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REYE CONGRUENCES

BY

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ABSTRACT. This paper studies the congruences of lines which are included in two distinct quadrics of a given generic three-dimensional projective space of quadrics in $\mathbf{P}^3$.

Introduction. A Reye congruence is classically defined as the set $R(W)$ of lines included in at least two distinct quadrics of a given projective space W of dimension three of quadrics in $\mathbf{P}^3$.

These congruences were introduced by T. Reye [**R**] and studied by G. Fano [$\mathbf{F_1}$, $\mathbf{F_2}$] who showed, in particular, that the generic Reye congruence is an Enriques surface. Fano's proofs are not satisfactory from a modern point of view. More important perhaps is the fact that Fano seems to have believed that the generic Enriques surface is a Reye congruence. A simple count of parameters shows that this is not the case. Indeed, the choice of W depends on $\dim(G(3,9))$ parameters—where $G(3,9)$ denotes the Grasmannian of projective spaces of dimension three in $\mathbf{P}^9$—so that after dividing by the action of $\mathbf{P}GL(3)$, it appears that the family of Reye congruences depends on at most $\dim G(3,9) - \dim \mathbf{P}GL(3) = 9$ parameters, i.e., one less than for the family of all Enriques surfaces.

The purpose of this paper is to understand the special features held by an Enriques surface which is a Reye congruence.

A careful description of the Picard group of the Reye congruence associated to a generic web W leads to the following results:

THEOREM 1. *The Reye congruences form a nine-dimensional family of Enriques surfaces which coincides generically with the family of Enriques surfaces of special type, i.e., which contains an elliptic pencil* $|P|$ *and a smooth rational curve* θ *such that* $P \cdot \theta = 2$.

THEOREM 2. *The family of Reye congruences coincides generically with the family of Enriques surfaces which contains an elliptic pencil* $|P|$ *and a smooth rational curve* θ *such that* $P \cdot \theta = 6$.

The $K3$ surfaces which arise as the ètale double cover of the generic Reye congruence have a history of their own. They can be realized in $\mathbf{P}^3$ as quartic surfaces, an equation of which is given by the vanishing of a 4×4 symmetrical

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determinant with linear entries in four variables. These quartic surfaces are classically called *symmetroids*. They were investigated for the first time by A. Cayley **[Ca]**. Our paper contains, in particular, a geometrical proof of a result due to Cayley.

THEOREM 3. *A $K3$ is the étale double cover of a generic Reye congruence if and only if it can be realized as the double cover of $\mathbf{P}^2$ branched over a sextic curve which splits into two cubics which intersect transversally and have a totally tangent smooth conic which does not contain any of the points of intersection of the cubics.*

It appears that the symmetroids are the quartic surfaces that M. Artin and D. Mumford **[A-M]** considered in order to construct a counterexample to the Luröth problem in dimension three.

1. The space of quadrics in $\mathbf{P}^2$ and $\mathbf{P}^3$. This preliminary section gives a description of the space of quadrics in $\mathbf{P}^2$ and $\mathbf{P}^3$. Proofs can be found in **[T]**. We let $\mathbf{P}^9 = \mathbf{P}H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2))$ be the space of quadrics in $\mathbf{P}^3$ and let $\mu_i \subset \mathbf{P}^9$ be the set of quadrics of $\mathbf{P}^3$ of rank $\leqslant 4 - i$. This gives a stratification $\mathbf{P}^9 \supset \mu_1 \supset \mu_2 \supset \mu_3$ which we now describe.

1.1. *The degree and dimension of μ_i.* A system of coordinates (X_i) of $\mathbf{P}^3$ is chosen in order to identify a quadric $q \in \mathbf{P}^9$ with a symmetrical matrix (q_{ij}) and its corresponding quadratic form

$$q(X) = \sum_{i,j} q_{i,j} X_i X_j \quad \text{where } X = (X_i).$$

(1.1.1) Since $\mu_1 = \{q = (q_{ij}), \det q = 0\}$, μ_1 is a quartic hypersurface μ_4^8 of $\mathbf{P}^9$.

(1.1.2) A quadric $q \in \mu_2$ is an unordered couple of planes. Let $\mathbf{P}^3$ be the dual space of $\mathbf{P}^3$. There is a map

$$\mathbf{P}^{3\vee} \times \mathbf{P}^{3\vee} \to \mathbf{P}H^0\big(\mathbf{P}^{3\vee}, \mathcal{O}_{\mathbf{P}^{3\vee}}(2)\big)^{\vee} \simeq \mathbf{P}H^0\big(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2)\big)$$

whose image is μ_2 and branch locus μ_3. This map is defined by the linear system of symmetrical divisors of bidegree $(1, 1)$ in $\mathbf{P}^{3\vee} \times \mathbf{P}^{3\vee}$.

The following easy lemma allows us to conclude that μ_2 is a variety μ_{10}^6 of dimension six and degree 10 in $\mathbf{P}^9$.

LEMMA 1.1.2. *The intersection number of $2n$ divisors of bidegree $(1, 1)$ in $\mathbf{P}^n \times \mathbf{P}^4$ is* $\binom{2n}{n}$.

(1.1.3) The variety μ_3 is the Veronese image of $\mathbf{P}^{3\vee}$. It is a smooth variety μ_8^3 of dimension three and degree eight in $\mathbf{P}^9$.

(1.1.4) The stratification of $\mathbf{P}^9$ by rank will henceforth be denoted by $\mathbf{P}^9 \supset \mu_4^8 \supset \mu_{10}^6 \supset \mu_8^3$.

1.2. *The singularities of μ_i.* As an algebraic variety, μ_4^8 carries the stratification $\mu_4^8 \supset \operatorname{sing} \mu_4^8 \supset \operatorname{sing}(\operatorname{sing} \mu_4^8) \supset \cdots$. It is well known that

$$\mu_{10}^6 = \operatorname{sing} \mu_4^8, \quad \mu_8^3 = \operatorname{sing} \mu_{10}^6, \quad \mu_8^3 \text{ is smooth.}$$

The following properties give a description of the singularities of μ_4^8 and μ_{10}^6.

