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UNIQUENESS OF TORSION FREE CONNECTION ON SOME INVARIANT STRUCTURES ON LIE GROUPS

BY

MICHEL NGUIFFO BOYOM AND GEORGES GIRAUD

ABSTRACT. Let $\mathcal{G}$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathrm{Int}(\mathfrak{g})$ be the group of inner automorphisms of $\mathfrak{g}$. The group $\mathcal{G}$ is naturally equipped with $\mathrm{Int}(\mathfrak{g})$-reductions of the bundle of linear frames on $\mathcal{G}$. We investigate for what kind of Lie group the 0-connection of E. Cartan is the unique torsion free connection adapted to any of those $\mathrm{Int}(\mathfrak{g})$-reductions.

1. Definitions and main results. Let M be an n-dimensional manifold and let G be a Lie subgroup of the linear group $Gl(\mathbf{R}^n)$, with Lie algebra $\mathfrak{G}$. All manifolds we shall consider are smooth and connected. Let us consider a G-reduction $E(M, G)$ of the frame bundle $E^0(M, Gl(\mathbf{R}^n))$ and two linear connections ∇_1 and ∇_2 adapted to $E(M, G)$. Suppose these connections have the same torsion tensor, so that

$$(\nabla_1)_X Y - (\nabla_1)_Y X - [X, Y] = (\nabla_2)_X Y - (\nabla_2)_Y X - [X, Y]$$

or

$$(\nabla_1 - \nabla_2)_X Y - (\nabla_1 - \nabla_2)_Y X = 0$$

for any vector fields X, Y on M. Then if one identifies the tangent space $T_x(M)$ for $x \in M$, with $\mathbf{R}^n$, the difference $\nabla_1 - \nabla_2$ appears as an element of the space $\mathbf{R}^{n^*} \otimes \mathfrak{G} \cap S^2\mathbf{R}^{n^*} \otimes \mathbf{R}^n$ which is known to be the first prolongation of $\mathfrak{G}$ (see **[4]**).

A G-structure $E(M, G)$ is said to be 1-flat if it can be equipped with a torsion free linear connection. Thus any 1-flat G-structure can be equipped with at most one torsion free linear connection if and only if the first prolongation of $\mathfrak{G}$ is zero.

We are concerned with the following problem. Let (M, ω) be a differentiable manifold equipped with a torsion free linear connection ω. We wish to describe those linear subgroups G such that the connection ω is the unique linear connection adapted to some G-reduction of the frame bundle of M. Obviously a necessary condition is that the first prolongation of the holonomy algebra of ω be zero. So if $\mathcal{H}_\omega$ is the holonomy algebra of ω, the problem of finding all linear Lie groups with the previous properties is equivalent to that of finding all Lie subalgebras $\mathfrak{G}$ of $\mathrm{End}(\mathbf{R}^n)$ such that

(p_1) $$\mathcal{H}_\omega \subset \mathfrak{G},$$

(p_2) $$\mathfrak{G}^{(1)} = 0.$$

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In what follows we deal with differentiable manifolds $(\mathcal{G}, \omega)$ which are Lie groups equipped with the 0-connection of Cartan. Our interest in this particular case is due to the fact that the 0-connection of Cartan describes the local structure of the Lie group $\mathcal{G}$. In particular, the 0-connection of Cartan is invariant by the left translations of $\mathcal{G}$, while if ∇ is the covariant derivative associated to the 0-connection then $\nabla_X Y = \frac{1}{2}[X, Y]$ for any two left invariant vector fields X and Y on $\mathcal{G}$.

As a direct consequence of the above facts one deduces that the holonomy algebra, $\mathcal{H}_0$, of the 0-connection is included in the Lie algebra $\mathrm{ad}(\mathfrak{g})$ of inner derivations of $\mathfrak{g}$ ($\mathfrak{g}$ being the Lie algebra of $\mathcal{G}$). Let $\mathrm{Int}(\mathfrak{g})$ be the connected Lie subgroup of $Gl(\mathfrak{g})$ associated to $\mathrm{ad}(\mathfrak{g})$. Let us extend the holonomy fiber bundles of the 0-connection to $\mathrm{Int}(\mathfrak{g})$-reductions of the frame bundle of $\mathcal{G}$ to get left invariant $\mathrm{Int}(\mathfrak{g})$-structures. Any two such extensions are conjugate.

Our main results give a characterization of those Lie groups $\mathcal{G}$ on which the $\mathrm{Int}(\mathfrak{g})$-structures constructed as above belong to the set of $\mathrm{Int}(\mathfrak{g})$-reductions of the frame bundle of $\mathcal{G}$ which satisfy the properties (p_1) and (p_2), so that $\mathcal{H}_0 \subset \mathrm{ad}(\mathfrak{g})$ and $(\mathrm{ad}(\mathfrak{g}))^{(1)} = 0$. For such a Lie group $\mathcal{G}$, the 0-connection of Cartan is the unique torsion free linear connection adapted to its holonomy bundles. We make technical use of a Lie subalgebra $\mathfrak{h}_{\mathfrak{g}}$ of the linear Lie algebra $\mathrm{End}(\mathfrak{g})$, which is defined as follows. A linear endomorphism φ of the vector space $\mathfrak{g}$ belongs to $\mathfrak{h}_{\mathfrak{g}}$ if it satisfies the identity

$$[\varphi(X), Y] + [X, \varphi(Y)] = 0$$

for any pair (X, Y) in $\mathfrak{g} \times \mathfrak{g}$. Such a φ is called a symmetric operator of $\mathfrak{g}$. In the present work we restrict ourselves to the case of nonsolvable Lie groups.

Now let us denote by $\mathfrak{r}$ the radical of the Lie algebra $\mathfrak{g}$, i.e., $\mathfrak{r}$ is the maximal solvable ideal in $\mathfrak{g}$. Taking a Levi subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, the vector space $\mathfrak{g}$ becomes a direct sum: $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$. Let us consider $\mathfrak{r}$ with its $\mathfrak{s}$-module structure given by the extension $0 \to \mathfrak{r} \to \mathfrak{g} \to \mathfrak{s} \to 0$. The subspace of $\mathfrak{r}$ consisting of $\mathfrak{s}$-invariant elements is denoted by $\mathfrak{r}^{\mathfrak{s}}$. As $\mathfrak{s}$ is a semisimple Lie algebra, the subspace $[\mathfrak{s}, \mathfrak{r}]$ is a submodule of the $\mathfrak{s}$-module $\mathfrak{r}$, and one gets the direct sum of $\mathfrak{s}$-modules

$$\mathfrak{r} = \mathfrak{r}^{\mathfrak{s}} \oplus [\mathfrak{r}, \mathfrak{s}].$$

The maximal ideal of $\mathfrak{g}$ contained in $\mathfrak{r}^{\mathfrak{s}}$ is denoted $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ and the center of the subalgebra $\mathfrak{r}^{\mathfrak{s}}$ is denoted $Z(\mathfrak{r}^{\mathfrak{s}})$. The subspace of $\mathfrak{r}^{\mathfrak{s}}$ denoted by $h_{\mathfrak{r}^{\mathfrak{s}}}(\mathfrak{r}^{\mathfrak{s}})$ is that obtained by the evaluation map of $h_{\mathfrak{r}^{\mathfrak{s}}} \otimes \mathfrak{r}^{\mathfrak{s}}$ in $\mathfrak{r}^{\mathfrak{s}}$.

