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STRONG FATOU-1-POINTS OF BLASCHKE PRODUCTS

BY

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ABSTRACT. This paper shows that to every countable set M on the unit circle there corresponds a Blaschke product whose set of strong Fatou-1-points contains M. It also shows that some Blaschke products have an uncountable set of strong Fatou-1-points.

1. Introduction. Let f, ζ and ω denote a complex-valued function in the unit disk D, a point on the unit circle C, and a complex number, possibly the point at infinity. We call ζ a *Fatou-ω-point* of f if in each Stolz angle at ζ the value of $f(z)$ tends to ω as $z \to \zeta$. We call ζ a *strong Fatou-ω-point* if, in addition, $|\omega| = 1$ and f maps each Stolz angle at ζ into a Stolz angle at ω.

The motivation for our investigation arose in a study of the Fatou points of strongly annular functions, that is, of holomorphic functions in D whose minimum modulus on a sequence of concentric circles C_n tends to ∞ as $n \to \infty$. An example of a strongly annular function g having a Fatou-∞-point was known. If a Blaschke product B has a strong Fatou-1-point at each point of a set E on the circle C, and if B maps certain circles with center 0 into sufficiently small neighborhoods of the circles C_n, then the composite function $g \circ B$ is strongly annular and has a Fatou limit ∞ at each point of E.

For the construction of annular functions with many Fatou points, we found a method more appropriate than composition of functions [**1**]; but the problem of strong Fatou-1-points of a Blaschke product appears to be worthy of analysis in its own right.

A holomorphic function f in D with a Fatou-1-point ζ on C has a *finite angular derivative* at ζ if the difference quotient $[f(z) - 1]/[z - \zeta]$ approaches a finite limit as $z \to \zeta$ in Stolz angles. Evidently, if χ is a chord of D with endpoints η and ζ, then the existence of a finite angular derivative of f at ζ implies

$$\arg[f(z) - 1] \to \arg[\eta/\zeta - 1] \quad \text{as } z \to \zeta \text{ along } \chi \tag{1.1}$$

and the convergence is uniform as long as η is bounded away from ζ. On the other hand, even if f is a Blaschke product, it need not have a finite angular derivative at each Fatou-1-point. To see this let $a_1, a_2, \ldots$ be an infinite Blaschke sequence lying on the oricycle $(1 - |z|^2)/|1 - z|^2 = 1$, and let B denote the Blaschke product with

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zeros $a_1, \bar{a}_1, a_2, \bar{a}_2, \ldots$. Because the sum of the terms $(1 - |a_n|^2)/|1 - a_n|^2$ is not finite, it follows from a result of O. Frostman [**4**, p. 177] that B does not have a finite angular derivative at 1. Because $B(z)$ is real on the interval (0, 1), B has the radial limit 1 at 1 if $a_n \to 1$ rapidly. Then, since B shrinks hyperbolic distances, 1 is a strong Fatou-1-point of B.

Corresponding to each holomorphic function f in D, we denote by $S(f)$ the set of strong Fatou-1-points of f and by $S^*(f)$ the subset of $S(f)$ where, in addition, f has a finite angular derivative. §2 of this paper deals with the size of $S^*(f)$ in the case where $|f(z)| < 1$ throughout D. We introduce an extended notion of the finite angular derivative, and we show that to each countable set $\{\zeta_n\}$ on C there corresponds a Blaschke product B having at each point of $\{\zeta_n\}$ a strong Fatou-1-point and a finite angular derivative in the extended sense. If $\{\zeta_n\}$ is of type G_δ we can construct the function B so that $\{\zeta_n\} = S^*(B)$. Considering the function $g(z) = \arg[B(z) - 1]$ and using (1.1), we may apply a result of H. Blumberg [**2**, p. 18] to deduce that the set $S^*(B)$ is countable for every Blaschke product B. By other methods M. Heins proved a theorem (see [**5**, Theorem 8.1]) from which it follows that *if f is a holomorphic mapping of D into D, then f has at most countably many Fatou-*1*-points where the angular derivative exists.* For the sake of completeness we insert a sketch of a slightly different proof of this. At each point ζ of $S^*(f)$ we construct a triod consisting of the line segment with endpoints ζ and 2ζ and of two segments that terminate at ζ and make an angle $3\pi/4$ with the first segment. Since f has a finite angular derivative at each point ζ of $S^*(f)$, the relation $f(D) \subset D$ requires that for each point ζ in $S^*(f)$ there exists a positive number k_ζ such that