PROPOSITION 1.2.1. (i) *μ_{10}^6 is the singular locus of μ_4^8.*
(ii) *$\mu_{10}^6 - \mu_8^3$ is the set of double points of μ_4^8.*
(iii) *μ_8^3 is the set of triple points of μ_4^8.*
(iv) *The tangent space of μ_4^8 at $q_0 \in \mu_4^8 - \mu_{10}^6$ is*

$$T_{q_0}(\mu_4^8) = \{q \in \mathbf{P}^9, \operatorname{sing} q_0 \in q\}.$$

(v) *The tangent cone of μ_4^8 at $q_0 \in \mu_{10}^6$ is*

$$C_{q_0}(\mu_4^8) = \{q \in \mathbf{P}^9; \, q \text{ tangent to } \operatorname{sing} q_0\}.$$

PROPOSITION 1.2.2. (i) *μ_8^3 is the singular locus of μ_{10}^6.*
(ii) *The tangent space of μ_{10}^6 at $q_0 \in \mu_{10}^6 - \mu_8^3$ is*

$$T_{q_0}(\mu_{10}^6) = \{q \in \mathbf{P}^9; \operatorname{sing} q_0 \subset q\}.$$

(iii) *The tangent cone of μ_{10}^6 at $q_0 \in \mu_8^3$ is*

$$C_{q_0}(\mu_{10}^6) = \{q \in \mathbf{P}^9; \, q \text{ tangent to } q_0 \text{ along a line}\}.$$

This is a quartic cone over a Veronese surface.
(iv) *The tangent space of μ_8^3 at $q_0 \in \mu_8^3$ is*

$$T_{q_0}(\mu_8^3) = \{q \in \mu_{10}^6; \, q \text{ contains the plane defining } q_0\}.$$

1.3. *The spaces of conics in* $\mathbf{P}^2$. The space of conics of $\mathbf{P}^2$ has a natural stratification

$$\mathbf{P}^5 = \mathbf{P}H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2)) \supset \mu_3^4 \supset \mu_4^2 = \operatorname{sing} \mu_3^4,$$

where μ_3^4 is the cubic fourfold of singular conics and μ_4^2, the Veronese image of $\mathbf{P}^{2\vee}$, is the quartic surface of double lines.

With obvious notations:

$$\begin{aligned} T_{q_0}(\mu_3^4) &= \{q \in \mathbf{P}^5; \operatorname{sing} q_0 \in q\} && \text{for any } q_0 \in \mu_3^4 - \mu_4^2.\\ T_{q_0}(\mu_4^2) &= \{q \in \mathbf{P}^5; \operatorname{sing} q_0 \subset q\} && \text{for any } q_0 \in \mu_4^2.\\ C_{q_0}(\mu_3^4) &= \{q \in \mathbf{P}^5; q \text{ tangent to } \operatorname{sing} q_0\} && \text{for any } q_0 \in \mu_4^2.\end{aligned}$$

2. Hessian and Steinerian surfaces of a web. This section describes the generic section of μ_4^8 by a three-dimensional projective space.

2.1. *The Hessian surface.*

DEFINITION 2.1.1. The Hessian surface $H = H(W)$ of a web W (i.e. a three-dimensional projective space) of quadrics in $\mathbf{P}^3$ is the set of singular quadrics of W.

$$H(W) = W \cap \mu_4^8.$$

If $W \subset \mu_4^8$, either there exists a point $x \in \mathbf{P}^3$ which is a singular point of all the quadrics of W or the base locus of W is of dimension at least one. From now on, we assume that W is not included in μ_4^8, so that $H(W)$ is a quartic surface in W.

The surface $H(W)$ is also classically called a *symmetroid* for the following reason: choose a basis $(q_0, \ldots, q_3)$ of W and let $q(\lambda) = \Sigma_i \lambda_i q_i$ for $\lambda = (\lambda_i)$ a system of coordinates of W. Then an equation of H is $\det(\Sigma_i \lambda_i q_i) = 0$, i.e., H is given by the vanishing of a 4×4 symmetrical determinant whose entries are linear forms in the four variables λ_i.

The main properties of these quartic surfaces were described by Cayley [**Ca**]. We hope that our approach will show more clearly the geometric properties of these symmetroids.

(2.1.2) *The singularities of* $H(W)$. As a corollary of the description of sing μ_4^8 given in §1, we have:

(2.1.2.1) W intersects transversally μ_4^8 at $q \in W \cap (\mu_4^8 - \mu_{10}^6)$ if and only if sing q is not a basepoint of W. Conversely, if x is a basepoint of W, there exists at least one quadric $q \in W$ which is singular at x and any such quadric is a singular point of $H(W)$.

(2.1.2.2) Any quadric $q \in W \cap (\mu_{10}^6 - \mu_8^3)$ is a double point of $H(W)$ unless $W \subset C_q(\mu_{10}^6)$, i.e., W has a basepoint $x \in \operatorname{sing} q$ at which all the quadrics of W are tangent to sing q, in which case q is a point of $H(W)$ of multiplicity $\geqslant 3$. Moreover, W intersects transversally μ_{10}^6 at q if and only if there is no quadric $q' \neq q$, $q' \in W$, such that sing $q \subset q'$.

(2.1.2.3) A quadric $q \in W \cap \mu_8^3$ is a triple point of $H(W)$ unless $W \subset C_q \mu_4^8$, i.e., all quadrics of W are tangent to q at a fixed point $x \in q$.

In particular, we have

PROPOSITION 2.1.2. *Let W be a web of quadrics in* $\mathbf{P}^3$. *Assume that*

(i) *W is basepoint free.*

(ii) *If l is a double line of a quadric $q \in W \cap \mu_{10}^6$, there is no other quadric $q' \in W$, $q' \neq q$, such that $l \subset q'$.*

Then $H(W)$ is a quartic surface with exactly ten ordinary double points.

DEFINITION 2.1.2. A *regular* web is a web satisfying (i) and (ii) of Proposition 2.1.2.

(2.1.3) This subsection describes a smooth model of $H(W)$ when W is regular. We first recall a desingularization of μ_4^8 defined by A. Tjurin [**T**]. We let

$$\tilde{\mu}_4^8 = \{(x, q);\ x \in \operatorname{sing} q\} \subset \mathbf{P}^3 \times \mu_4^8.$$

Clearly $\tilde{\mu}_4^8$ is the complete intersection of four divisors of bidegree $(1, 1)$ in $\mathbf{P}^3 \times \mathbf{P}^9$. The smoothness of $\tilde{\mu}_4^8$ follows from Lemma 2.1.3, which is an easy consequence of the Jacobian criterion.

LEMMA 2.1.3. *Let $X = \{(x, y);\ \varphi_i(x, y) = 0,\ i = 1, \ldots, p\}$ be the intersection of p divisors of bidegree $(1, 1)$ in $\mathbf{P}^n \times \mathbf{P}^m$. Then (x, y) is a smooth point of X if and only if there is no $\lambda = (\lambda_i)$ such that $\varphi(\lambda)(x, \circ) = \varphi(\lambda)(y, \circ) = 0$ where $\varphi(\lambda) = \Sigma_i \lambda_i \varphi_i$.*

We let $\pi_1\colon \tilde{\mu}_4^8 \to \mathbf{P}^3$ and $\pi_2\colon \tilde{\mu}_4^8 \to \mu_4^8$ be the natural projection maps.

