

REAL VS. COMPLEX RATIONAL CHEBYSHEV APPROXIMATION ON AN INTERVAL

BY

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ABSTRACT. If $f \in C[-1, 1]$ is real-valued, let $E^r(f)$ and $E^c(f)$ be the errors in best approximation to f in the supremum norm by rational functions of type (m, n) with real and complex coefficients, respectively. It has recently been observed that $E^c(f) < E^r(f)$ can occur for any $n \geq 1$, but for no $n \geq 1$ is it known whether $\gamma_{mn} = \inf_f E^c(f)/E^r(f)$ is zero or strictly positive. Here we show that both are possible: $\gamma_{01} > 0$, but $\gamma_{mn} = 0$ for $n \geq m + 3$. Related results are obtained for approximation on regions in the plane.

1. Introduction. Let I be the unit interval $[-1, 1]$, C^r the set of continuous real functions on I , and $\|\cdot\|$ the supremum norm $\|f\| = \sup_{x \in I} |f(x)|$. For nonnegative integers m and n , let R_{mn} and $R_{mn}^c \subseteq R_{mn}$ be the spaces of rational functions of type (m, n) with coefficients in \mathbf{C} and \mathbf{R} , respectively. For $f \in C^r$, let $E^c(f)$ and $E^r(f)$ denote the infima

$$(1) \quad E^c(f) = \inf_{r \in R_{mn}^c} \|f - r\|, \quad E^r(f) = \inf_{r \in R_{mn}} \|f - r\|.$$

It is known that both limits are attained, and a function that does so is called a *best approximation (BA)* to f . In the real case the BA is unique [8], and in the complex case for $n \geq 1$ in general it is not [7, 10, 11, 14, 15].

Obviously $E^c \leq E^r$ for any f , but since f is real, it is not at first obvious whether a strict inequality can occur. However in 1971 Lungu [7], following a proposal of Goňčar [16], published a class of examples showing that $E^c(f) < E^r(f)$ is indeed possible if $n \geq 1$. Independently, Saff and Varga [10, 11] made the same discovery in 1977, and obtained more general sufficient conditions for $E^c(f) < E^r(f)$ and also a sufficient condition for $E^c(f) = E^r(f)$. The former was later sharpened by Ruttan [18] to the following statement: $E^c(f) < E^r(f)$ must hold if the best real approximation to f attains its maximum error on no alternation set of length greater than $m + n + 1$ points. For a survey of such results, see [14].

But is E^c ever *much* less than E^r ? If γ_{mn} denotes the infimum

$$(2) \quad \gamma_{mn} = \inf_{f \in C^r \setminus R_{mn}^c} E^c(f)/E^r(f),$$

then one would like to know whether γ_{mn} can be zero or is always positive, and if the latter, how small it is. In all of the examples devised to date, $E^c(f)/E^r(f)$ has fallen

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in the range $(\frac{1}{2}, 1]$, suggesting that $\gamma_{mn} = \frac{1}{2}$ might be the minimum value. Saff and Varga posed in particular the question, is γ_{nn} positive or zero [10, 11]? Ellacott has suggested that $\gamma_{mn} = \frac{1}{2}$ may hold for $m \geq n$ [3]. (For more on his argument see §2.) Some partial results for $(m, n) = (1, 1)$ have been obtained by Bennet, et al. [1, 2] and by Ruttan [9].

In this paper we resolve some of these questions, as follows. First, not only can $\gamma_{mn} < \frac{1}{2}$ occur, but $\gamma_{mn} = 0$ for all $m \geq 0, n \geq m + 3$ (Theorem 1). Second, $\gamma_{01} > 0$ (Theorem 2). We conjecture that $\gamma_{mn} > 0$ holds whenever $n < m + 3$. Finally, at least some of our arguments extend to approximation on complex regions, and we show: $\gamma_{0n}^\Delta = 0$ for $n \geq 4$ in approximation on the unit disk Δ (Theorem 3). A similar result is obtained for approximation on a symmetric Jordan region.