The main geometrical results to be proved are the following.

$(\mathfrak{R}_1)$ Let $\mathcal{G}$ be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Then the 0-connection ∇_0 of Cartan is the unique torsion free connection on each $\mathrm{Int}(\mathfrak{g})$-extension of any holonomy bundle of ∇_0 if and only if the ideal $h_{\mathfrak{r}^{\mathfrak{s}}}(\mathfrak{r}^{\mathfrak{s}}) \cap D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ is included in the center of $\mathfrak{r}^{\mathfrak{s}}$.

$(\mathfrak{R}_2)$ Let $\mathcal{G}$ be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Suppose that $\mathfrak{r}^{\mathfrak{s}}$ is a commutative subalgebra of $\mathfrak{g}$. Then one gets uniqueness of torsion free connection adapted to each $\mathrm{Int}(\mathfrak{g})$-extension of any holonomy bundle of ∇_0 if and only if the Lie group $\mathcal{G}$ has discrete center.

($\mathcal{R}_3$) Take $\mathcal{G}$ to be a Lie group, the radical of which is nilpotent subgroup. If $\mathcal{G}$ has discrete center, then there is a unique torsion free connection on each Int($\mathfrak{g}$)-extension of the holonomy bundle of ∇_0.

($\mathcal{R}_4$) Given a Lie group $\mathcal{G}$, let $\mathcal{R}^{\mathfrak{s}}$ be the connected Lie subgroup of $\mathcal{G}$ associated to the Lie subalgebra $\mathfrak{r}^{\mathfrak{s}}$. If $\mathcal{R}^{\mathfrak{s}}$ is a normal subgroup, then one gets uniqueness of the torsion free connection on Int($\mathfrak{g}$)-extension of the holonomy bundle of ∇_0 if and only if the same result holds on the Lie group $\mathcal{R}^{\mathfrak{s}}$.

2. Algebraic results. Because of the left invariant character of the previous results we shall deal with their infinitesimal versions. Thus, at the Lie algebra level we are concerned with finite-dimensional Lie algebras on a field K of characteristic zero.

THEOREM 1. *For any linear endomorphism* φ *of* $\mathfrak{g}$ *which belongs to the Lie algebra* $h_{\mathfrak{g}}$ *the following assertions hold*:

(i) *The restriction of* φ *to the subspace* $[\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}$ *takes its values in the center of the Lie algebra* $\mathfrak{g}$.

(ii) *The restriction of* φ *to the Lie algebra* $\mathfrak{r}^{\mathfrak{s}}$ *is an element of the Lie algebra* $h_{\mathfrak{r}^{\mathfrak{s}}}$ *and takes its values in the subspace* $\mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$ *of elements in* $\mathfrak{r}$ *which commute with the subspace* $[\mathfrak{r}, \mathfrak{s}]$.

Before starting the proof of Theorem I, let us remark that our interest in the Lie algebra $h_{\mathfrak{g}}$ arises from the following facts. Let $\mathfrak{g}$ be a Lie algebra and let $h_{\mathfrak{g}}^0$ be the vector space of all linear maps of $\mathfrak{g}$ into its center $Z(\mathfrak{g})$. Consider the linear map π of $h_{\mathfrak{g}}$ into $(\mathrm{ad}(\mathfrak{g}))^{(1)}$ given by $\pi(\varphi) = \mathrm{ad} \circ \varphi$, $\varphi \in h_{\mathfrak{g}}$, so that for any element X in $\mathfrak{g}$ one gets $\pi(\varphi)(X) = \mathrm{ad}_{\varphi(X)}$. It is clear that the bilinear map $(X, Y) \to [\varphi(X), Y]$ of $\mathfrak{g} \times \mathfrak{g}$ in $\mathfrak{g}$ is symmetric. Thus the previous map π takes its values in the first prolongation of ad($\mathfrak{g}$). This map is onto because of the definition of $(\mathrm{ad}(\mathfrak{g}))^{(1)}$. The kernel of π is $h_{\mathfrak{g}}^0$. So one obtains the following exact sequence of vector spaces:

$$0 \to h_{\mathfrak{g}}^0 \to h_{\mathfrak{g}} \to (\mathrm{ad}(\mathfrak{g}))^{(1)} \to 0.$$

Since the Cartan-Killing form $(X, Y) \mapsto \Phi(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$ of $\mathfrak{g}$ is invariant by the inner derivations of $\mathfrak{g}$, for each φ in $h_{\mathfrak{g}}$ one gets

$$\begin{aligned}\Phi([\varphi(X), Y], Z) &= -\Phi(Y, [\varphi(X), Z]) = -\Phi(Y, [\varphi(Z), X]) \\ &= \Phi([\varphi(Z), Y], X) = \Phi([\varphi(Y), Z], X) = -\Phi(Z, [\varphi(Y), X]) \\ &= -\Phi([\varphi(Y), X], Z) = -\Phi([\varphi(X), Y], Z).\end{aligned}$$

Thus $\Phi([\varphi(X), Y], Z) = \Phi(\varphi(X), [Y, Z]) = 0$, and the image $\varphi(\mathfrak{g})$ is perpendicular to $[\mathfrak{g}, \mathfrak{g}]$ under Φ. As is well known this implies that $\varphi(\mathfrak{g})$ lies in the radical $\mathfrak{r}$ of $\mathfrak{g}$.

For an element φ in $h_{\mathfrak{g}}$ let us denote by A and B the restriction of φ to $\mathfrak{r}$ and to $\mathfrak{s}$, respectively. Let (r, s) and (r', s') be two elements in $\mathfrak{g} \approx \mathfrak{r} \times \mathfrak{s}$. With respect to above notation one gets

$$[(A(r) + B(s), 0), (r', s')] = [(A(r') + B(s'), 0), (r, s)].$$

This last identity gives rise to the system

$$[A(r), r'] = [A(r'), r], \tag{1}$$

$$[B(s), r'] = [A(r'), s], \tag{2}$$

$$[B(s), s'] = [B(s'), s]. \tag{3}$$

To prove Theorem I, we need two technical lemmas.