$$\frac{f(z) - 1}{z/\zeta - 1} \to k_\zeta$$

as $z \to \zeta$ in a Stolz angle. This implies that near the point ζ the imaginary part of $f(z)$ is positive (negative) on the segment lying to the left (right) of the radius vector of ζ as seen from the origin. Elementary computations show that if none of our segments in D has length greater than $1/2$, then no two left segments and no two right segments intersect in D. If, in addition, we shorten each of the segments in D sufficiently so that the imaginary part of $f(z)$ has constant sign on the segment, then no left segment intersects any of the right segments, and therefore the triods of our system are disjoint. By a theorem of R. L. Moore [**6**], the set $S^*(f)$ is countable.

In §3 we show that if $\{\zeta_n\}$ is a countable set on C and $\{\omega_n\}$ is a sequence of points on C, then some Blaschke product B has for each index n a strong Fatou-ω_n-point at ζ_n. This result gives a partial solution to a problem mentioned by G. T. Cargo [**3**, p. 288]. Our construction involves successive adjustments of the positions of the zeros of B. We hope that our treatment of the technical difficulties will be applicable in other contexts.

In §4 we show that there exists a Blaschke product B with uncountably many strong Fatou-1-points. Explicitly, $B = (G + 1)/(G - 1)$, where

$$G(z) = \int_{-\pi}^{\pi} \frac{z + e^{it}}{z - e^{it}} \, d\mu(t)$$

and $d\mu$ is the standard singular distribution of unit mass on the image M of the classical Cantor set under the obvious mapping of the interval [0, 1] onto the arc of C whose length is 1 and whose midpoint is 1. The set M is precisely the set $S(B)$. The proof makes no appeal to the existence of finite angular derivatives at the points of M (obviously), but rather to the fact that the images under G of the radii terminating at the points of M all lie in a certain wedge in the left half-plane.

2. Equal Fatou limits. By a *boundary domain* at 1 we shall mean a convex domain H whose boundary lies in D except for the point 1. If $\zeta \in C$ and H is a boundary domain, we denote by ζH the image of H under the rotation $z \to \zeta z$. A boundary domain is *tangent* if its boundary is tangent to C. We shall say that the *restricted derivative of* f relative to a boundary domain ζH (briefly: the *H-derivative* of f) exists if for some number w the quotient $[f(z) - w]/[z - \zeta]$ approaches a finite limit as $z \to \zeta$ in ζH. Our discussion in §1 shows that if H is a tangent boundary domain, then the set of Fatou-1-points where an inner function has an H-derivative is at most countable.

THEOREM 1. *Corresponding to each countable set* $\{\zeta_n\}$ *on* C *and each tangent boundary domain* H *at* 1, *there exists a Blaschke product whose Fatou limit is* 1 *and whose H-derivative exists at each point* ζ_n. *If the set* $\{\zeta_n\}$ *is of type* G_δ, *there exists a Blaschke product* B *such that* $\{\zeta_n\} = S(B)$ *and the H-derivative of* B *exists at each point* ζ_n.

In the first part of the proof we merely suppose that $\{\zeta_n\}$ is a countable set on C.

Let $G(z) = \sum a_n(z + \zeta_n)/(z - \zeta_n)$, where $\{a_n\}$ is a positive sequence such that $\sum a_n < \infty$. Then G maps D into the left half-plane L; because the function $F(w) = (w + 1)/(w - 1)$ maps L into D, the composite function $B = F \circ G$ maps D into D.

In the domain $\zeta_m H$ each term of index less than m in the series for G is bounded, the mth term tends to infinity near ζ_m, and the sum of terms of index greater than m is bounded if $a_n \to 0$ rapidly enough. The technical meaning of "rapidly enough" depends on the distribution of the set $\{\zeta_n\}$ and the shape of the domain H near its boundary point 1. Henceforth we shall tacitly assume that the sequence $\{a_n\}$ meets all requirements of rapid convergence that emerge in the proof.

Because $G(z) \to \infty$ as $z \to \zeta_n$ in $\zeta_n H$, we see from the definition of B that $B(z) \to 1$ as $z \to \zeta_n$ in $\zeta_n H$.