2. $\gamma_{mn} = 0$ for $n \geq m + 3$.

THEOREM 1. $\gamma_{mn} = 0$ for all $m \geq 0, n \geq m + 3$.

PROOF. The idea of the construction is indicated in Figure 1, where crosses represent poles and circles represent zeros.

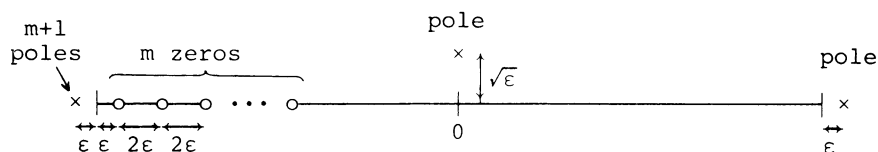


FIGURE 1

Given $m \geq 0$, let $\phi \in R_{m, n+3}$ be defined by

$$(3) \quad \phi(x) = \frac{\epsilon \prod_{j=1}^m [(-1 + (2j-1)\epsilon) - x]}{[x + (1 + \epsilon)]^{m+1} [i\sqrt{\epsilon} - x] [(1 + \epsilon) - x]}$$

and as the function in C^r to be approximated take $f(x) = \operatorname{Re} \phi(x)$. We will show that f has the following two properties:

(a) $\|f - \phi\| = \|\operatorname{Im} \phi\| = O(\sqrt{\epsilon})$ as $\epsilon \rightarrow 0$.

(b) There exists a constant $C > 0$ such that for all sufficiently small ϵ ,

$$(4) \quad (-1)^j f(-1 + 2j\epsilon) \geq C, \quad 0 \leq j \leq m,$$

and

$$(5) \quad (-1)^{m+1} f(1) \geq C.$$

Condition (b) states that the error function for the zero approximation to f approximately equioscillates at $m + 2$ points, and by the de la Vallée Poussin theorem for real rational approximation [8, Theorem 98], this implies $E^r \geq C$. (For the purposes of this theorem $r \equiv 0$ has rational type $(\mu, \nu) = (-\infty, 0)$, so the "defect" $d = \min\{m - \mu, n - \nu\}$ is n , which means one needs approximate equioscillation at $m + n + 2 - d = m + 2$ points.) On the other hand if $n \geq m + 3$, then $\phi \in R_{mn}$, so (a) implies $E^c = O(\sqrt{\epsilon})$. Thus since ϵ can be arbitrarily small, the theorem will be proved once (a) and (b) are established.

PROOF OF (a). Let us write ϕ as a product of three functions ϕ_1, ϕ_2, ϕ_3 corresponding to the poles and zeros near $-1, 0,$ and $1,$ respectively. Of these functions only ϕ_2 has a nonzero imaginary part on $I,$ and we bring this into the numerator. The factor ϕ_1 gets the constant ϵ from (3):

$$(6) \quad \phi(x) = \phi_1(x)\phi_2(x)\phi_3(x) \\ = \left(\frac{\epsilon \prod_{j=1}^m [(-1 + (2j - 1)\epsilon) - x]}{[x + (1 + \epsilon)]^{m+1}} \right) \left(\frac{-i\sqrt{\epsilon} - x}{x^2 + \epsilon} \right) \left(\frac{1}{(1 + \epsilon) - x} \right).$$

Since $(f - \phi)(x) = -i \operatorname{Im} \phi(x),$ we compute

$$(f - \phi)(x) = -i\phi_1(x)\operatorname{Im} \phi_2(x)\phi_3(x) = \phi_1(x) \frac{i\sqrt{\epsilon}}{x^2 + \epsilon} \phi_3(x).$$

It is not hard to see that on $[-1, -\frac{1}{2}]$ these factors have magnitude $O(1), O(\sqrt{\epsilon}),$ and $O(1),$ so their product is $O(\sqrt{\epsilon}).$ Similarly in $[-\frac{1}{2}, \frac{1}{2}]$ one has $O(\epsilon)O(1/\sqrt{\epsilon})O(1) = O(\sqrt{\epsilon}),$ and in $[\frac{1}{2}, 1], O(\epsilon)O(\sqrt{\epsilon})O(1/\epsilon) = O(\sqrt{\epsilon}).$ Together these estimates give $(f - \phi)(x) = O(\sqrt{\epsilon})$ for all $x \in I,$ as claimed.