(2.1.4) We define $\tilde{H} = \tilde{H}(W) = \pi_2^{-1}(H(W))$ or, equivalently,

$$\tilde{H} = \{(x, q);\ x \in \operatorname{sing} q\} \subset \mathbf{P}^3 \times W.$$

PROPOSITION 2.1.4. *Assume W is regular. Then $\tilde{H}(W)$ is a minimal desingularization of $H(W)$. It is a $K3$ surface.*

PROOF. We apply Lemma 1.3. First, we choose a basis (φ_i) of W, and let $\varphi(\lambda) = \sum_i \lambda_i \varphi_i$ so that

$$H(W) = \left\{(x, \lambda);\ \frac{\partial \varphi(\lambda)}{\partial x_i}(x) = 0 \text{ for any } i\right\}.$$

Assume that (x_0, λ_0) is a singular point of $H(W)$. Then there is $a = (a_i)$ such that

$$\sum_{i,j} a_i \frac{\partial \varphi(\lambda)}{\partial x_j}(x_0) = 0 \quad \text{for all } \lambda,$$

$$\sum_{i,j} a_i \frac{\partial \varphi(\lambda_0)}{\partial x_j}(x) = 0 \quad \text{for all } x,$$

or, equivalently,

$$\sum_i a_i \frac{\partial \varphi_j}{\partial x_i}(x_0) = 0, \quad \frac{\partial \varphi(\lambda_0)}{\partial x_j}(a) = 0 \qquad \text{for all } j.$$

Then if $a = x_0$, x_0 is a basepoint of W. If $a \neq x_0$, $\varphi(\lambda_0)$ is singular along the line l containing a and x_0; moreover, the orthogonality of a and x_0 for all the quadrics of W implies that there exists $\lambda_1 \neq \lambda_0$ such that $\varphi(\lambda_1)$ contains l. In both cases, W would not be regular. Therefore $H(W)$ is smooth. Since it is a complete intersection of divisors of bidegree $(1, 1)$ in $\mathbf{P}^3 \times W$, it is a $K3$ surface.

2.2 *The Steinerian surface.*

DEFINITION 2.2.1. The *Steinerian surface* $S(W)$ of a web W is defined by

$$S(W) = \pi_1 \circ \pi_2^{-1}(H(W)) \subset \mathbf{P}^3.$$

This is the set of singular points of singular quadrics of W. It is also called the *Jacobian surface* of W for the following reason:

$$\begin{aligned} S(W) &= \left\{x \in \mathbf{P}^3 \text{ such that } \exists\, \lambda,\ \frac{\partial \varphi(\lambda)}{\partial x_i}(x) = 0 \text{ for } i = 0, \ldots, 3\right\} \\ &= \left\{x \in \mathbf{P}^3 \text{ such that } \frac{\partial(\varphi_0, \ldots, \varphi_3)}{\partial(x_0, \ldots, x_3)}(x) = 0\right\}. \end{aligned}$$

In particular, $S(W)$ is a quartic surface in $\mathbf{P}^3$. The *Steiner map* is $\sigma\colon H(W) \to S(W)$ which associates to a singular quadric of W its singular locus. It is a birational map.

PROPOSITION 2.2.2. *The Steinerian surface of a regular web W is a smooth quartic surface if and only if there is no line on $H(W)$.*

PROOF. Since we are making a local study of $S(W)$ at one of its points x_0, we can choose coordinates (x, y, z, t) in $\mathbf{P}^3$ such that $x_0 = (0, 0, 0, 1)$. We can also choose a basis (φ_i) of W such that

$$\begin{aligned} \varphi_0 &= P(x, y, z), & \varphi_2 &= \beta y t + P_2(x, y, z), \\ \varphi_1 &= \alpha x t + P_1(x, y, z), & \varphi_3 &= \gamma t^2 + t Q_3(x, y, z) + P_3(x, y, z), \end{aligned}$$

where α, β, γ are constants and P_0, P_1, P_2, P_3 (resp. Q_3) are homogeneous polynomial in x, y, t of degree two (resp. one). Then it is easily checked that

$$\partial(\varphi_0,\ldots,\varphi_3)/\partial(x_0,\ldots,x_3) = -2t^3\alpha\beta\gamma\partial_z P_0 + R,$$

where R is a homogeneous polynomial in (x, y, z, t) of partial degree in t less than two. Therefore $x_0 \in \operatorname{sing}(S(W))$ if and only if

(i) $\gamma = 0$, or (ii) $\alpha\beta = 0$, or (iii) $\partial_t P_0 = 0$.

Case (i). $\gamma_0 = 0$ if and only if x_0 is a basepoint of W.

Case (ii). Say $\alpha = 0$. Then φ_0, φ_1 generate a pencil of quadrics of W which are singular at x_0. This defines a line l on $H(W)$. Consider the restriction of this pencil to a plane which does not contain x_0. It is a pencil of conics which, by the fact that W is regular, contains exactly three singular conics so that l is a line going through three of the double points of $H(W)$. Conversely, it is easily checked that for any regular web W, the only possible lines on $H(W)$ are lines through three of its double points, and to each such line is associated a singular point of $S(W)$.

Case (iii). $\partial_z P_0 = 0$, $\gamma \neq 0$: P_0 is a quadric of rank less than two which is singular along the line $l = \{x = y = 0\}$. The intersection of the polar hyperplanes of x_0 with respect to the quadrics of W is given by

$$\alpha x = \beta y = 0, \qquad 2\gamma t + Q_3(x, y, z) = 0,$$

hence there exists $x_1 \neq x_0$, $x_1 \in l$ such that (x_0, x_1) are orthogonal for all the quadrics of W. This line l contradicts the regularity of W.

(2.3) We will now introduce another surface which will help in understanding the properties held by $S(W)$. We further identify a quadric $q = (q_{ij})$ with its associated bilinear form

$$q(X, Y) = \sum_{i,j} q_{ij} X_i, X_j, \qquad X = (X_i), Y = (Y_j).$$

We define

$$\tilde{S} = \tilde{S}(W) = \{(x, y) \in \mathbf{P}^3 \times \mathbf{P}^3, q(x, y) = 0 \text{ for all } q \in W\}.$$

A direct consequence of Lemma 2.1.3 is

PROPOSITION 2.3.1. *$\tilde{S}(W)$ is a $K3$ surface if and only if W is regular.*

COROLLARY 2.3.2. *For any regular web W, $\tilde{S}(W)$ is the unramified double cover of an Enriques surface.*

PROOF. If W is a basepoint free, i: $(x, y) \to (y, x)$ is a fixed-point free involution of $\tilde{S}$. The corollary follows from the fact that the quotient of a $K3$ surface by a fixed-point free involution is an Enriques surface.