LEMMA 1. *Let* $\mathfrak{g}$ *be a Lie algebra such that its Levi subalgebras* $\mathfrak{s}$ *are* 3*-dimensional, and let* $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ *be a Levi decomposition of* $\mathfrak{g}$. *Suppose* $\mathfrak{r}$ *is an irreducible* $\mathfrak{s}$*-module of dimension greater than one. Then for any element* φ *of* $h_{\mathfrak{g}}$ *the restriction* B *of* $\mathfrak{s}$ *to* φ *is zero.*

PROOF. One can suppose the ground field is algebraically closed. (This is done without loss of generality.) Let $m + 1$ be the dimension of the radical of $\mathfrak{g}$. Since $\mathfrak{s}$ is a 3-dimensional semisimple Lie algebra, we can choose a basis (X, Y, H) in $\mathfrak{s}$ such that

$$[X, Y] = H, \qquad [H, X] = 2X, \qquad [H, Y] = -2Y. \tag{4}$$

Let v be a primitive element of the $\mathfrak{s}$-module $\mathfrak{r}$. Then the sequence $v_0 = v$, $v_1 = Yv, \ldots, v_m = Y^m v$ is a basis of the vector space $\mathfrak{r}$ which satisfies the system

$$\begin{gathered} H.v_i = (m - 2i)v_i, \qquad i = 0, 1, \ldots, m, \\ Y.v_i = v_{i+1}, \qquad i = 0, 1, \ldots, m-1 \text{ and } Y.v_m = 0, \\ X.v_0 = 0 \quad \text{and} \quad X.v_i = (-mi + i(i-1))v_{i-1}, \qquad i = 1, \ldots, m, \end{gathered} \tag{5}$$

where, for any $s \in \mathfrak{s}$ and $r \in \mathfrak{r}$ we write $s.r$ for $[s, r]$. Now from the relations (3) and the system (4) one obtains

$$Y.B(H) = H.B(Y), \qquad Y.B(X) = X.B(Y), \qquad H.B(X) = X.B(H).$$

If one writes these in terms of the basis (v_i), one gets

$$\begin{gathered} \sum_{i=0}^{m} B_i(H)Y.v_i = \sum_{i=0}^{m} B_i(Y)H.v_i, \\ \sum_{i=0}^{m} B_i(X)Y.v_i = \sum_{i=0}^{m} B_i(Y)X.v_i, \\ \sum_{i=0}^{m} B_i(X)H.v_i = \sum_{i=0}^{m} B_i(H)X.v_i, \end{gathered} \tag{6}$$

the v_i-components in (6) for $i = 0, 1, \ldots, m$, we have the relations

$$B_0(Y) = 0, \qquad B_1(Y) = 0, \qquad B_{m-1}(X) = 0, \qquad B_m(X) = 0,$$

and for $1 \leqslant i \leqslant m - 1$,

$$\begin{gathered} B_{i-1}(H) = (m - 2i)B_i(Y), \\ B_{i-1}(X) = (i + 1)(-m + i)B_{i+1}(Y), \\ (m - 2i)B_i(X) = (i + 1)(-m + i)B_{i+1}(Y). \end{gathered}$$

The last three equalities give

$$(i + 2)(m - 2i)(-m + i + 1)B_{i+2}(Y) = (i + 1)(-m + i)(m - 2i - 4)B_{i+2}(Y).$$

Therefore, we get either $B_{i+2}(Y) = 0$ or

$$(i+2)(m-2i)(m-i-1) = (i+1)(m-i)(m-2i-4).$$

The ultimate equality implies $m(m+2) = 0$; that cannot hold because m is positive. If i is an integer such that $2 < i+2 < m$ one gets $B_{i+2}(Y) = 0$. This proves that $B(Y) = 0$ and we conclude that $B(X) = B(H) = 0$. Now we show that we can drop the condition that $\mathfrak{r}$ is an irreducible $\mathfrak{s}$-module.

LEMMA 2. *Let $\mathfrak{g}$ be a Lie algebra such that its Levi subalgebras $\mathfrak{s}$ are 3-dimensional and let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ be a Levi decomposition of $\mathfrak{g}$. For any element φ of $h_{\mathfrak{g}}$ the restriction B of φ to $\mathfrak{s}$ takes its values in the center $Z(\mathfrak{g})$.*

PROOF. As in Lemma 1, let us suppose that the ground field is algebraically closed. Because of the simplicity of $\mathfrak{s}$, the radical $\mathfrak{r}$ is a direct sum of irreducible $\mathfrak{s}$-modules

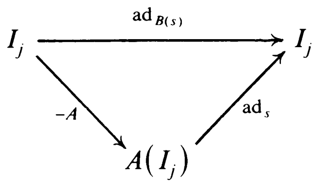
$$\mathfrak{r} = I_1 \oplus \cdots \oplus I_t. \tag{7}$$

We know that if φ is an element in $h_{\mathfrak{g}}$, the linear map B of $\mathfrak{s}$ into $\mathfrak{r}$ which is deduced from φ satisfies the relation $[B(s), s'] = [B(s'), s]$. Take B_{I_j} to be the I_j-component of B. Then Lemma 1 tells us that, for any I_j which has dimension greater than one, we get $B_{I_j} = 0$, so that B takes its values in the subalgebra $\mathfrak{r}^{\mathfrak{s}}$. The relation $[B(s), r] = [A(r), s]$ implies that $[B(s), r]$ lies in the $\mathfrak{s}$-module $[\mathfrak{s}, \mathfrak{r}]$. The subspace $\mathfrak{r}^{\mathfrak{s}}$ being a subalgebra of $\mathfrak{g}$, the term $[B(s), r_0]$ also lies in $\mathfrak{r}^{\mathfrak{s}}$ for any (s, r_0) in $\mathfrak{s} \times \mathfrak{r}^{\mathfrak{s}}$, so that we get $[B(s), r_0] = 0$. Therefore, we see that B takes its values in the center of $\mathfrak{r}^{\mathfrak{s}}$. Thus if s and s' are elements of $\mathfrak{s}$ and if $r \in \mathfrak{r}$ we get

$$[s', [B(s), r]] = [[s', B(s)], r] + [B(s), [s', r]] = [B(s), [s', r]],$$

so that the inner derivation $\mathrm{ad}_{B(s)}$ of $\mathfrak{r}$ is compatible with the action of $\mathfrak{s}$. This means that $\mathrm{ad}_{B(s)}$ is a $\mathfrak{s}$-module morphism.