PROOF OF (b). Again we use the factorization $\phi = \phi_1\phi_2\phi_3$ of (6). Let $\{x_j\}_{j=0}^m$ be the set of points $x_j = -1 + 2j\epsilon$ that appear in condition (4). At each $x_j,$ ϕ_1 evidently takes the form $\alpha_j\epsilon^{m+1}/\beta_j\epsilon^{m+1}$ for some constants α_j and $\beta_j,$ and thus $\phi_1(x_j)$ is independent of $\epsilon.$ Moreover these quantities obviously alternate in sign, i.e.

$$\phi_1(x_0) = \tau_0 > 0, -\phi_1(x_1) = \tau_1 > 0, \dots, (-1)^m \phi_1(x_m) = \tau_m > 0,$$

with τ_j independent of $\epsilon.$ In addition since all of the points x_j are contained in $[-1, -1 + 2m\epsilon],$ we have $\phi_2(x_j) = 1 + O(\sqrt{\epsilon}), \phi_3(x_j) = \frac{1}{2} + O(\epsilon)$ on $\{x_j\}.$ Together these facts establish (4) for some $C = C_1 > 0.$

For condition (5) we compute

$$\phi(1) = \phi_1(1)\phi_2(1)\phi_3(1) \\ = \left(\frac{\epsilon}{2} (-1)^m (1 + O(\epsilon)) \right) (-1 + O(\sqrt{\epsilon})) \frac{1}{\epsilon} = \frac{1}{2} (-1)^{m+1} + O(\sqrt{\epsilon}),$$

which implies that (5) holds for $C = C_2$ with any $C_2 < \frac{1}{2}.$ Taking $C = \min\{C_1, C_2\}$ now yields (b). \square

REMARK ON AN ARGUMENT OF ELLACOTT. As alluded to in the Introduction, Ellacott has observed that one can conclude from the *CF method* [13, 4] that if p is a polynomial of degree $m + 1,$ then

$$(7) \quad E^c(p)/E^r(p) \geq \frac{1}{2}$$

for $n \leq m$ [3]. This is one of his arguments for suggesting that $\gamma_{mn} = \frac{1}{2}$ or at least $\gamma_{mn} > 0$ may hold for $n \leq m.$ However we claim that (7) is valid in fact for all $n \leq 2m + 1,$ which by Theorem 1 means that it holds even in many cases with $\gamma_{mn} = 0.$ Therefore although Ellacott's conjecture is plausible, it appears that (7) does not provide very strong support for it.

To demonstrate that (7) holds for $n \leq 2m + 1$, let p be transplanted to the unit circle by defining a function \hat{p} for $z \in \mathbb{C}$ as follows:

$$x = \frac{1}{2}(z + z^{-1}), \quad \hat{p}(z) = p(x) = p\left(\frac{1}{2}z + \frac{1}{2}z^{-1}\right) = \sum_{k=-m-1}^{m+1} \alpha_k z^k.$$

For $n \leq 2m + 1$, the BA to p in R_{mn}^r on I was obtained explicitly by Talbot [12, 5], and its deviation from p is

$$(8) \quad E^r(p) = 2\sigma_n,$$

where σ_n is the smallest singular value of the $(n + 1) \times (n + 1)$ Hankel matrix $(\alpha_{m-n+1+i+j})_{i,j=0}^n$. On the other hand if $r \in R_{mn}$ is any complex approximation to p on I , consider the transplanted function \hat{r} defined by $\hat{r}(z) = r(x)$. It is readily verified that \hat{r} has $\nu \leq n$ poles in $1 < |z| < \infty$ and is of order $O(z^{m-\nu})$ at ∞ . Therefore \hat{r} lies in the space \tilde{R}_{mn} defined in [13, 4], and by the theory given there this implies

$$\sigma_n \leq \sup_{|z|=1} |(\hat{p} - \hat{r})(z)| = \sup_{|x|=1} |(p - r)(x)|.$$

Thus

$$(9) \quad E^c(p) \geq \sigma_n,$$

which together with (8), establishes (7).