To establish the existence of the H-derivative of B at ζ_n, we write

$$g_n(z) = a_n(z + \zeta_n)/(z - \zeta_n)$$

and $G = g_n + \gamma_n$. Then

$$B - 1 = \frac{G+1}{G-1} - 1 = \frac{2}{G-1} = \frac{2}{g_n - (1 - \gamma_n)}.$$

Because γ_n is bounded in $\zeta_n H$, it follows that

$$\frac{B(z) - 1}{z - \zeta_n} = \frac{2}{a_n(z + \zeta_n) - (z - \zeta_n)(1 - \gamma_n(z))} \to (a_n\zeta_n)^{-1}$$

as $z \to \zeta_n$ through $\zeta_n H$.

To see that B is an inner function, let I_n denote the arc of length $1/n^2$ with midpoint ζ_n on C. If $a_n \to 0$ rapidly enough, then on each radius of D whose endpoint lies in at most finitely many of the arcs I_n and does not belong to the set $\{\zeta_n\}$, the series for $G(z)$ converges uniformly to a function whose value at the endpoint is a finite imaginary number. Therefore the radial limit of B has modulus 1 almost everywhere on C.

To show that for an appropriate choice of $\{a_n\}$ the function B is a Blaschke product, we denote by d_n a circular disk tangent from the inside to C at ζ_n, and we suppose that no two of the disks intersect. If a_n is small enough, then the real part of $g_n(z)$ lies in the interval $(-3^{-n}, 0)$ for all z in $D \setminus d_n$. Therefore we may assume that in the set $\Delta = D - \bigcup d_n$ the real part of $G(z)$ lies in the interval $(-1/2, 0)$. Because $G(z)$ is bounded away from -1 in Δ, $B(z)$ is bounded away from 0 in Δ. Because every singular function has the radial limit 0 at some point of C, this implies that F is a Blaschke product [7].

REMARK. For each n the real part of the radial limit of g_n is 0 at each point ζ in $C \setminus \{\zeta_n\}$. By a slight refinement of our argument, the radial cluster set of B at each point on C is a subset of C.

Finally, suppose $\{\zeta_n\}$ is a countable set of type G_δ. Then $C \setminus \{\zeta_n\}$ is the union of an increasing sequence $\{F_n\}$ of closed sets; the distance ρ_n between ζ_n and F_n is positive. The mapping $w = (z + \zeta_n)/(z - \zeta_n)$ carries the radius vector of a point z_0 on C onto the circular arc in the left half-plane that is orthogonal to the imaginary axis and has the endpoints -1 and $(z_0 + \zeta_n)/(z_0 - \zeta_n)$. On this arc, the maximum modulus of w is either 1 or $|z_0 + \zeta_n|/|z_0 - \zeta_n|$. Therefore, if $a_n < 2^{-(n+1)}\rho_n$, then $|g_n(z)| < 2^{-n}$ on every radius of D that terminates on F_n. It follows that at each point of $C \setminus \{\zeta_n\}$ the function G has a finite radial limit whose real part is 0. Consequently, B has a Fatou limit of modulus 1 at each point of C, and it has the Fatou limit 1 only at the points ζ_n.

3. Preassigned Fatou limits. The theorem in this section includes the first part of Theorem 1 as a special case. Because for some nontrivial Blaschke products each point of C is a strong Fatou point, Theorem 2 contains no analogue to the second part of Theorem 1.

THEOREM 2. *Let* $\{\zeta_n\}$, $\{\omega_n\}$ *and* H *be a countable set on* C, *a sequence on* C, *and a tangent boundary domain at* 1. *Then there exists a Blaschke product possessing an* H*-derivative at each point of* $\{\zeta_n\}$ *and having the Fatou limit* ω_n *at* ζ_n *for* $n = 1, 2, \ldots$.

Our theorem overlaps with recent results of Cargo [3], who uses the weaker hypothesis $\omega_n \in D \cup C$; but his theorems allow only a finite set of points ζ_n.

For each index n let $\zeta_n = \exp i\psi_n^*$. By an inductive process, we shall construct a rapidly decreasing sequence $\{\delta_n\}$ of positive numbers. For each n we shall define a short arc Δ_n on C with midpoint ζ_n. The elementary building block in our construction will be a two-factor Blaschke product with zeros at the points $(1 - \delta_n)\exp i(\psi_n \pm \delta_n)$. Each of the parameters ψ_n will vary continuously as our construction proceeds, in such a way that $\exp i\psi_n$ approaches a limit $\exp i\tilde{\psi}_n$ on Δ_n. The following geometric sketch of the construction will put the analytic details in perspective.