(2.3.3) If (φ_i) is a basis of W, and if π_0: $\tilde{S}(W) \to \mathbf{R}^3$ is obtained by projection into one of its factors, the image of $\tilde{S}(W)$ is

$$\left\{x \in \mathbf{P}^3 \text{ such that } \exists y = (y_i) \text{ with } \sum_i y_i \frac{\partial\varphi_j}{\partial x_i}(x) = 0 \text{ for all } j\right\},$$

which is precisely $S(W)$.

(2.3.4) The following diagram summarizes the different surfaces which have been introduced:

$$\begin{array}{ccc} \tilde{S}(W) & & \tilde{H}(W) \\ \downarrow \pi_0 & \pi_1 \swarrow & \pi_2 \downarrow \\ S(W) & \overset{\sigma}{\leftarrow} & H(W) \end{array}$$

where π_0, π_1, π_2 and σ are birational maps and, when W is regular, $\tilde{S}(W)$ and $\tilde{H}(W)$ are isomorphic $K3$ surfaces.

(2.4) *On the Picard group of $S(W)$.*

(2.4.1) This paragraph will assume that W is a regular web for which $S(W)$ is smooth. Such webs will be called *excellent*. The surface $S(W)$ is then isomorphic to $\tilde{S}(W)$ and contains:

1. the rational curves θ_i, singular loci of the ten quadrics of $W \cap \mu_{10}^6$;
2. the system of hyperplane sections $|\eta_S|$ of S;
3. the image $|\eta_H|$ of the system of hyperplane sections of $H(W)$;
4. ten pencils of elliptic curves $|E_i|$ obtained as residual intersection of S with the planes containing θ_i.

The following relations should be clear:

(i) $\eta_H^2 = \eta_S^2 = 4$;
(ii) $|\eta_S| = |\theta_i + E_i|$;
(iii) $\eta_H \cdot \theta_i = 0$, $\theta_i \cdot \theta_j = 0$ for $i \neq j$;
(iv) $\eta_S \cdot \theta_i = 1$, $\eta_S \cdot E_i = 3$ for all i;
(v) $E_i \cdot E_j = 2$ if $i \neq j$.

PROPOSITION 2.4.1. $2\eta_S = 3\eta_H - \Sigma_i \theta_i$ *in* $\mathrm{Pic}(S)$.

PROOF (cf. [T]). Let P be the linear system of polar cubics of H. For any $q_0 \in W$ the polar cubic $P(q_0)$ of q_0 with respect to H satisfies the following characteristic property:

$$P(q_0) \cap H = \left\{ \overline{q \in H;\, q \neq q_0,\, qq_0 \text{ is tangent to } H(W) \text{ at } q} \right\}.$$

From the description of T_qH given in §1, we see that

$$P(q_0) \cap H = \sigma^{-1}(q_0 \cap S(W)),$$

from which the proposition follows.

(2.4.2) The proof of (2.4.1) shows more precisely that $P = W_{|H}$. Therefore, since a generic web is not invariant under any projective automorphism of $\mathbf{P}^3$, we have

PROPOSITION 2.4.2. *The smooth Steinerian surfaces form a nine-dimensional locally closed set of the Hilbert scheme of smooth polarized $K3$ surfaces of degree four in* $\mathbf{P}^3$.

(2.4.3) We define the *enveloping cone* Γ_q of a point $q \in H(W)$ to be the closure in $\mathbf{P}^3$ of the sets of lines tangent to $H(W)$ at a point $q' \neq q$ and containing q. Let $q_0 \in \mathrm{sing}(H(W))$. After an appropriate choice of coordinates in $\mathbf{P}^3$, we can assume that $q = (0, 0, 0, 1)$ and an equation of $H(W)$ is

$$x_0^2 A_2(x_1, x_2, x_3) + 2x_0 B_3(x_1, x_2, x_3) + C_4(x_1, x_2, x_3) = 0,$$

where A_2, B_3, C_4 are homogeneous polynomials of degrees 2, 3, 4 in (x_1, x_2, x_3). Then Γ_q is the cone of equation $B_3^2 - A_2C_4 = 0$. It is a sextic cone totally tangent to the tangent cone $A_2 = 0$.

PROPOSITION 2.4.3 (CAYLEY'S PROPERTY). *Let W be a regular web. Then the enveloping cone of the double point $q_i \in \operatorname{sing} H(W)$ splits into two cubic cones totally tangent to the (irreducible) tangent cone of $H(W)$ at q_i.*

PROOF. By Proposition 1.2.1, Γ_{q_i} is tangent to $H(W)$ along a curve γ_i which is the inverse image of $q_i \cap S(W)$ under the Steiner map. Since q_i splits into two planes, $q_i \cap S = E_i + E_i' + 2\theta_i$, where E_i, E_i' are the residual cubics cut by the planes of q_i. Therefore, γ_i splits into the images of E_i, E_i' on H. The irreducibility of the tangent cone is a consequence of the regularity of W.

COROLLARY 2.4.4. *The Hessian surface of an excellent web is a double cover of $\mathbf{P}^2$ branched along a sextic curve which is totally tangent to a smooth conic and which splits into two cubic curves which intersect transversally at points not lying on the conic.*

THEOREM 2.4.5. *Let H be a quartic surface in $\mathbf{P}^3$ with ten rational double points p_i. Let $f\colon \tilde{H} \to H$ be a minimal desingularization of H and assume that $\tilde{H}$ is a $K3$ surface. Let $|\eta_H| = |f^*\mathcal{O}_H(1)|$ and let θ_i be the fundamental cycle of $f^{-1}(p_i)$. Assume, moreover, that there exists an effective divisor η_S such that*

$$2\eta_S = 3\eta_H - \sum_i \theta_i \quad \textit{in } \operatorname{Pic}(\tilde{H}).$$

Then H is the Hessian surface of a web.