By applying [4, Lemma 5.1 in Part II] (7) can be seen to hold even for some rational functions f , namely for those of exact type (M, N) where either $M \leq m + 1$, $N = n + 1$, $n \leq m$ or $M = m + 1$, $N \leq n + 1$, $n \leq 2m + 1 - N$; details will be given in [5].

3. $\gamma_{01} > 0$.

THEOREM 2. $\gamma_{01} > 0$.

PROOF. Let $f \in C^r$ be arbitrary, and let c^* be a BA to f in R_{mn} . Then for any $r \in R_{mn}^r$ one has $\|\text{Im } c^*\| \leq \|f - c^*\| = E^c(f)$ and $E^r(f) \leq E^c(f) + \|c^* - r\|$, and therefore

$$(10) \quad E^r(f) \leq E^c(f) + \|\text{Im } c^*\| \frac{\|c^* - r\|}{\|\text{Im } c^*\|} \leq E^c(f) \left(1 + \frac{\|c^* - r\|}{\|\text{Im } c^*\|}\right).$$

Now suppose that for any $c \in R_{mn} \setminus R_{mn}^r$ with no poles on I , one can find $r^{(c)} \in R_{mn}^r$ such that

$$(11) \quad \|c - r^{(c)}\| / \|\text{Im } c\| \leq M$$

for some fixed M . Then $r^{(c^*)}$ can be inserted in (10), independent of f , and one obtains $\gamma_{mn} \geq 1/(1 + M)$. Our proof of $\gamma_{01} > 0$ consists of exhibiting a mapping $c \mapsto r^{(c)}$ for the case $(m, n) = (0, 1)$ that satisfies (11).

Thus let $c(z) = a/(1 - z/z_0)$ be given, where z_0 lies in the region $C^0 = \mathbb{C} \cup \{\infty\} \setminus I$. Let $\theta \in (0, \pi/2)$ and $\rho \in (1, \infty)$ be arbitrary fixed constants (say,

$\theta = \pi/4, \rho = 2$). Our choice of $r^{(c)}$ depends on which of four domains A^+, A^-, B, C the pole lies in:

$$\begin{aligned} A^\pm &= \{z \in \mathbb{C} : |\arg(-1 \pm z)| < \theta\}, \\ B &= \{z \in \mathbb{C} - A^+ - A^- : |z| \leq \rho\}, \\ C &= \mathbb{C}^0 - A^+ - A^- - B. \end{aligned}$$

The configuration is indicated in Figure 2.

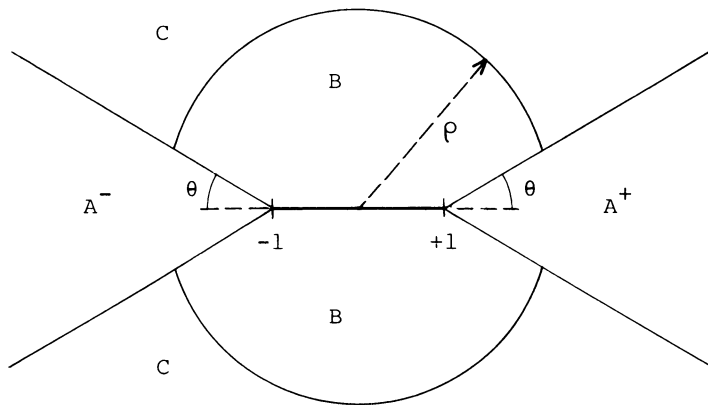


FIGURE 2

We define $r^{(c)}$ as follows:

$$\text{For } z_0 \in A^\pm : r^{(c)}(z) = \frac{1 - 1/|z_0|}{1 \mp z/|z_0|} \operatorname{Re} c(\pm 1).$$

$$\text{For } z_0 \in B : r^{(c)} \equiv 0.$$

$$\text{For } z_0 \in C : r^{(c)} \equiv \operatorname{Re} a.$$

The proof can now be completed by showing that there exist constants M_A, M_B, M_C such that (11) holds for z_0 restricted to each domain $A^+ \cup A^-, B, C$. The global constant M can then be taken as $M = \max\{M_A, M_B, M_C\}$. The algebra involved is unfortunately quite tedious, so we will omit these verifications. However, details of a similar argument for the case of approximation on certain Jordan regions in \mathbb{C} are given in [17]. \square

4. $\gamma_{0n}^\Delta = 0$ for $n \geq 4$.

Let Δ be the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$, and let f be continuous in Δ and analytic in the interior and satisfy $f(\bar{z}) = f(z)$. Let $\|f\|_\Delta$ denote $\sup_{z \in \Delta} |f(z)|$, and define $E^c(f; \Delta), E^r(f; \Delta)$, and γ_{mn}^Δ as in (1) and (2). Until recently it was not even known whether $\gamma_{mn}^\Delta < 1$ is possible, but in a separate paper we show that this inequality holds at least for all pairs (m, n) with $m = 0, n \geq 1$ or $m \geq 0, n = 1$ [6].

By a variation of the argument of §2, we will now prove

THEOREM 3. $\gamma_{0n}^\Delta = 0$ for $n \geq 4$.

PROOF. Let $\zeta = e^{i\theta}$ for some fixed $\theta \in (0, \pi)$, and for any $\varepsilon > 0$, define

$$\phi(z) = \frac{\varepsilon(1 - \zeta)^2}{[z + (1 + \varepsilon)][(1 + \varepsilon) - z][z - (1 + \varepsilon^{1/3})\zeta]^2}$$

and

$$f(z) = \frac{1}{2}(\phi(z) + \overline{\phi(\bar{z})}).$$

In analogy to the proof of Theorem 1, $\gamma_{0n}^\Delta = 0$ for $n \geq 4$ will follow from the properties

- (a) $\|f - \phi\|_\Delta = O(\varepsilon^{1/3})$;
- (b) there exists a constant $C > 0$ such that for all sufficiently small ε , $f(-1) \leq -C$, $f(1) \geq C$.

Both (a) and (b) can be readily derived by observing that the term

$$(1 - \zeta)^2 / [z - (1 + \varepsilon^{1/3})\zeta]^2$$

behaves like $1 + O(\varepsilon^{1/3})$ near $z = 1$ and like $-|(1 - \zeta)/(1 + \zeta)|^2 + O(\varepsilon^{1/3})$ near $z = -1$. We omit the details. \square

This argument can be extended to show $\gamma_{0n}^\Omega = 0$ for $n \geq 4$ for approximation on any Jordan region Ω with $\Omega = \bar{\Omega}$, provided $\partial\Omega$ is differentiable at its two points of intersection with \mathbf{R} , say z_1 and z_2 , hence forms a right angle to \mathbf{R} at these points. Again one introduces a complex double pole, slightly above the point z_1 (analogous to taking $\xi = e^{i\theta}$ with θ small above), and this generates an approximate sign change between $\phi(z_1)$ and $\phi(z_2)$.

One can also prove $\gamma_{01}^\Omega > 0$ for the same class of regions Ω . See [17].