The first element of our Blaschke product is the two-zero product b_1; for practical purposes its zeros might at the beginning lie at $(1-\delta_1)\exp i(\psi_1^* \pm \delta_1)$; but to establish as early as possible our induction ritual, we ask the reader to consider first the two-zero product with zeros at $(1-\delta)\exp i(\psi_1^* \pm \delta)$, and to regard δ as a parameter in the interval $[0, \frac{1}{2})$. As $\delta \to 0$, the product b_1 approaches the value 1 uniformly in the complement of every disk $|z - \zeta_1| < \varepsilon$. We allow δ to begin at 0 and to grow until it reaches some small positive number whose precise value is not relevant to the proof. Thereafter, δ remains fixed, in the context of b_1, and we denote it by δ_1. This completes the first half of the first step.

In the second half we replace the constant ψ_1^* with the variable ψ_1 and allow ψ_1 to increase or decrease until the two-factor Blaschke product with zeros at $(1-\delta_1)\exp i(\psi_1 \pm \delta_1)$ takes the value ω_1 at ζ_1. This completes the first step.

Suppose that the $(n-1)$st step has given us $n-1$ two-factor products b_ν $(\nu = 1,\dots,n-1)$ whose product P_{n-1} takes the value ω_ν at ζ_ν $(\nu = 1,\dots,n-1)$. We construct P_n by introducing the two-factor product b_n whose zeros lie at the points $(1-\delta)\exp i(\psi_n^* \pm \delta)$. As $\delta \to 0$ the function b_n tends to the constant 1 uniformly in the complement of each neighborhood of ζ_n. We shall show that as δ increases from 0 to δ_n, we can adjust the parameters $\psi_1,\dots,\psi_{n-1}$ continuously so that $P_n(\zeta_\nu) = \omega_\nu$ for $\nu = 1,\dots,n-1$. Thereafter, as the two zeros of b_n slide along the circle $|z| = 1 - \delta_n$, we can again adjust the parameters $\psi_1,\dots,\psi_{n-1}$ so as to keep $P_n(\zeta_\nu)$ constant for $\nu = 1,\dots,n-1$. When $P_n(\zeta_n)$ reaches the value ω_n, the nth step of the construction is completed.

To ensure that the formal limit of our product satisfies the Blaschke condition, we must choose the positive numbers δ_n so that $\sum \delta_n < \infty$. In our description we shall focus on a more severe criterion of smallness. Suppose at the end of the $(n-1)$st step of the construction each of the points $\exp i\psi_\nu$ $(\nu = 1,\dots,n-1)$ lies in the interior of the corresponding arc Δ_ν. We shall show that if δ_n is small enough, then at the end of the nth stage each of the points $\exp i\psi_\nu$ $(\nu = 1,\dots,n)$ still lies in the interior of the corresponding arc Δ_ν.

Suppressing temporarily the index n, we observe that our two-factor products b have the form

$$b(\psi,\delta,z) = \frac{(1-\delta) - e^{-i(\psi+\delta)}z}{1-(1-\delta)e^{-i(\psi+\delta)}z} \cdot \frac{(1-\delta) - e^{-i(\psi-\delta)}z}{1-(1-\delta)e^{-i(\psi-\delta)}z}$$

and, consequently, for all θ,

$$b(\psi,\delta,z) = b(\psi+\theta,\delta,ze^{i\theta});$$

in particular,

$$b(\psi_n^* + \theta, \delta_n, z) = b(\psi_n^*, \delta_n, ze^{-i\theta}).$$

From the last equation we deduce that the behavior of $b(\psi_n, \delta_n, \zeta_n)$ as $\exp i\psi_n$ traverses the arc Δ_n is the same as the behavior of $b(\psi_n^*, \delta_n, z)$ as z traverses the arc Δ_n in the opposite direction. Taking partial derivatives with respect to θ and writing

$$\frac{\partial}{\partial z} b(\psi,\delta,z) = b'(\psi,\delta,z),$$

we obtain the relation

$$\frac{\partial}{\partial\theta} b(\psi_n^* + \theta, \delta_n, z) = ie^{-i\theta} z b'(\psi_n^*, \delta_n, ze^{-i\theta}).$$

As z traverses the circle C, the value $b(\psi, \delta, z)$ traverses C twice; moreover, most of the movement of $b(\psi, \delta, z)$ takes place while z traverses a short arc of C with midpoint $\exp i\psi$. Invoking the convention that $\arg b(\psi, \delta, e^{i\psi}) = 0$, we define $\Delta(\psi, \delta)$ as the subarc of C along which $\arg b(\psi, \delta, z)$ increases from $-3\pi/2$ to $3\pi/2$. Let $2u(\delta)$ denote the length of $\Delta(\psi, \delta)$. It is easy to verify that there exist positive constants K_1 and K_2 such that $u(\delta) < K_1\delta$ and

$$|b'(\psi, \delta, z)| > K_2/\delta \tag{3.1}$$

whenever $z = \exp i(\psi + \theta)$ and $|\theta| < 2u(\delta)$.