Suppose that I_j is an irreducible factor of the decomposition (7) with $\dim I_j > 1$. The classical Schur lemma tells us that either $\mathrm{ad}_{B(s)}(I_j)$ is $\{0\}$ or $\mathrm{ad}_{B(s)|I_j}$ is an isomorphism. In the latter case the subspace $J_j = [B(s), I_j]$ is an irreducible $\mathfrak{s}$-module which is not zero. According to the formula (2), one gets the following commutative diagram:



If J_j is different from $\{0\}$ the above diagram implies that A is an isomorphism of I_j on $A(I_j)$ and idem for the restriction to $A(I_j)$ of ad_s. We conclude that $A(I_j)$ is exactly the submodule J_j. Moreover, J_j does not depend to the choice of s in $\mathfrak{s}$. As we deal only with restrictions, the map $s \to \mathrm{ad}_{B(s)} | I_j$ is a linear map of $\mathfrak{s}$ in the space of $\mathfrak{s}$-morphisms of I_j into J_j, so that the Schur lemma implies that the above map has rank one. Finally, we deduce from the relation $-\mathrm{ad}_{B(s)|I_j} \circ A^{-1}_{|J_j} = \mathrm{ad}_s | I_j$ that the kernel of $s \to \mathrm{ad}_{B(s)}|I_j$ is a nonzero ideal of $\mathfrak{s}$ different from $\mathfrak{s}$. The Lie algebra $\mathfrak{s}$ being simple, we get a contradiction.

PROOF OF THEOREM I. Let us keep in mind that the ground field is algebraically closed. Let C be a Cartan subalgebra of the Lie algebra $\mathfrak{s}$. Fix a simple system of roots $\mathcal{Q} = (\alpha_1, \dots, \alpha_k)$ associated to C. We write (X_i, Y_i, H_i) for the Weyl system $(X_{\alpha_i}, Y_{\alpha_i}, H_{\alpha_i})$ corresponding to the system $\mathcal{Q}$. As vector space, the Lie algebra $\mathfrak{s}$ is generated by the system (X_i, Y_i, H_i), $i = 1, 2, \dots, k$. Let (n_{ij}), $i, j = 1, 2, \dots, k$, be the Cartan matrix which is associated to $\mathcal{Q}$. For any $i = 1, 2, \dots, k$, let $\mathfrak{s}_i$ be the Lie algebra $KX_i \oplus KY_i \oplus KH_i$ (see [**2**, Chapter IV, §3]). Now let φ be an element of $h_{\mathfrak{g}}$. Lemma 2 guarantees that the vector subspace $\varphi(\mathfrak{s}_i) = B(\mathfrak{s}_i)$ is contained in the center of the subalgebra $\mathfrak{r} \oplus \mathfrak{s}_i$ of $\mathfrak{g}$. If we consider the 2-cochain $X, Y \to -B[X, Y]$, then (3) is equivalent to the fact that the above 2-cochain is the coboundary of the 1-cochain $X \to B(X)$. Thus the 2-cochain $X, Y \to -B[X, Y]$ must be closed, so that

$$(8) \qquad [X, B[Y, Z]] - [Y, B[X, Z]] + [Z, B[X, Y]] = 0$$

for any (X, Y, Z) in $\mathfrak{s} \times \mathfrak{s} \times \mathfrak{s}$. Now take i, j in $[1, 2, \dots, k]$. According to the Weyl relations we may deduce from (8) that $[X_i, B[H_i, Y_j]] - [Y_j, B[H_i, X_i]] = -n_{ij}[X_i, B(Y_j)] - 2[Y_j, B(X_i)] = -(n_{ij} + 2)[X_i, B(Y_j)] = 0$. On the other hand, the relation (3) gives

$$(n_{ij} + 2)\left[X_i, B(Y_j)\right] = (n_{ji} + 2)\left[X_i, B(Y_i)\right] = 0.$$

For these last equalities to hold, the necessary condition is

$$(9) \qquad \left[X_i, B(Y_j)\right] = 0.$$

Now let us compute the quantity $[H_i, B(Y_j)]$, taking

$$\begin{aligned}\left[H_i, B(Y_j)\right] &= \left[[X_i, Y_i], B(Y_j)\right] = \left[\left[X_i, B(Y_j)\right], Y_i\right] + \left[X_i, \left[Y_i, B(Y_j)\right]\right] \\ &= \left[X_i, \left[Y_i, B(Y_j)\right]\right] = \left[X_i, \left[Y_j, B(Y_i)\right]\right] \\ &= \left[\left[X_i, Y_j\right], B(Y_i)\right] + \left[Y_j, \left[X_i, B(Y_i)\right]\right].\end{aligned}$$

If $i \neq j$, the Weyl relations together with Lemma 2 give

$$\left[\left[X_i, Y_j\right], B(X_i)\right] = 0 \quad \text{and} \quad \left[Y_j, \left[X_i, BY_i\right]\right] = 0,$$

so that, for any i, j in $[1, 2, \dots, k]$,

$$(10) \qquad \left[H_i, B(Y_j)\right] = 0.$$

Finally, (9) and (10) tell us that for any $j = 1, 2, \dots, k$ the element $\varphi(Y_j) = B(Y_j)$ (when it is not zero) is a primitive element in the $\mathfrak{s}$-module $\mathfrak{r}$ with the weight $0 \in C^*$. Therefore let us denote by $\mathfrak{m}_j$ the irreducible $\mathfrak{s}$-module generated by $\varphi(Y_j)$. It is well known that $\mathfrak{m}_j$ is generated as a vector space by the system $Y_1^{m_1} Y_2^{m_2} \cdots Y_k^{m_k}.B(Y_j)$ where one identifies Y_i with the operator $B(Y_j) \to [Y_i, B(Y_j)]$. On the other hand,

$Y_1^{m_1} \cdots Y_k^{m_k}.B(Y_j)$ has the weight $-\Sigma_{i=1}^k m_i\alpha_i$. In particular, let us compute the quantity $[H_t, [Y_i, BY_j]]$, taking

$$\begin{aligned}
[H_t, [Y_i, B(Y_j)]] &= [[X_t, Y_t], [Y_i, B(Y_j)]] \\
&= [X_t, [Y_t, [Y_i, B(Y_j)]]] - [Y_t, [X_t, [Y_i, B(Y_j)]]] \\
&= [X_t, [Y_t, [Y_i, B(Y_j)]]] = [X_t, [[Y_t, Y_i], B(Y_j)]] + [X_t, [Y_i, [Y_t, B(Y_j)]]] \\
&= [X_t, [[Y_t, Y_i], B(Y_j)]] = [X_t, [Y_j, B([Y_t, Y_i])]] = [Y_j, [X_t, B([Y_t, Y_i])]] \\
&= [Y_j, [[Y_t, Y_i], B(X_t)]] = [Y_j, [[Y_t, B(X_t)], Y_i]] + [Y_j, [Y_t, [Y_i, B(X_t)]]] \\
&= [Y_j, [Y_t, [Y_i, B(X_t)]]] = [Y_j, [Y_t, [X_t, B(Y_i)]]] = 0.
\end{aligned}$$

