Continuum Hypothesis (abbreviated CH) we can do better. The groups we constructed then have the property that every self-map is either constant or a translation. We will sketch the proof. Our notation and terminology is the same as in the proof of Theorem 3.1. Let $f: H \rightarrow H$ be a self-map, and let S be a G_δ -subset of G such that f can be extended to a map $\tilde{f}: S \rightarrow \mathcal{G}$. Fix $\alpha < \mathfrak{c}$ such that $\tilde{f} = f_\alpha$. Just as in the proof of Theorem 3.1, we find that if $T = \{x \in S: f_\alpha(x) \notin (H^\alpha \cup (x + H^\alpha))\}$ then $|T| < \mathfrak{c}$, whence by CH, $|T| \leq \aleph_0$. For each $h \in H^\alpha$ define

$$A_h = \{x \in S: f_\alpha(x) = h\} \quad \text{and} \quad B_h = \{x \in S: f_\alpha(x) = x + h\},$$

respectively. Observe that if $h, g \in H^\alpha$ are distinct then

$$(1) \quad A_h \cap A_g = \emptyset, \quad B_h \cap B_g = \emptyset \quad \text{and}$$

$$(2) \quad A_h \cap B_g \subset H^\alpha.$$

If $h \in H^\alpha$, define $A'_h = A_h \setminus H^\alpha$ and $B'_h = B_h \setminus H^\alpha$. Then the collection

$$\mathcal{C} = \{A'_h: h \in H^\alpha\} \cup \{B'_h: h \in H^\alpha\}$$

is pairwise disjoint and

$$\left| G \setminus \bigcup_{h \in H^\alpha} (A'_h \cup B'_h) \right| \leq |G \setminus S| + |T| + |H^\alpha| \leq \aleph_0.$$

Consequently, by using Sierpiński's Lemma 2.1(b), we can conclude by precisely the same construction as in the proof of Theorem 3.1 that exactly one of the collection \mathcal{C} is nonempty. If $A'_h \neq \emptyset$, then f is constant and if $B'_h \neq \emptyset$ then f is a translation.

We have completed the proof of the following

4.1. THEOREM (CH). *There is a connected and locally connected topological group H such that each self-map $f: H \rightarrow H$ (not necessarily onto) is either constant or a translation.*

We do not know whether the above theorem is true without the assumption of the CH. Notice that for the case of homeomorphisms we found a trick which was used to avoid the CH. We have tried for some time to find a similar trick for self-maps, but although we believe that there should be one handling this more general case, we did not succeed.

4.2. Question. *Does there exist a homogeneous continuum X such that each self-map $f: X \rightarrow X$ is either constant or a homeomorphism?*

5. Unique homogeneity. Burgess [4] asked whether uniquely homogeneous continua exist and Ungar [22], using deep results of Effros [8], showed that if X is uniquely homogeneous and either compact or locally compact and locally connected, then X is an Abelian topological group, and the group of topological automorphisms of X is trivial. Barit and Renaud [2] in their paper, *There are no uniquely homogeneous spaces*, show that if G is a locally compact topological group with $\text{Aut}(G) = \{\text{id}\}$, then $G = \{0\}$ or $G = \mathbb{Z}_2$, thus answering Burgess' question in the negative. Since the examples we get from Theorem 3.1 are Boolean groups, it easily follows

that they are uniquely homogeneous. This shows that the title of the Barit and Renaud paper is somewhat misleading. In addition, it also shows that in their result the local compactness is essential, since our group H has no nontrivial topological automorphisms. In addition, since H intersects all Cantor sets in \mathcal{G} , it easily follows that H is Baire. We therefore conclude that on the one hand, compact or locally compact and locally connected uniquely homogeneous spaces do not exist, but on the other hand, Baire uniquely homogeneous spaces do exist. This suggests the following question.

5.1. *Question. Does there exist a uniquely homogeneous topologically complete space (containing more than one point)?*

All examples we have of uniquely homogeneous spaces are topological groups. The way we constructed these examples shows that we cannot avoid this. This suggests the following question.

5.2. *Question. Does there exist a uniquely homogeneous space that does not admit the structure of a topological group?*

6. Rigid subspaces. We will proceed to show that H has a very curious additional property.

6.1. **THEOREM.** *If $x \in H$, then $H \setminus \{x\}$ is rigid.*

PROOF. Let $h: H \setminus \{x\} \rightarrow H \setminus \{x\}$ be a homeomorphism. By 3.3, there is a translation $g: \mathcal{G} \rightarrow \mathcal{G}$ such that $g|_{H \setminus \{x\}} = h$. Assume that $g(y) = y + a$ for all $y \in \mathcal{G}$. If a is not the identity, then $x + a \neq x$; consequently, $x + a \in H \setminus \{x\}$. Therefore,

$$h(x + a) = g(x + a) = x + a + a = x,$$

which is a contradiction of course. So a must be the identity of \mathcal{G} , whence h is the identity homeomorphism of $H \setminus \{x\}$. \square

The above result shows that there are rigid spaces that can be made into a topological group by adding just one point.

The reader is encouraged to check that under CH, the space $H \setminus \{x\}$ has the property that any self-map is either constant or the identity. Again, we do not know whether CH is essential here.

The reader can easily cook up many more improbable properties of H in the same spirit.

7. Remarks. R. D. Anderson [1] showed that the autohomeomorphism group $\text{Auth}(X)$ of many topological spaces X is algebraically simple. The easiest examples of spaces X for which $\text{Auth}(X)$ is not simple are spaces for which $\text{Auth}(X)$ is Abelian. There are such spaces, since de Groot and Wille [11] showed that each countable group is isomorphic to the autohomeomorphism group of some 1-dimensional Peano continuum: see also de Groot [10]. While discussing this with Eric van Douwen, we observed that we did not know of an example of a *homogeneous* space the autohomeomorphism group of which is Abelian. Our example H is isomorphic to $\text{Auth}(H)$ and consequently $\text{Auth}(H)$ is Abelian.

REMARK (added in February 1983). In a forthcoming paper entitled *A uniquely homogeneous space need not be a topological group* (to appear in *Fund. Math.*), I will answer Question 5.2.

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