PROOF. We want to apply Lemma 6.22, p. 374 of **[B]** to the map

$$S^2H^0(\tilde{H}, \mathcal{O}_{\tilde{H}}(\eta_S)) \to H^0\left(\tilde{H}, \mathcal{O}_{\tilde{H}}\left(3\eta_H - \sum_i \theta_i\right)\right).$$

Let $|\eta_S| = |M| + F$ be the decomposition of $|\eta_S|$ into its moving $|M|$ and its fixed part F. Then $\dim|M| \geqslant 3$ since $\eta_S^2 = 4$. If $|M|$ is reducible, there exists an elliptic pencil $|E|$ and an integer $k \geqslant 3$ such that $|M| = |kE|$ (cf. **[S-D]**). But $\eta_S \cdot \eta_H = 6 \geqslant kE \cdot \eta_H$ implies $E \cdot \eta_H = 2$. Hence, $|\eta_H|$ is hyperelliptic, which is absurd. Therefore $|M|$ is irreducible and, in particular, $M^2 \geqslant 4$.

Let us show that $M^2 = 4$. We first remark that $|M - \eta_H|$ is empty, otherwise let $G \in |M - \eta_H|$. Then

$$2\eta_S = 2M = 2F = 2\eta_H + 2G + 2F = 3\eta_H - \sum_i \theta_i,$$

hence $\eta_H = \sum_i \theta_i + 2F + 2G$, which is absurd since no hyperplane section of H can be a double curve. This implies $\eta_H \cdot (M - \eta_H) > 0$, otherwise $\eta_H \cdot M \leqslant 4$ and, by the Hodge index theorem, $M - \eta_H = 0$. But then by Riemann-Roch, $(M - \eta_H)^2 \leqslant -4$, hence $M^2 = 4$ and $\eta_H \cdot F = 0$ as desired.

To conclude that $F = 0$, notice that $\eta_S^2 = (M + F)^2 = 4$ gives $2M \cdot F + F^2 = 0$, and $2F \cdot \eta_S = 2F \cdot (M + F) = F \cdot (3\eta_H - \sum \theta_i)$ gives $F^2 + F \cdot \sum \theta_i = 0$, which is impossible since $\eta_H \cdot F = 0$ implies $F^2 < 0$, $\theta_i \cdot F \leqslant 0$ unless $F = 0$.

We claim now that $|\eta_S|$ is nonhyperelliptic. Otherwise, let $|E|$ be an elliptic pencil such that $E \cdot \eta_S = 2$. Then it is easily checked that the nonhyperellipticity of $|\eta_H|$ implies $E \cdot \eta_H = 3$, hence $E \cdot (\Sigma \theta_i) = 5$. Using the fact that $|E|$ is obtained by moving a plane about a line of H, there would be five nodes of H on a line, which is absurd. Therefore $|\eta_S|$ is a nonhyperelliptic system of genus three.

It is now easy to complete the proof of Theorem 2.4.5 by using Beauville's lemma. This part of the proof is omitted.

COROLLARY 2.4.6. *Let H be the double cover of $\mathbf{P}^2$ branched along two cubic curves which intersect transversally and have a totally tangent conic which does not contain any of the intersection points of the cubic curves. Then H is a Hessian surface.*

PROOF. Let $H \to \mathbf{P}^2$ be the covering map. Let (θ_i), $i = 1, \ldots, 9$, be the inverse image of the intersection points of the two cubic curves, let θ_0 be the inverse of the totally tangent conic, and let $|E|$ be the proper transform of the obvious pencil of elliptic curves. Finally let $|M| = |\sigma^* \mathcal{O}_{\mathbf{P}^2}(1)|$. Then

$$2\eta_S = 3\eta_H - \sum_i \theta_i \quad \text{where } |\eta_S| = |E + \theta_0|, \ |\eta_H| = |M + \theta_0|$$

and apply Theorem 2.4.5.

REMARK. (2.4.6) shows that the quartic surfaces considered by M. Artin and D. Mumford in [**A-M**] are precisely Hessian surfaces.

PROPOSITION 2.4.5. *Let H be the Hessian surface of a regular web and let h be a smooth hyperplane section of H. Then there exists a double cover $f\colon \Sigma \to M$ branched along $h \cup \operatorname{sing} H$. Moreover, Σ is a surface of general type such that $p_g = 1$, $q = 0$, $K_\Sigma^2 = 2$.*

PROOF. Proposition 2.4.1 clearly holds for regular webs. Therefore

$$h + \sum_i \theta_i = 2(2\eta_H - \eta_S) \quad \text{in } \operatorname{Pic} \tilde{S}.$$

Therefore there exists a double cover $\overline{\Sigma}$ of $\tilde{S}$ branched along $h + \Sigma_i \theta_i$. By the projection formula,

$$\begin{aligned} p_g(\overline{\Sigma}) &= h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2\eta_H - \eta_S)) + h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 1 + h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2\eta_H - \eta_S)), \\ q(\overline{\Sigma}) &= h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(2\eta_H - \eta_S)) + h^1(\tilde{S}, \mathcal{O}_S) = h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(2\eta_H - \eta_S)). \end{aligned}$$

Assume there exists $F \in |2\eta_H - \eta_S|$. Then $\eta_S + \Sigma_i \theta_i = \eta_H + F$, and, multiplying by θ_j, $-1 = F \cdot \theta_j$ for $j = 1, \ldots, 10$. Therefore $F = \Sigma_i \theta_i + F$, $F_1 > 0$, and $\eta_S = \eta_H + F_1$, which is clearly absurd. This shows that $p_g(\overline{\Sigma}) = 1$, and $q(\overline{\Sigma}) = 0$ follows by Riemann-Roch. Moreover, one has

$$K_{\overline{\Sigma}} = 2(2\eta_H - \eta_S)^2 = -8.$$

The surface Σ is obtained from $\overline{\Sigma}$ by blowing-down the ten exceptional curves of the first kind defined by the θ_i's. This concludes the proof.

3. Reye congruences. This section describes the generic section of $\mu_{10}^6 \subset \mathbf{P}^9$ by a projective space of dimension five.

3.1. *Definition and the Picard group of a Reye congruence.*

PROPOSITION 3.1.2. *Let W be a web without basepoints. Then $R(W)$ is the set of lines of $\mathbf{P}^3$ which are included in two distinct quadrics of W.*

PROOF. Let l be the line defined by a point $(x, y) \in \tilde{S}$. A quadric of W of equation $\varphi = 0$ contains l if and only if $\varphi(x) = \varphi(y) = 0$, so there exist at least two distinct quadrics of W containing l.

Conversely, let l be a line of $\mathbf{P}^3$ included in two distinct quadrics of W. Let $N \subset W$ be a pencil generating W together with the previous quadrics. Then the restriction of N to l is a pencil of quadrics of l. By Lemma 1.1.2, there exists a unique couple of points (x, y) of l orthogonal for all the quadrics of $N|l$ such that $(x, y) \in \tilde{S}$.