This gives the identity

$$[H_t, [Y_i, BY_j]] = 0 \tag{11}$$

for any i, j, t in $[1, 2, \ldots, k]$. From the formulas (9) and (10) one gets $[X_t, [Y_i, B(Y_j)]] = 0$ for any i, j, t in $[1, 2, \ldots, k]$. Thus (11) implies that $[Y_i, B(Y_j)]$ (if not zero) is a primitive element in $\mathfrak{r}$ with weight $0 \in K^*$. This contradicts the fact that any $Y_i B(Y_j) = [Y_i, B(Y_j)]$ is associated to the weight $-\alpha_i$. We see that B takes its values in the center of the Lie algebra $\mathfrak{g}$, which proves part of (i). Let A_0 (resp. A_1) be the restriction to $\mathfrak{r}^{\mathfrak{s}}$ (resp. to $[\mathfrak{r}, \mathfrak{s}]$) of $\varphi \in h_{\mathfrak{g}}$. It is a consequence of the exact sequence $0 \to h_{\mathfrak{g}}^0 \to h_{\mathfrak{g}} \to \mathrm{ad}(\mathfrak{g})^{(1)} \to 0$ that the subspace $h_{\mathfrak{g}}(\mathfrak{g})$ generated by all the $\varphi(X)$, $\varphi \in h_{\mathfrak{g}}$, $X \in \mathfrak{g}$, is an ideal of the Lie algebra $\mathfrak{g}$. In fact, take (X, φ) in $\mathfrak{g} \times h_{\mathfrak{g}}$ and define $X\varphi$ to be the element of $\mathrm{End}(\mathfrak{g})$ defined by $Y \to (X\varphi)(Y) = [X, \varphi(Y)] - \varphi[X, Y]$. One easily verifies that the map $X\varphi$ belongs to $h_{\mathfrak{g}}$, so that for any X and X' in $\mathfrak{g}$ and for any φ in $h_{\mathfrak{g}}$ the element $[X, \varphi(X')]$ lies in $h_{\mathfrak{g}}(\mathfrak{g})$. Now take r and r' in $\mathfrak{r} = \mathfrak{r}^{\mathfrak{s}} + [r, s]$. We may write

$$r = r_0 + r_1, \qquad r' = r_0' + r_1'$$

where r_0 and r_0' (resp. r_1 and r_1') belong to $\mathfrak{r}^{\mathfrak{s}}$ (resp. to $[\mathfrak{r}, \mathfrak{s}]$), to get

$$[A_0 r_0 + A_1 r_1, r_0' + r_1'] = [A_0 r_0' + A_1 r_1', r_0 + r_1].$$

This equation yields the three identities

$$[A_1 r_1, r_1'] = [A_1 r_1', r_1], \tag{12}$$

$$[A_0 r_0, r_0'] = [A_0 r_0', r_0], \tag{13}$$

$$[A_0 r_0, r_1'] = [A_1 r_1', r_0]. \tag{14}$$

Given an element s in $\mathfrak{s}$, (14) implies

$$[s, [A_1 r_1, r_1']] = [[s, A_1 r_1], r_1'] + [A_1 r_1, [s, r_1']].$$

Relation (2) together with Lemma 2 implies that the ideal $h_{\mathfrak{g}}(\mathfrak{g})$ lies in the subalgebra $\mathfrak{r}^{\mathfrak{s}}$, so that we get

$$[s, [A_1 r_1, r_1']] = [A_1 r_1, [s, r_1']].$$

The first member $[s,[A_1r_1, r_1']]$ lies in the subspace $[\mathfrak{r}, \mathfrak{s}]$, while the second member lies in the ideal $h_{\mathfrak{g}}(\mathfrak{g})$, so that $[A_1r_1,[s, r_1']] = 0$, and we obtain the equality

$$[A_1r_1,[\mathfrak{r}, \mathfrak{s}]] = \{0\}.$$

Bracketing $s \in \mathfrak{s}$ with both sides of (14) one gets

$$[s,[A_0r_0, r_1']] = [A_0r_0,[s, r_1']] = [s,[A_1r_1', r_0]] = 0.$$

Our conclusion is

$$[A_0r_0,[s, r_1']] = [A_1[s, r_1'], r_0] = 0.$$

That ends the proof of (i). Proving (i), we established (13) and $[A_0r_0,[\mathfrak{r}, \mathfrak{s}]] = \{0\}$, so that (ii) holds and Theorem I is proved.

Applying Theorem I to a particular situation, we get the following

COROLLARY I.1. *Let $\mathfrak{g}$ be a Lie algebra. Keeping the previous notations, suppose that the subalgebra $\mathfrak{r}^{\mathfrak{s}}$ is commutative. Then the Lie algebra $h_{\mathfrak{g}}$ is zero if and only if the center of $\mathfrak{g}$ is zero.*

PROOF. First, suppose that $h_{\mathfrak{g}}$ is zero. Then because of the inclusion of $h_{\mathfrak{g}}^0 = Z(\mathfrak{g}) \otimes \mathfrak{g}^*$ in $h_{\mathfrak{g}}$ the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is zero. Second, suppose the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is zero. Let φ be an element of $h_{\mathfrak{g}}$. For any element r_0 in $\mathfrak{r}^{\mathfrak{s}}$ the assertion (ii) of Theorem I tells us that the element $\varphi(r_0)$ commutes with the subspace $[\mathfrak{s}, \mathfrak{r}]$. Since $\mathfrak{r}^{\mathfrak{s}}$ is supposed to be commutative, $\varphi(r_0)$ lies in the center of $\mathfrak{g}$, which implies that the map φ is identically zero.

EXAMPLE 2.1. Let $\mathcal{G}$ be any semisimple connected Lie group with Lie algebra $\mathfrak{g}$. Theorem I tells us that the Lie algebra $h_{\mathfrak{g}}$ is zero, so that the exact sequence $0 \to h_{\mathfrak{g}}^0 \to h_{\mathfrak{g}} \to (\mathrm{ad}(\mathfrak{g}))^{(1)} \to 0$ gives $\mathrm{ad}(\mathfrak{g})^{(1)} = \{0\}$.

Keeping our previous notations, we have the following result.

THEOREM II. *Let $\mathfrak{g}$ be a Lie algebra and let us denote by $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ the largest ideal of $\mathfrak{g}$ contained in $\mathfrak{r}^{\mathfrak{s}}$. For any decomposition $\mathfrak{g} = \mathfrak{r} \oplus [\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}$ we have* $h_{\mathfrak{g}} = \mathrm{Hom}_K([\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}, Z(\mathfrak{g})) \oplus (h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}_K(\mathfrak{r}^{\mathfrak{s}}, D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})))$.

