TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 281, Number 1, January 1984

M-STRUCTURE IN THE BANACH ALGEBRA OF OPERATORS ON $C_0(\Omega)$

BY

P. H. FLINN AND R. R. SMITH

ABSTRACT. The M-ideals in $B(C_0(\Omega))$, the space of continuous linear operators on $C_0(\Omega)$, are determined where Ω is a locally compact Hausdorff countably paracompact space. A one-to-one correspondence between M-ideals in $B(C_0(\Omega))$, open subsets of the Stone-Čech compactification of Ω, and lower semicontinuous Hermitian projections in $B(C_0(\Omega))^{**}$ is established.

1. Introduction. The M-ideal theory of Banach algebras has been developed in the last few years by various authors [**10–12**, **14**, **15**, **18–21**] and is now reasonably well understood. The requirement that a subspace be an M-ideal is very restrictive, and so, in general, an algebra contains few M-ideals. For example, if H is a separable Hilbert space, the only nontrivial M-ideal in $B(H)$ is the ideal of compact operators. The object of this paper is to present a class of Banach algebras which have complex but completely determined M-ideal structures. Let $C_0(\Omega)$ denote the algebra of continuous complex valued functions vanishing at infinity on a locally compact Hausdorff space Ω. The spaces to be considered are the algebras $B(C_0(\Omega))$ of bounded linear operators on $C_0(\Omega)$. It is well known that a closed subspace M of $C_0(\Omega)$ is an M-ideal if and only if $M = \{f \in C_0(\Omega)\colon \chi_U f = f\}$ where U is an arbitrary but fixed open subset of Ω. We will show that if Ω is compact and Hausdorff, an analogous result holds in $B(C(\Omega))$; that is, M-ideals in $B(C(\Omega))$ correspond to open subsets of Ω. The case in which Ω is locally compact and Hausdorff is more complicated, and we will show that under an extra topological assumption on Ω the M-ideals in $B(C_0(\Omega))$ correspond to open subsets of the Stone-Čech compactification $\beta\Omega$ of Ω. Finally, we characterize the M-summands of $B(C_0(\Omega))$ as being those M-ideals for which the associated U is both open and closed in Ω.

The first step is to establish a one-to-one correspondence between the sets of M-summands and Hermitian projections in $B(C_0(\Omega))^{**}$. Only certain M-summands are the double annihilators of M-ideals, and so it is necessary to characterize the associated projections. This is accomplished in terms of a semicontinuity condition, which then allows a complete description of the M-ideal structure of $B(C_0(\Omega))$ in terms of the open subsets of $\beta\Omega$.

Received by the editors January 4, 1983. Presented at the 802nd meeting of the AMS, March 1983.
1980 *Mathematics Subject Classification*. Primary 46H10; Secondary 46E15.
Key words and phrases. M-ideal, M-summand, Banach algebra, Hermitian projections, lower semicontinuous projections, countably paracompact space.

It will be helpful to recall some relevant definitions and facts concerning M-ideals in Banach algebras needed in the sequel. For the general theory of M-ideals we refer the reader to [**2, 3**].

Let A be a Banach algebra with identity I. The state space S of A is the w^*-compact convex subset of A^* given by $S = \{\rho \in A^*\colon\ \rho(I) = 1 = \|\rho\|\}$. In dealing with complex scalars it is often useful to consider the w^*-compact convex set $K = \mathrm{co}\{S \cup -iS\}$. Denote by $A(K)$ and $A^b(K)$ the continuous real valued affine and bounded real valued affine functions, respectively, on K. The map $\theta\colon A \to A(K)$ given by $\theta(a)(k) = \operatorname{Re} k(a)$ is easily seen to be a surjective real isomorphism. Since, for any $a \in A$ and $s \in S$, $s(a) = \operatorname{Re} s(a) + i \operatorname{Re}(-is(a))$, we see that $\theta(a)|_S$ and $\theta(a)|_{-iS}$ are just the real and imaginary parts of $\theta(a)$. θ extends to an isomorphism between A^{**} and $A^b(K)$ which we will also denote by θ. When we say that $z \in A^{**}$ is lower semicontinuous we mean that $\theta(z)$ is lower semicontinuous as a function on K with its w^*-topology.