As a corollary of (3.1.2) and (2.3.1), we have

PROPOSITION 3.1.3. *Let W be a web of quadrics in $\mathbf{P}^3$. Then $R(W)$ is an Enriques surface if and only if W is regular.*

(3.2) We will now proceed to describe the Picard group of $R(W)$ when W is excellent. Let $p\colon \tilde{S} \to R$ be the quotient map and define, after 2.4,

$$F_i = p(E_i), \quad F_i' = p(E_i') \quad \text{and} \quad D_i = p(\theta_i).$$

The elliptic pencils $|E_i|$ are i-invariant, the i-invariant fibres being precisely E_i and E_i'. The following lemma is a consequence of (2.4.1):

LEMMA 3.2.1. *There exist on $R(W)$ ten smooth rational curves D_i and ten elliptic pencils $|2F_i| = |2F_i'|$ such that:*

(i) $D_i \cdot F_j = 1$ *for* $i \neq j$;

(ii) $D_i \cdot F_i = 3$ *for* $i = 1, \ldots, 10$;

(iii) $F_i \cdot F_j = 1$ *for* $i \neq j$;

(iv) $D_i \cdot D_j = 2$ *for* $i \neq j$;

(v) $\mathrm{Pic}(R) \otimes Q$ *is generated by the* F_i*'s.*

(3.2.2) The first intersection relation says that $R(W)$ is an *Enriques surface of special type*, i.e., there exist a smooth irreducible rational curve θ and an elliptic pencil $|P|$ such that $P \cdot \theta = 2$. Actually, using results of **[Co]**, we have

THEOREM 3.2.2. *The generic Enriques surface of special type is the Reye congruence of an excellent web.*

(3.2.3) The second intersection relation of (3.2.1) shows that there exist a rational curve θ and an elliptic pencil $|P|$ such that $P \cdot \theta = 6$. Conversely, we have

THEOREM 3.2.3. *Let R be an Enriques surface which contains an elliptic pencil $|P|$ and a smooth rational curve θ such that $P \cdot \theta = 6$. Then R is the minimal desingularization of a Reye congruence.*

PROOF. Let $p\colon S \to R$ be the unramified double cover of R. If $|P| = |2F|$, then $|p^{-1}(F)| = |E|$ is an elliptic pencil on S. Let θ_0, θ_1 be the components of $p^{-1}(\theta)$ and define

$$|\eta_S| = |E + \theta_0|, \qquad |i(\eta_S)| = |E + \theta_1|,$$

where i is the canonical involution of S. Then consider the map $\varphi = \varphi_{|\eta_S|} \times \varphi_{|i(\eta_S)|}$:

$$\varphi\colon S \to \mathbf{P}H^0(S, \mathcal{O}_S(\eta_S))^\vee \times \mathbf{P}H^0(S, \mathcal{O}_S(i(\eta_S)))^\vee .$$

The idea of the proof is to show that $\varphi(S)$ is the complete intersection of four symmetrical divisors of bidegree (1, 1) so that $\varphi_{|\eta_S|}(S)$ is the surface of the corresponding web of quadrics, from which the theorem is clear.

The morphisms $\varphi_{|\eta_S|}$ and $\varphi_{|i(\eta_S)|}$ are easily checked to be of degree one. Let a, b be the canonical generators of the Chow ring of $\mathbf{P}^3 \times \mathbf{P}^3$. Then the class of $\varphi(S)$ is equal to $4a^3b + 6a^2b^2 + 4ab^3$, as follows from $\eta_S \cdot i(\eta_S) = 6$, $\eta_S^2 = 4$, $i(\eta_S)^2 = 4$. Therefore if we let $L = \varphi^*\mathcal{O}_{\mathbf{P}^3\times\mathbf{P}^3}(1, 1)$, $L^2 = 20$ and $h^0(S, L) = 12$ by Riemann-Roch. Since $h^0(\mathbf{P}^3 \times \mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3\times\mathbf{P}^3}(1, 1)) = 16$, there exist at least four independent divisors of bidegree (1, 1) containing $\varphi(S)$. Since the class of $\varphi(S)$ is precisely $(a + b)^4$, it follows that $\varphi(S)$ is equal to the intersection of these divisors.

Let (q_i) be the bilinear forms associated to these divisors. Let $q_i = q_i^s + q_i^a$ be their decomposition into symmetric and antisymmetric parts. We want to prove that $q_i^a = 0$. Since $\varphi(S)$ lies symmetrically in $\mathbf{P}^3 \times \mathbf{P}^3$, $q_i^s(x, y) = q_i^a(x, y) = 0$ for any $(x, y) \in \varphi(S)$. Let us assume, for example, that q_i^s, $i = 1, 2, 3$, are linearly independent and $q_0^a \neq 0$. Then $\varphi_{|\eta_S|}(S)$ is a quartic surface in $\mathbf{P}^3$ included in the quartic surface, an equation of which is

$$\partial(q_0, \ldots, q_3)/\partial(x_0, \ldots, x_3) = 0.$$

These quartic surfaces must coincide, which is absurd since a basepoint of q_1, q_2, q_3 would give a fixed-point of the involution induced by i on $\varphi_{|\eta_S|}(S)$, hence a fixed-point on S.

(3.2.4) Using the third intersection relation of (3.2.1), we have

PROPOSITION 3.2.4. *Let* (F_1, F_2, F_3) *be any three of the "half-elliptic pencils" defined in* (3.2.1). *Then* $|F_1 + F_2 + F_3|$ *or* $|F_1 + F_2 + F_3 + K_R|$ *defines a morphism of degree one onto a sextic surface in* $\mathbf{P}^3$ *which is double along the edges of a tetrahedron.*

PROOF. This follows directly from [**Co**].

REMARK. After tedious computations, one checks that the elliptic pencils $|2F_i|$ have no reducible fibres when no four nodes of $H(W)$ are coplanar. Under this additional assumption, $|F_1 + F_2 + F_3|$ and $|F_1 + F_2 + F_3 + K_R|$ are both of degree one (cf. [**Co**]).