PROOF OF THEOREM II. Let us recall the construction of $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ as it is given in [**1**]. We define the sequence $D_{\mathfrak{g}}^{i}(\mathfrak{r}^{\mathfrak{s}})$ by setting $D_{\mathfrak{g}}^{0}(\mathfrak{r}^{\mathfrak{s}}) = \mathfrak{r}^{\mathfrak{s}}$ and $D_{\mathfrak{g}}^{i+1}(\mathfrak{r}^{\mathfrak{s}}) = D^1(D_{\mathfrak{g}}^{i}(\mathfrak{r}^{\mathfrak{s}})) = \{X \in D_{\mathfrak{g}}^{i}(\mathfrak{r}^{\mathfrak{s}})/[X, \mathfrak{g}] \subset D_{\mathfrak{g}}^{i}(\mathfrak{r}^{\mathfrak{s}})\}$, $i \geqslant 0$. The ideal $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ is the limit of the sequence $D_{\mathfrak{g}}^{i}(\mathfrak{r}^{\mathfrak{s}})$.

First let us observe that $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ is equal to $\mathfrak{r}^{\mathfrak{s}} \cap \mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$. Indeed, if (r_0, r_1, s) is an element of $\mathfrak{r}^{\mathfrak{s}} \times \mathfrak{r} \times \mathfrak{s}$, we have $[s,[r_0, r_1]] = [r_0,[s, r_1]]$. The first member $[s,[r_0, r_1]]$ lies in the subspace $[\mathfrak{r}, \mathfrak{s}]$, so that we have the inclusion $[\mathfrak{r}^{\mathfrak{s}},[\mathfrak{r}, \mathfrak{s}]] \subset [\mathfrak{r}, \mathfrak{s}]$. Now, if X is an element of $\mathfrak{r}^{\mathfrak{s}} \cap \mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$ and $(r_0, r_1) \in \mathfrak{r}^{\mathfrak{s}} \times [\mathfrak{r}, \mathfrak{s}]$, we get

$$[[X, r_0], r_1] = [[X, r_1], r_0] + [X,[r_0, r_1]] = 0.$$

We conclude that $\mathrm{ad}_X(\mathfrak{g})$ is included in $\mathfrak{r}^{\mathfrak{s}} \cap \mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$, so that $\mathfrak{r}^{\mathfrak{s}} \cap \mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$ is included in $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$. Conversely, let (x, y, s) be an element of $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}) \times \mathfrak{r} \times \mathfrak{s}$, so we have $[[x, y], s] - [x,[y, s]] = 0$. The term $[[x, y], s]$ belongs to $[\mathfrak{r}, \mathfrak{s}]$ while $[x,[y, s]]$

belongs to $[\mathfrak{r}, \mathfrak{s}]$, so that $\mathrm{ad}_x([\mathfrak{r}, \mathfrak{s}]) = \{0\}$. Once we get $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}) = \mathfrak{r}^{\mathfrak{s}} \cap \mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$, Theorem I implies the inclusion

$$h_{\mathfrak{g}} \subset \mathrm{Hom}_K([\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}, Z(\mathfrak{g})) \oplus h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}_K\big(\mathfrak{r}^{\mathfrak{s}}, D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})\big).$$

Conversely, any element (φ_0, φ_1) of $\mathrm{Hom}_K([\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}, Z(\mathfrak{g})) \oplus h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}_K(\mathfrak{r}^{\mathfrak{s}}, D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}))$ defines a unique element $\hat{\varphi}$ of $h_{\mathfrak{g}}$ by setting $\hat{\varphi}(r_0 + r_1 + s) = \varphi_0(r_1 + s) + \varphi_1(r_0)$. Indeed, according to the previous results we have

$$\big[\varphi_0(r_1 + s) + \varphi_1(r_0), r_0' + r_1' + s'\big] = \big[\varphi_1(r_0), r_0'\big]$$

and

$$\big[\varphi_0(r_1' + s') + \varphi_1(r_0'), r_0 + r_1 + s\big] = \big[\varphi_1(r_0'), r_0\big]$$

where (r_0, r_1, s) and (r_0', r_1', s') are elements of $\mathfrak{r}^{\mathfrak{s}} \times [\mathfrak{r}, \mathfrak{s}] \times \mathfrak{s} \simeq \mathfrak{g}$. Since φ_1 is an element of $h_{\mathfrak{r}^{\mathfrak{s}}}$ we have

$$[\hat{\varphi}(X), Y] = [\hat{\varphi}(Y), X]$$

for any pair (X, Y) in $\mathfrak{g} \times \mathfrak{g}$. That proves the inclusion

$$\mathrm{Hom}_K([\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}, Z(\mathfrak{g})) \oplus h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}_K\big(\mathfrak{r}^{\mathfrak{s}}, D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})\big) \subset h_{\mathfrak{g}}$$

which ends the proof of Theorem II.

Corollary II.1. *For a Lie algebra $\mathfrak{g} \simeq \mathfrak{r}^{\mathfrak{s}} \oplus [\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}$, the space $h_{\mathfrak{g}}$ is zero if and only if the ideal $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ is zero.*

Proof. The sufficient condition is trivial. Conversely, let us suppose that $h_{\mathfrak{g}}$ is zero. As we did before, we may suppose that the ground field K is algebraically closed. If $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ were different from zero, by applying a classical Lie theorem to the solvable Lie algebra $\mathfrak{r}$, one could find a nonzero element v_0 in $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ and a linear form $\lambda \in \mathfrak{r}^*$ such that for any $X \in \mathfrak{r}$ one gets

$$[X, v_0] = \lambda(X) v_0.$$

Since $h_{\mathfrak{g}}$ is zero, so is $Z(\mathfrak{g})$, so that the linear form λ is different from zero. Let us define the linear map φ of $\mathfrak{g}$ into itself by putting

$$\varphi(r + s) = \lambda(r) v_0$$

for all $(r, s) \in \mathfrak{r} \times \mathfrak{s}$. Thus, given (r, s) and (r', s') in $\mathfrak{r} \times \mathfrak{s}$ we have

$$\big[\varphi(r + s), r' + s'\big] = \big[\lambda(r) v_0, r' + s'\big] = \lambda(r)[v_0, r'] = -\lambda(r)\lambda(r') v_0$$

and

$$\big[\varphi(r' + s'), r + s\big] = \big[\lambda(r') v_0, r + s\big] = \lambda(r')[v_0, r] = -\lambda(r')\lambda(r) v_0.$$

We must conclude that the linear map φ is a nonzero element of $h_{\mathfrak{g}}$, which is contrary to our assumption. Corollary II.1 is proved.