Note added in proof. After studying the present paper, E. Saff has pointed out to us that the existence of arbitrarily small numbers γ_{mn} is implied by a result of Walsh in 1934 [19, Theorem IV], although this consequence was never recognized. Walsh showed that for any $m \geq 0$, the family $\bigcup_{n=0}^\infty R_{mn}$ is dense in $C[I]$ (or indeed in the space of continuous functions on any Jordan arc in \mathbf{C}), so that $\lim_{n \rightarrow \infty} E_{mn}(f) = 0$ for $f \in C[I]$. On the other hand, as we have seen, if f has $m + 1$ zeros, then it cannot be approximated arbitrarily closely in $\bigcup_{n=0}^\infty R_{mn}^r$, i.e. $\lim_{n \rightarrow \infty} E_{mn}^r(f) > 0$. It follows that for any $m \geq 0$, $\lim_{n \rightarrow \infty} \gamma_{mn} = 0$.

REFERENCES

1. C. Bennet, K. Rudnick and J. D. Vaaler, *On a problem of Saff and Varga concerning best rational approximation*, Padé and Rational Approximation (E. B. Saff and R. S. Varga, eds.), Academic Press, New York, 1977, pp. 483–488.
2. ———, *Best uniform approximation by linear fractional transformations*, J. Approximation Theory **25** (1979), 204–224.
3. S. W. Ellacott, *A note on a problem of Saff and Varga concerning the degree of complex rational approximation to real valued functions*, Bull. Amer. Math. Soc. (N. S.) **6** (1982), 218–220.
4. M. H. Gutknecht, *On complex rational approximation*, Computational Aspects of Complex Analysis (H. Werner, et al., eds.), Reidel, Dordrecht, 1983.
5. ———, *Algebraically solvable Chebyshev approximation problems*, Approximation Theory IV (L. Schumaker, ed.), Academic Press, New York (in press).
6. M. H. Gutknecht and L. N. Trefethen, *Nonuniqueness of rational Chebyshev approximations on the unit disk*, J. Approximation Theory (in press).

7. K. N. Lungu, *Best approximation by rational functions*, Mat. Zametki **10** (1971), 11–15. (Russian)
8. G. Meinardus, *Approximation of functions: Theory and numerical methods*, Springer-Verlag, Berlin and New York, 1967.
9. A. Ruttan, *On the cardinality of a set of best complex rational approximations to a real function*, Padé and Rational Approximation (E. B. Saff and R. S. Varga, eds.), Academic Press, New York, 1977, pp. 303–319.
10. E. B. Saff and R. S. Varga, *Nonuniqueness of best approximating complex rational functions*, Bull. Amer. Math. Soc. **83** (1977), 375–377.
11. _____, *Nonuniqueness of best complex rational approximations to real functions on real intervals*, J. Approximation Theory **23** (1978), 78–85.
12. A. Talbot, *On a class of Tchebysheffian approximation problems solvable algebraically*, Proc. Cambridge Philos. Soc. **58** (1962), 244–267.
13. L. N. Trefethen, *Rational Chebyshev approximation on the unit disk*, Numer. Math. **37** (1981), 297–320.
14. R. S. Varga, *Topics in polynomial and rational interpolation and approximation*, Les Presses de l'Université de Montréal, Montréal, 1982.
15. J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloq. Publ., vol. 20, Amer. Math. Soc., Providence, R. I., 1935 (5th ed., 1969).
16. A. A. Gončar, *The rate of approximation by rational fractions and the properties of functions*, Proc. Internat. Congress of Mathematicians (Moscow, 1966), Izdat. "Mir", Moscow, 1968. (Russian)
17. M. H. Gutknecht and L. N. Trefethen, *Real vs. complex rational Chebyshev approximation on complex domains*, Numerische Methoden der Approximationstheorie, vol. 7 (L. Collatz and H. Werner, eds.), Birkhäuser (in press).
18. A. Ruttan, *The length of the alternation set as a factor in determining when a best real rational approximation is also a best complex rational approximation*, J. Approx. Theory **31** (1981), 230–243.
19. J. L. Walsh, *On approximation to an analytic function by rational functions of best approximation*, Math. Z. **38** (1934), 163–176.

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