The numerical range $W(a)$ of $a \in A$ is defined as a compact convex subset of the complex plane by $W(a) = \{\rho(a)\colon\ \rho \in S\}$. An element $h \in A$ is Hermitian if $W(a) \subseteq \mathbf{R}$, and we denote the real Banach space of Hermitian elements of A by $\mathcal{H}(A)$. In the case where Ω is a compact Hausdorff space, for example, $\mathcal{H}(C(\Omega)) = C_{\mathbf{R}}(\Omega)$ (the real valued continuous functions on Ω) whereas $\mathcal{H}(B(C(\Omega)))$ is the set of operators T given by multiplication by elements of $C_{\mathbf{R}}(\Omega)$. That is, $T \in \mathcal{H}(B(C(\Omega)))$ if and only if there exists an $h \in C_{\mathbf{R}}(\Omega)$ for which $Tf = hf$ for any $f \in C(\Omega)$ [**5**]. If we call this operator M_h, then the map $\pi\colon C_{\mathbf{R}}(\Omega) \to B(C(\Omega))$, defined by $\pi(h) = M_h$, is an isometric isomorphism of $C_{\mathbf{R}}(\Omega)$ onto $\mathcal{H}(B(C(\Omega)))$. If Ω is a locally compact Hausdorff space, then $T \in \mathcal{H}(B(C_0(\Omega)))$ if and only if $T = M_h$ where now $h \in C^b_{\mathbf{R}}(\Omega)$, the algebra of bounded real valued continuous functions on Ω. Again, $\pi\colon C^b_{\mathbf{R}}(\Omega) \to \mathcal{H}(B(C_0(\Omega)))$ is a surjective isometric isomorphism.

If A is a unital Banach algebra, then A^{**} endowed with the Arens multiplication is also a Banach algebra whose identity is jI where I is the identity element of A, and j denotes the canonical injection of A into A^{**} (cf. [**5, 6**] for details). We observe here only a few facts concerning the Arens multiplication.

(1) $F \in A^{**}$ is Hermitian if and only if $F|_S \subseteq \mathbf{R}$ where S is the state space of A.

(2) If $a \in A$ and $F_\alpha \xrightarrow{w^*} F \in A^{**}$, then $aF_\alpha \xrightarrow{w^*} aF$.

(3) If $F \in A^{**}$ and $G_\alpha \xrightarrow{w^*} G$, then $G_\alpha F \xrightarrow{w^*} GF$.

See [**5**] for a proof of (1). In (2) we have identified a with $ja \in A^{**}$ and will do this when no confusion is likely to arise. It is straightforward to prove (2) and (3) using the definition of the Arens multiplication. We note, in passing, that (2) is not generally true if $a \in A^{**} \setminus A$, and so left multiplication need not be w^*-continuous [**7**].

A closed subspace M of a Banach space X is an L-summand (respectively, an M-summand) if there exists a closed subspace $\tilde{M}$ for which $X = M \oplus \tilde{M}$ and $\|m + \tilde{m}\| = \|m\| + \|\tilde{m}\|$ (respectively, $\|m + \tilde{m}\| = \max\{\|m\|, \|\tilde{m}\|\}$) for every $m \in M$ and $\tilde{m} \in \tilde{M}$. The natural projection $P\colon X \to M$ is called an L-projection (M-projection), and obviously we could characterize L-summands and M-summands as ranges

of L- and M-projections. A closed subspace $M \subseteq X$ is an M-ideal if its annihilator $M^{\perp} \subseteq X^*$ is an L-summand in X^*. Clearly, M-summands are M-ideals although the reverse need not be true; consider c_0 as an M-ideal in l_∞. It is also easy to see that if M is an M-ideal in X then the double annihilator $M^{\perp\perp}$ is an M-summand in X^{**}.

2. M-summands in $B(C_0(\Omega))^{**}$. Suppose that A is a unital Banach algebra with identity I, that J and $\tilde{J}$ are complementary L-summands in A^* (i.e. $A^* = J \oplus_{l_1} \tilde{J}$), and that P is the associated L-projection of A^* onto $\tilde{J}$. Then P^*: $A^{**} \to A^{**}$ is an M-projection with range $J^{\perp}$. Let $P^*(I) = z$. Since the state space S of A is a convex direct sum of $F_1 = S \cap J$ and $F_2 = S \cap \tilde{J}$, $z|_{F_1} = 0$ and $z|_{F_2} = 1$ it follows that $z(S) \subseteq [0, 1]$. It is shown in **[20]** that $z = z^2$, so by (1) z is an Hermitian projection. Conversely, if M is an M-summand in A^{**} then M is w^*-closed **[8]**, so the associated M-projection P is the transpose of an L-projection Q: $A^* \to A^*$. Thus, $P(I) = z$ is an Hermitian projection, and this establishes the connection between M-summands and certain Hermitian projections. In general, not every Hermitian projection on A^{**} gives rise to an M-summand in A^{**}, but it will be shown in this section that this is the case for $B(C_0(\Omega))$.