Corollary II.2. *Let $\mathfrak{g}$ be a Lie algebra with nilpotent radical $\mathfrak{r}$. Then if the center $Z(\mathfrak{g})$ is zero so is the Lie algebra $h_{\mathfrak{g}}$.*

Proof. By Corollary II.1, if $h_{\mathfrak{g}}$ were not zero, the same would hold for the ideal $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$. Applying the theorem of Engel, one would have a nonzero element X_0 in $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}) \cap Z(\mathfrak{r})$. Such an element X_0 would lie in the center $Z(\mathfrak{g})$.

Keeping in mind our geometrical interest in the prolongation $\mathrm{ad}(\mathfrak{g})^{(1)}$, the previous results lead to this result.

THEOREM III. *Let* $\mathfrak{g}$ *be a Lie algebra with a decomposition* $\mathfrak{g} \simeq \mathfrak{r}^{\mathfrak{s}} \oplus [\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}$. *The first prolongation* $(\mathrm{ad}(\mathfrak{g}))^{(1)}$ *of the linear space* $\mathrm{ad}(\mathfrak{g})$ *is isomorphic to the factor space* $h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}_K(\mathfrak{r}^{\mathfrak{s}}, D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}))/h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}_K(\mathfrak{r}^{\mathfrak{s}}, Z(\mathfrak{g}))$.

The proof is an immediate consequence of Theorem II together with the exact sequence $0 \to \mathrm{Hom}(\mathfrak{g}, Z(\mathfrak{g})) \to h_{\mathfrak{g}} \to (\mathrm{ad}(\mathfrak{g}))^{(1)} \to 0$.

COROLLARY III.1. *Let* $\mathfrak{g}$ *be a Lie algebra with a Levi decomposition* $\mathfrak{r} \oplus \mathfrak{s}$. *If* $\mathfrak{r}^{\mathfrak{s}}$ *is commutative then* $(\mathrm{ad}(\mathfrak{g}))^{(1)}$ *is zero.*

PROOF. We already proved that the ideal $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ is equal to $\mathfrak{r}^{\mathfrak{s}} \cap \mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$. Since $\mathfrak{r}^{\mathfrak{s}}$ is commutative we get $h_{\mathfrak{r}^{\mathfrak{s}}} = \mathrm{Hom}(\mathfrak{r}^{\mathfrak{s}}, \mathfrak{r}^{\mathfrak{s}})$ and $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}) = \mathfrak{g}^{\mathfrak{r}^{\mathfrak{s}}} = \mathfrak{g}^{\mathfrak{g}} = Z(\mathfrak{g})$. Therefore, we have $h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}(\mathfrak{r}^{\mathfrak{s}}, Z(\mathfrak{g})) = \mathrm{Hom}(\mathfrak{r}^{\mathfrak{s}}, Z(\mathfrak{g}))$.

COROLLARY III.2. *Let* $\mathfrak{g}$ *be a Lie algebra such that some* $\mathfrak{r}^{\mathfrak{s}}$ *is an ideal in* $\mathfrak{g}$. *Then* $(\mathrm{ad}(\mathfrak{g}))^{(1)}$ *is isomorphic to* $(\mathrm{ad}(\mathfrak{r}^{\mathfrak{s}}))^{(1)}$.

PROOF. Since $\mathfrak{r}^{\mathfrak{s}}$ is an ideal of $\mathfrak{g}$ we have $\mathfrak{r}^{\mathfrak{s}} = D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}) = \mathfrak{r}^{\mathfrak{s}} \cap \mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$. On the other hand, we can write $[\mathfrak{r}^{\mathfrak{s}}]^{\mathfrak{r}^{\mathfrak{s}}} = Z(\mathfrak{r}^{\mathfrak{s}})$ so that

$$Z(\mathfrak{r}^{\mathfrak{s}}) \subset [\mathfrak{r}^{\mathfrak{s}} \oplus [\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}]^{\mathfrak{r}^{\mathfrak{s}}} = Z(\mathfrak{g}).$$

That proves the equality $Z(\mathfrak{r}^{\mathfrak{s}}) = Z(\mathfrak{g})$. We apply Theorem III and we obtain $(\mathrm{ad}(\mathfrak{g}))^{(1)} \simeq h_{\mathfrak{r}^{\mathfrak{s}}}/\mathrm{Hom}(\mathfrak{r}^{\mathfrak{s}}, Z(\mathfrak{r}^{\mathfrak{s}})) \simeq (\mathrm{ad}(\mathfrak{r}^{\mathfrak{s}}))^{(1)}$.

PROPOSITION 2.1. *Let* $\mathfrak{g}$ *be a Lie algebra. Then for any Levi subalgebra* $\mathfrak{s}$, *the subspace* $h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}(\mathfrak{r}^{\mathfrak{s}}, D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}))$ *is an ideal of* $h_{\mathfrak{r}^{\mathfrak{s}}}$. *Furthermore, the subspace* $h_{\mathfrak{r}^{\mathfrak{s}}}(\mathfrak{r}^{\mathfrak{s}}) \cap D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ *is an ideal of* $\mathfrak{g}$ *which does not depend on the choice of* $\mathfrak{s}$.

PROOF. Let $\varphi \in h_{\mathfrak{r}^{\mathfrak{s}}}$ and let $\psi \in h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}(\mathfrak{r}^{\mathfrak{s}})$ so that $[\varphi, \psi]$ lies in $h_{\mathfrak{r}^{\mathfrak{s}}}$. It remains to prove that for all (r_0, r, s) in $\mathfrak{r}^{\mathfrak{s}} \times \mathfrak{r} \times \mathfrak{s}$ we have $[[\varphi, \psi](r_0), [r, s]] = 0$. Here

$$\begin{aligned} [[\varphi, \psi](r_0), [r, s]] &= [\varphi\psi(r_0) - \psi\varphi(r_0), [r, s]] \\ &= [\varphi\psi(r_0), [r, s]] = [[\varphi\psi(r_0), r], s]. \end{aligned}$$

If $\mathfrak{i}$ is an ideal of a Lie algebra $\mathfrak{g}$ and $\varphi \in h_{\mathfrak{g}}$, for all $v \in \mathfrak{g}$, we get $[\varphi(\mathfrak{i}), v] = [\varphi(v), \mathfrak{i}] \subset \mathfrak{i}$ so that in the previous case $[\varphi\psi(r_0), r]$ lies in $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ and the first statement holds. Now let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be two Levi subalgebras of $\mathfrak{g}$. A theorem of Malcev and Harish-Chandra tells us that there is an element X_0 of the nilpotent radical of $\mathfrak{g}$ such that $\mathfrak{s}_2 = e^{\mathrm{ad}(X_0)}(\mathfrak{s}_1)$. Since $e^{\mathrm{ad}(X_0)}$ preserve every ideal of $\mathfrak{g}$ we have $e^{\mathrm{ad}(X_0)}(D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}_1})) = D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}_1})$. Since $\mathfrak{r}^{\mathfrak{s}_1}$ and $\mathfrak{r}^{\mathfrak{s}_2}$ must be conjugated by $e^{\mathrm{ad}(X_0)}$, so must the ideals $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}_1})$ and $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}_2})$, and one concludes that $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}_1}) = D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}_2})$.