Once δ_n is chosen, we define the arc Δ_n by the rule

$$\Delta_n = \Delta(\psi_n^*, \delta_n) = \{e^{i\psi}: |\psi - \psi_n^*| < u(\delta_n)\}.$$

From our definition of Δ_1 it follows that at the end of the first step we can choose the point $\exp i\psi_1$ so that it lies in Δ_1.

Suppose now that at the end of the $(n-1)$st step the points $\exp i\psi_\nu$ $(\nu = 1,\ldots,n-1)$ lie in the interiors of the corresponding arcs Δ_ν and $P_{n-1}(\zeta_k) = \omega_k$ for $k = 1,\ldots,n-1$. During the first half of the nth step, the variable ψ in our new factor b_n has the fixed value ψ_n^* and the variable δ increases from 0 to some δ_n. Meanwhile, the behavior of the variables ψ_ν $(\nu = 1,\ldots,n-1)$ is governed by the $n-1$ equations $P_n(\zeta_k) = \omega_k$. If we write these equations in detail, they take the form

$$\left[\prod_{\nu=1}^{n-1} b(\psi_\nu, \delta_\nu, \zeta_k)\right] b(\psi_n^*, \delta, \zeta_n) = \omega_k.$$

We regard the parameters ψ_ν as dependent on δ, take the logarithmic partial derivative with respect to δ, and obtain the equations

$$\sum_{\nu=1}^{n-1} \frac{\partial b(\psi_\nu, \delta_\nu, \zeta_k)/\partial\psi_\nu}{b(\psi_\nu, \delta_\nu, \zeta_k)} \cdot \frac{\partial\psi_\nu}{\partial\delta} + \frac{\partial b(\psi_n^*, \delta, \zeta_k)/\partial\delta}{b(\psi_n^*, \delta, \zeta_k)} = 0. \tag{3.2}$$

The last term on the left has a bound that depends only on $\psi_n^* - \psi_k^*$. In the coefficient of $\partial\psi_\nu/\partial\delta$, the denominator has modulus 1; as long as $\exp i\psi_\nu$ lies on Δ_ν, it follows from (3.1) that the modulus of the numerator is greater than K_2/δ_k if $\nu = k$; if $\nu \neq k$, the numerator has a bound independent of δ. It follows that if we have chosen δ_{n-1} small enough, then the Jacobian of the system (3.2) is bounded away from 0 as long as δ is small enough and $\exp i\psi_\nu$ lies on Δ_ν for $\nu = 1,\ldots,n-1$. This implies that each of the partial derivatives $\partial\psi_\nu/\partial\delta$ is a bounded function of δ. We deduce that at the end of the first half of the nth step the points $\exp i\psi_\nu$ $(\nu = 1,\ldots,n-1)$ lie as near as we like to their positions at the beginning of the nth step, provided we have chosen δ_n small enough.

For the second half of the nth step we must replace the partial derivatives $\partial\psi_\nu/\partial\delta$ in (3.2) with $\partial\psi_\nu/\partial\psi_n$ and the last term with

$$\frac{\partial b(\psi_n, \delta_n, \zeta_k)/\partial\psi_n}{b(\psi_n, \delta_n, \zeta_k)}.$$

Again, the last term is a bounded function. The second half of the nth stage ends when $P_n(\zeta_n)$ reaches the value ω_n. Because this requires an increase or decrease in ψ_n less than $u(\delta_n)$, we see that if we have chosen δ_n small enough, then at the end of the nth stage the point $\exp i\psi_\nu$ lies in Δ_ν for $\nu = 1,\dots,n$.

It is now obvious that if $\delta_n \to 0$ rapidly enough, then, as the construction continues, each of our parameters ψ_n approaches a limit $\tilde{\psi}_n$; the formal product

$$B(z) = \prod_1^\infty b(\tilde{\psi}_n, \delta_n, z)$$

is a Blaschke product, and $B(\zeta_k) = \omega_k$ for $k = 1, 2, \dots$.