(3.2.5) The fourth intersection of (3.2.1) can be used to show

PROPOSITION 3.2.5. *The unramified double cover of the Reye congruence of an excellent web can be realized as the intersection of three quadrics in* $\mathbf{P}^5$ *on which the involution is projective.*

PROOF. The inverse image of $D_i + D_j$, $i \neq j$, on $S(W)$ splits into two disjoint components $\theta_i + i(\theta_j)$, $\theta_j + i(\theta_i)$ so that $|D_i + D_j|$ is an elliptic pencil (cf. [**Co**]). Let $|D_i + D_j| = |2F_{ij}|$ and let E_{ij} be the inverse image of F_{ij} on $S(W)$ so that $|E_{ij}|$ is an elliptic pencil such that $E_j \cdot E_{ij} = E_i \cdot E_{ij} = 4$. Then the linear system $|E_i + E_{ij}|$ gives a morphism

$$\varphi_{i,j}\colon S(W) \to \mathbf{P}H^0\big(S(W), \mathcal{O}_{S(W)}(E_i + E_{ij})\big)^\vee \subset \mathbf{P}^5.$$

This morphism is of degree one since $|E_i + E_{ij}|$ is obviously not hyperelliptic. Moreover, $\varphi_{i,j}(S(W))$ is the intersection of three quadrics, as follows from the fact that $|E_i + E_{ij}|$ is not trigonal (cf. [**S-D**]). The involution induced by i is projective because $|E_i + E_{ij}|$ is i-invariant.

PROPOSITION 3.2.6. *The map associated to* $|D_1 + D_2 + D_3|$ *gives a morphism of degree two onto a Cayley cubic surface* $\mathcal{C}$ *which is branched along* $(\operatorname{sing} \mathcal{C}) \cup \gamma$, *where* γ *is generically a smooth canonical curve of genus four such that* $\gamma \cap (\operatorname{sing} \mathcal{C}) = \varnothing$.

COROLLARY 3.2.7. *The generic minimal desingularization of the double cover of* $\mathcal{C}$ *branched along* $(\operatorname{sing} \mathcal{C}) \cup \gamma$, *where* γ *is a canonical curve of genus four such that* $\gamma \cap (\operatorname{sing} \mathcal{C}) = \varnothing$, $\gamma \subset \mathcal{C}$, *coincides with the generic Reye congruence.*

This follows from [**Co**].

3.3. *A first projective model of* $R(W)$ *in* $\mathbf{P}^5$.

(3.3.1) Let W be a web of quadrics in $\mathbf{P}^3$. Let $W^{\perp}$ be the orthogonal W in $\mathbf{P}H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2))^{\vee}$. Then $W^{\perp}$ can be identified with a five-dimensional projective space of quadrics in $\mathbf{P}^3$.

Consider the filtration of $\mathbf{P}H^0(\mathbf{P}^{3\vee}, \mathcal{O}_{\mathbf{P}^{3\vee}}(2))$ by the rank

$$\mathbf{P}H^0(\mathbf{P}^{3\vee}, \mathcal{O}_{\mathbf{P}^{3\vee}}(2)) \supset \nu_4^8 \supset \nu_{10}^6 \supset \nu_8^3.$$

Then $W^{\perp} \cap \nu_{10}^6$ is equal to

$$\{(x, y);\ \varphi(x, y) = 0 \text{ for all } \varphi \in W\} \subset \mathbf{P}H^0(\mathbf{P}^{3\vee}, \mathcal{O}_{\mathbf{P}^{3\vee}}(2)).$$

This means that $R(W)$ lies naturally in $\mathbf{P}^5$:

$$R(W) = W^{\perp} \cap \nu_{10}^6 \subset W = \mathbf{P}\big(H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2))/W\big) \subset \mathbf{P}H^0(\mathbf{P}^{3\vee}, \mathcal{O}_{\mathbf{P}^{3\vee}}(2)).$$

This allows to give another proof of

(3.3.2) $R(W)$ is smooth if and only if W is regular.

PROOF. $R(W)$ is smooth if and only if $W^{\perp}$ does not intersect ν_8^3 and intersects transversally $\nu_{10}^6 - \nu_8^3$. Clearly, $W^{\perp} \cap \nu_8^3 = \varnothing$ if and only if W has a basepoint x. Moreover, for a generic basepoint x of W, $\{x, x\} \in W^{\perp} \cap \nu_8^3$ is a quadruple point of $R(W)$, as expected from the fact that x is an ordinary double point of $S(W)$. Then one checks, using the description of the tangent space to $\nu_{10}^6 - \mu_8^3$, that W intersects transversally $\nu_{10}^6 - \nu_8^3$ at (x, y) if and only if there exists no pencil of quadrics of W containing the line l joining x and y and containing a quadric singular along l.

PROPOSITION 3.3.3. *Let W be a regular web. The linear system* $|\delta|$ *of hyperplane sections of* $R(W) \subset W^{\perp} \cong R^5$ *is numerically equivalent to* $\sum_i F_i/3$.

PROOF. The images of the curves F_i are plane cubics on $R(W) \subset \mathbf{P}^5$. Indeed let P_i be the plane of the quadric of rank two q_i containing E_i. Let N be a net of quadrics generating W with q_i, and let $N^{\perp}$ be the orthogonal of N in $\mathbf{P}H^0(P_i, \mathcal{O}_{P_i}(2))^{\vee}$. Consider the stratification of $\mathbf{P}H^0(P_i, \mathcal{O}_{P_i}(2))^{\vee}$ by the rank

$$\mathbf{P}H^0(P_i, \mathcal{O}_{P_i}(2))^{\vee} \supset \nu_3^4 \supset \nu_4^2.$$

Then our claim follows from

$$N^{\perp} \cap \nu_3^4 = \{(x, y) \in P_i \times P_i;\ \varphi(x, y) = 0 \text{ for all } \varphi \in N\}.$$

Since $\Delta^2 = 10$, $\Delta \cdot F_i = 3$ for all i, it is easy to check that Δ is numerically equivalent to $\Sigma_i F_i/3$.

COROLLARY 3.3.4. *The Picard group of* $R(W)$ *is generated by* (Δ, F_i) *for an appropriate choice of* F_i*'s.*

PROOF. Changing one F_i into $F_i + K_R$, if necessary, we can assume $3\Delta = \Sigma_i F_i + K_R$. Therefore, Pic R is generated (over $\mathbf{Z}$) by Δ and the F_i's: One checks that for any divisor D, D and $(D\Delta)\Delta - \Sigma_i(DF_i)F_i$ are numerically equivalent.

COROLLARY 3.3.5. *Let* W *be an excellent web. Then* $|\Delta| = |2F_i + D_i|$ *for all* i.

PROOF. The numerical equivalence of $|\Delta|$ and $|2F_i + D_i|$ results from intersection relations. To conclude one needs to notice that $h^0(R, \mathcal{O}_R(\Delta - D_i)) \geqslant 2$ because the image of D_i is included in the two hyperplanes defined by the quadrics of rank one corresponding to the two planes of q_i.

3.4. *A second projection model of* $R(W)$ *is* $\mathbf{P}^5$.