Let us illustrate the main results by a few examples.

EXAMPLE 1. Let $\mathfrak{g}$ be a semisimple Lie algebra. The radical $\mathfrak{r}$ of $\mathfrak{g}$ being zero, Theorem I gives $h_{\mathfrak{g}} = \{0\}$, so that we get $(\mathrm{ad}(\mathfrak{g}))^{(1)} = \{0\}$.

EXAMPLE 2. If $\mathfrak{g}$ is a reductive Lie algebra then we get $Z(\mathfrak{g}) = \mathfrak{r}^{\mathfrak{s}} = \mathfrak{r}$. By the Corollary III.1, we have $(\mathrm{ad}(\mathfrak{g}))^{(1)} = \{0\}$.

EXAMPLE 3. Let $\mathfrak{g}$ be the affine Lie algebra $\mathbf{R}^2 \times sl(2, \mathbf{R})$ and let (u, X) be an element of $\mathfrak{g}$. Then we get $\mathrm{ad}(u, X) = [\begin{smallmatrix} X & -\delta u \\ 0 & \mathrm{ad}\, X \end{smallmatrix}]$ where $\delta u(Y) = Yu$ for $Y \in sl(2, \mathbf{R})$. Since $sl(2, \mathbf{R})$ is irreducible on $\mathbf{R}^2$, we have $\mathfrak{r}^{sl(2,\mathbf{R})} = \{0\}$, and Theorem III gives $(\mathrm{ad}(\mathfrak{g}))^{(1)} = \{0\}$.

EXAMPLE 4. Let $\mathfrak{g}$ be the Lie algebra $\mathbf{R}^5 \# sl(2, \mathbf{R})$, with the bracket given by

$$\begin{aligned}&[((a, b, c, \alpha, \beta), X), ((a', b', c', \alpha', \beta'), X')] \\ &\qquad = (bc' - b'c + \alpha\beta' - \alpha'\beta, 0, 0, X(\alpha', \beta') - X'(\alpha, \beta), [X, X']).\end{aligned}$$

Let us take $\mathfrak{s}$ to be the subalgebra $\{0,0,0,0,0\} \# sl(2, \mathbf{R})$. It is clear that

$$\begin{aligned}\mathfrak{r}^{\mathfrak{s}} &= \mathbf{R}^3 \times \{(0,0)\} \# \{0\}, \\ [\mathfrak{r}, \mathfrak{s}] &= \{(0,0,0)\} \times \mathbf{R}^2 \# \{0\}, \\ Z(\mathfrak{g}) &= Z(\mathfrak{r}^{\mathfrak{s}}) = \mathbf{R} \times \{0,0,0,0\} \# \{0\}.\end{aligned}$$

Since $\mathfrak{r}^{\mathfrak{s}}$ is an ideal in $\mathfrak{g}$, by Corollary III.2, $(\mathrm{ad}(\mathfrak{g}))^{(1)}$ is isomorphic to the first prolongation of the inner derivations of the Heisenberg algebra $\mathfrak{r}^{\mathfrak{s}}$, which is the set of those $S \in \mathrm{Hom}(\mathfrak{r}^{\mathfrak{s}} \times \mathfrak{r}^{\mathfrak{s}}, \mathfrak{r}^{\mathfrak{s}})$ defined by

$$S((a, b, c), (a', b', c')) = ((\lambda b + \mu c)b' + (\mu b + \nu c)(c', 0, 0)$$

where $(\lambda, \mu, \nu) \in \mathbf{R}^3$.

3. Return to differential geometry. We begin by explaining the geometric interest of the ideal $\mathcal{I} = h_{\mathfrak{r}^{\mathfrak{s}}}(\mathfrak{r}^{\mathfrak{s}}) \cap D^{\infty}_{\mathfrak{g}}(\mathfrak{r}^{\mathfrak{s}})$. One easily verifies that $\mathcal{I}$ is the minimal ideal of $\mathrm{ad}(\mathfrak{g})$ such that the first prolongation of $\mathrm{ad}(\mathfrak{g})$ coincides with that of $\mathcal{I}$, so that

$$(\mathrm{ad}(\mathfrak{g}))^{(1)} = \mathcal{I}^{(1)}.$$

This gives another understanding of Proposition 2.1. Moreover, the geometrical statement $(\mathfrak{R}_1)$ is a direct consequence of the above remark. The geometrical statements $(\mathfrak{R}_2)$, $(\mathfrak{R}_3)$ and $(\mathfrak{R}_4)$ are consequences of Corollaries I.1, II.2 and III.2, respectively.

Take a left invariant torsion free connection ∇ on a Lie group $\mathcal{G}$ and assume that its holonomy group is a subgroup of $\mathrm{Int}(\mathfrak{g})$. One observes that the space $\mathrm{ad}(\mathfrak{g})^{(1)}$ provides a parametrization of the set of all left invariant torsion free connections which are adapted to the $\mathrm{Int}(\mathfrak{g})$-structure obtained from the holonomy bundle of ∇ (see §1).

Our last remark applies to the case of solvable Lie groups which cannot be handled by the techniques used in this work. We may observe that for such a Lie group $\mathcal{G}$ with Lie algebra $\mathfrak{g}$ the linear Lie algebra $h_{\mathfrak{g}}$ is always different from zero. Let $\mathfrak{g}$ be a solvable Lie algebra. If $Z(\mathfrak{g}) \neq 0$, $\mathrm{Hom}_K(\mathfrak{g}, Z(\mathfrak{g}))$ is included in $h_{\mathfrak{g}}$. If $Z(\mathfrak{g}) = 0$ then any ξ in $Z([\mathfrak{g}, \mathfrak{g}]) - \{0\}$ gives us a nonzero element ad_{ξ} in $h_{\mathfrak{g}}$. Thus, for any solvable Lie algebra with $Z(\mathfrak{g}) = 0$ the first prolongation $\mathrm{ad}(\mathfrak{g})^{(1)}$ is never zero.

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UNIVERSITÉ DES SCIENCES ET TECHNIQUES DU LANGUEDOC, PLACE EUGÈNE BARAILLON, 34060-MONTPELLIER CEDEX, FRANCE