Finally, let

$$B_n(z) = \prod_{\nu=1}^{n} b(\tilde{\psi}_\nu, \delta_\nu, z).$$

Then for each index k the sequence $\{[B_n(z) - B_n(\zeta_k)]/[z - \zeta_k]\}$ of difference quotients converges for each z in D, and the convergence is uniform in the domain $\zeta_k H$ if $\delta_n \to 0$ rapidly enough. This concludes the proof of Theorem 2.

4. Uncountable sets of strong Fatou-1-points. Because an inner function has a finite angular derivative at no more than countably many Fatou-1-points, we now abandon the angular derivative and its generalizations.

THEOREM 3. *There exists a Blaschke product with uncountably many strong Fatou-1-points.*

Let M denote the image of the classical Cantor set under the obvious mapping of the interval $[0, 1]$ onto the arc of C whose length is 1 and whose midpoint is 1, and let $d\mu$ denote the standard singular distribution of unit mass on M. We take the liberty of writing $d\mu(t)$ instead of $d\mu(e^{it})$, and we define the function

$$G(z) = u(z) + iv(z) = \int_{-\pi}^{\pi} \frac{z + e^{it}}{z - e^{it}}\, d\mu(t).$$

We shall show that each point of M is a strong Fatou-1-point of the function $B = (G+1)/(G-1)$. Because G has a holomorphic extension across the set $C \setminus M$ and its real part is 0 everywhere on this set, it will follow that B is a Blaschke product.

Because holomorphic mappings of D into D never incease non-Euclidean distances, it is sufficient to prove that B maps each radius terminating on M into a Stolz path teminating at 1. This is equivalent to proving that on each radius terminating on M the value of $G(z)$ tends to infinity in a wedge whose sides lie in the left half-plane.

We regard M as the intersection of closed sets M_n, each consisting of 2^n arcs of length 3^{-n}. Each of these arcs supports a mass 2^{-n} of the distribution $d\mu$.

Let $\zeta = e^{it_0}$ denote a point in M and write $z = r\zeta$ $(1/2 \leqslant r < 1)$. Then

$$u(z) = \int_{-\pi}^{\pi} \frac{(r^2 - 1)\, d\mu(t)}{1 - 2r\cos(t - t_0) + r^2} \quad \text{and} \quad v(z) = \int_{-\pi}^{\pi} \frac{-2r\sin(t - t_0)\, d\mu(t)}{1 - 2r\cos(t - t_0) + r^2}.$$

We recall the identity

$$(4.1) \qquad 1 - 2r\cos(t - t_0) + r^2 = (1 - r)^2 + 4r\sin^2(t - t_0)/2.$$

Let $n = n(r)$ denote the integer for which $3^{-n} < 1 - r \leq 3^{1-n}$, and let $I(z)$ denote the component of M_n that contains the point ζ. Then $|t - t_0| < 1 - r$ on $I(z)$, so the right side of (4.1) is less than $(1 + r)(1 - r)^2$. Sacrificing all contributions to $u(z)$ except that from $I(z)$, we obtain the inequality

$$|u(z)| > 1/2^n(1 - r) > 3^{-1}(3/2)^n.$$

To obtain a comparable upper bound on $|v(z)|$, we consider first the contribution from $I(z)$. Here

$$2r|\sin(t - t_0)| < 2r(1 - r) < (1 + r)(1 - r),$$

and therefore the contribution to $v(z)$ has absolute value less than $|u(z)|$.

To deal with the remainder of M_n, we consider for each of the indices $k = 1, 2, \ldots, n$ the $2^k - 1$ components of M_k that do not contain ζ. One of these $2^k - 1$ components is nearer to ζ than all the others; we denote it by I_k. Clearly,

$$M \subset M_n \subset I(z) \cup I_n \cup I_{n-1} \cup \cdots \cup I_1.$$

The angular distance between ζ and I_k is at least 3^{-k}. Therefore the estimate

$$\frac{2r|\sin(t - t_0)|}{4r\sin^2(t - t_0)/2} = \left|\cot\frac{t - t_0}{2}\right| < \frac{2}{|t - t_0|} < \frac{2}{3^{-k}}$$

is valid everywhere on I_k, and because the mass of $d\mu$ on I_k is 2^{-k}, the contribution to $v(z)$ from I_k has absolute value less than $2^{1-k}3^k = 2(3/2)^k$. The sum over k of these quantities is

$$2\left(3/2 + \cdots + (3/2)^n\right) < 6(3/2)^n < 18|u(z)|.$$

It follows that $|v(z)| < 19|u(z)|$, and Theorem 3 is proved.

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