If $R(W)$ is the Reye convergence of a basepoint free web W, $R(W)$ lies naturally in the Grasmannian $G(1,3)$ of lines in $\mathbf{P}^3$ by (3.1.2).

We will compute the homology class of $R(W)$ in $G(1,3)$ after a brief reminder of Schubert calculus in $G(1,3)$. After embedding $G(1,3)$ in $\mathbf{P}^5$ by Plücker coordinates, we will compute the corresponding system of hyperplane sections of $R(W)$.

(3.4.1) *Schubert calculus in* $G(1,3)$.

(3.4.1.1) Let $l \cap l'$ (resp. p, P) be a line (resp. a point, a plane) of $\mathbf{P}^3$. We have the following usual notations for Schubert cycles:

$$\sigma_l = \{l'; l \cap l' \neq \varnothing\}, \qquad \sigma_p = \{l'; p \in l'\},$$
$$\sigma_P = \{l'; l' \subset P\}, \qquad \sigma_{p,P} = \{l'; p \in l' \subset P\}.$$

The cohomology ring of $G(1,3)$ is generated by these cycles, $\sigma_{p,P} \in H_2(G(1,3), \mathbf{Z})$, $\sigma_p, \sigma_P \in H_4(G(1,3), \mathbf{Z})$ and $\sigma_l \in H_6(G(1,3), \mathbf{Z})$.

(3.4.1.2) $G(1,3)$ is embedded by Plücker coordinates as a smooth quadric hypersurface $\Omega \subset \mathbf{P}^5$. The two rulings of Ω are the two families of 2-planes σ_p, σ_P. Any line of Ω is of the form $\sigma_{p,P}$. The tangent space $T_l\Omega$ of Ω at l is such that $(T_l\Omega) \cap \Omega = \sigma_l$.

(3.4.1.3) For any cycle $[S] \in H_4(G(1,3), \mathbf{Z})$, we let a, b be the two integers such that $[S] = a\sigma_p + b\sigma_P$. Classically, a is called the *order* of $[S]$ and b its *class*. We will say that (a, b) is the *type* of $[S]$ and $a + b$ is its *degree*.

(3.4.2) We have the well-known result.

PROPOSITION 3.4.2. *The* 4*-cycle associated to a Reye congruence in* $G(1,3)$ *is of type* $(7,3)$.

PROOF. (Somewhat simpler than in [**G-H**].) The class of $R(W)$ is the number of lines of $R(W)$ in a generic plane P, i.e., half the number of points of

$$\{(x, y), \varphi(x, y) = 0, \text{ for all } \varphi \in W\} \subset P \times P.$$

By Lemma 1.1.2, the class of $R(W)$ is equal to three.

The order of $R(W)$ is the number of lines $R(W)$ through a generic point p. Let $N(p)$ be the set of quadrics of W through p. Let $p, x_1, \ldots, x_7$ be the basepoints of $N(p)$. Then obviously the seven lines px_i belong to $R(W)$. Conversely, let l be a line of $R(W)$ through p and let $\pi \subset N(p)$ be a pencil of quadrics of W containing l. Consider the residual twisted cubic γ of the base locus of π. Then γ does contain all the x_i's if $l \neq px_i$ for all i. Hence, γ is in the base locus of $N(p)$, which would imply that W has some basepoint, which is not true for a generic W.

COROLLARY 3.4.3. *The system of hyperplane sections of* $R(W) \subset G(1,3) \subset \mathbf{P}^5$ *is* $|\Delta + K_R|$.

PROOF. Let us first prove that $R(W)$ is not included in any hyperplane. From the description of a hyperplane tangent to Ω, it is clear that $R(W)$ can be included in a hyperplane only if this hyperplane is intersecting Ω transversally. Then $R(W)$ would be included in a smooth quadric in $\mathbf{P}^4$, hence a generic hyperplane section of $R(W)$ would be of type $(7,3)$ on a smooth quadric in $\mathbf{P}^3$. The genus of such a curve is $3 \cdot 7 - 3 - 7 + 1 = 12$ and, using the genus formula on $R(W)$, this genus should be $10/2 + 1 = 6$ since $R(W)$ is of degree 10 for the embedding in $G(1,3) = \Omega \subset \mathbf{P}^5$.

Therefore, $R(W)$ is embedded in $\mathbf{P}^5$ by a complete linear system. It is now easy to check that the images of the plane cubics F_i are plane cubics of $R(W)$. Hence the system of hyperplane sections $|\Delta'|$ is (as in (3.3.3)) numerically equivalent to $|\Delta|$. Assuming that $S(W)$ is smooth, we have proved that $|\Delta| = |2F_i + D_i|$, therefore $|\Delta'| = |2F_i + D_i + K_R|$ because there is a hyperplane of $\mathbf{P}^5$ containing simultaneously F_i and $F_i + K_R$, namely $T_{\mathrm{sing}(q_i)}(\Omega)$.

REMARK. Let $R(W) = \Omega \subset \mathbf{P}^5$ be the Reye congruence of a good web W embedded in $\mathbf{P}^5$ as a congruence of lines via Plücker coordinates. Then the generic hyperplane section h of $R(W)$ is a Prym-canonical curve of genus six (cf. **[D-S]**) where the 2-torsion sheaf is $\mathcal{O}_h \otimes \mathcal{O}_R(K_R)$. This Prym-canonical curve lies in the ramification locus of the Prym map $P_5\colon M_6^{(2)} \to \mathcal{Q}_5$ by a theorem of Beauville **[B]**.

This suggests the following question: Does the image of the Reye-congruences dominate the ramification locus B of P_5?

Notice that $\dim B = 14 =$ (dimension of the family of Reye congruences) $+ \dim \mathbf{P}^{5\vee}$.

Notice also that (dim of the family of Reye congruences) $+ \dim \Omega^\vee = 13$, where $\Omega^\vee$ is the dual variety of Ω. By the fact that $R(W)$ is of type $(7,3)$, one can show that a section of $R(W)$ by a hyperplane tangent to Ω is generically a smooth trigonal curve and 13 is precisely the dimension of the trigonal locus of M_6.

More generally, one should look at the Hilbert scheme $\mathfrak{H}$ of polarized smooth Enriques surfaces of degree ten in $\mathbf{P}^5$ and consider the obvious map, defined in an open set $\mathcal{V}$ of $\mathfrak{H}$, $\mathcal{V} \to M_6^{(2)}$. This map is known to be dominant **[M]**. A natural problem is the study of the induced map $\mathcal{V}/\mathbf{P}GL(5) \to M_6^{(2)}$ especially in its relation to the Prym map $P_5\colon M_6^{(2)} \to \mathcal{Q}_5$.

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