We will employ the technique of approximating the given Hermitian projection by operators of thin numerical range, and so some facts about such operators will be established. This set of lemmas is based on arguments presented in **[5]**.

For $\varepsilon > 0$ let R_ε denote the rectangle with vertices at $-\varepsilon \pm i\varepsilon$, $1 + \varepsilon \pm i\varepsilon$. In the following four lemmas it will be assumed that $T \in B(C_0(\Omega))$ is an operator whose numerical range $W(T)$ is contained in R_ε. Notice that any results obtained for such an operator T are also valid for $I - T$.

LEMMA 2.1. *Suppose that $f \in C_0(\Omega)$ and $x_0 \in \Omega$ satisfy $f(x_0) = \|f\| = 1$. Then $(Tf)(x_0) \in R_\varepsilon$.*

PROOF. Define a linear functional ψ on $B(C_0(\Omega))$ by $\psi(U) = (Uf)(x_0)$ for $U \in B(C_0(\Omega))$. The hypothesis on f and x_0 implies that $\psi \in S$ so that $(Tf)(x_0) \in W(T) \subseteq R_\varepsilon$.

LEMMA 2.2. *Suppose that $f \in C_0(\Omega)$ satisfies $f \geqslant 0$, $\|f\| \leqslant 1$ and $f(x_0) = 0$. Then $|(Tf)(x_0)| \leqslant 4\varepsilon$.*

PROOF. Choose a real valued function $g \in C_0(\Omega)$ of unit norm so that $g \geqslant f$ and $g(x_0) = 1$. Then the functions g, $g - f$, $(g - f^2)^{1/2}$ and $(g - f^2)^{1/2} + if$ all satisfy the hypothesis of Lemma 2.1 at the point $x_0 \in \Omega$. Thus,

$$|\mathrm{Im}(Tg)(x_0)| \leqslant \varepsilon \quad \text{and} \quad |\mathrm{Im}(Tg - Tf)(x_0)| \leqslant \varepsilon$$

from which it follows that $|\mathrm{Im}(Tf)(x_0)| \leqslant 2\varepsilon$. A similar computation for the second pair of functions, combined with the observation that $\mathrm{Im}(iTf)$ equals $\mathrm{Re}\, Tf$, leads to $|\mathrm{Re}(Tf)(x_0)| \leqslant 2\varepsilon$. It is now clear that $|(Tf)(x_0)| \leqslant 4\varepsilon$. $\square$

LEMMA 2.3. *Suppose that $f \in C_0(\Omega)$ satisfies $f(x_0) = 0$. Then $|(Tf)(x_0)| \leqslant 16\|f\|\varepsilon$.*

PROOF. Let $h = f/\|f\|$, and decompose h as $h = h_1 - h_2 + i(h_3 - h_4)$ where $h_n \geqslant 0$, $\|h_n\| \leqslant 1$ and $h_n(x_0) = 0$ for $n = 1, 2, 3, 4$. Lemma 2.2 can be applied to

each function h yielding $|(Th_n)(x_0)| \leqslant 4\varepsilon$ for each n. The result is now a simple consequence of the triangle inequality. □

LEMMA 2.4. *If f and g are elements of the unit ball of $C_0(\Omega)$ then $\|Tf + (I - T)g\| \leqslant 1 + 68\varepsilon$.*

PROOF. Fix an arbitrary point $x_0 \in \Omega$ and choose a positive function $h \in C_0(\Omega)$ for which $h(x_0) = \|h\| = 1$. The functions $f - f(x_0)h$ and $g - g(x_0)h$ are both zero at x_0 and have norm at most two. Lemma 2.3 may be applied to T and $I - T$, giving the inequalities

$$|(Tf)(x_0) - f(x_0)(Th)(x_0)| \leqslant 32\varepsilon,$$
$$|((I - T)g)(x_0) - g(x_0)((I - T)h)(x_0)| \leqslant 32\varepsilon.$$

Thus,

$$|(Tf)(x_0) + ((I - T)g)(x_0)| \leqslant |f(x_0)(Th)(x_0) + g(x_0)((I - T)h)(x_0)| + 64\varepsilon.$$

By Lemma 2.1, $(Th)(x_0) \in R_\varepsilon$, and so there exists $\lambda \in [0, 1]$ with $|\lambda - (Th)(x_0)| \leqslant 2\varepsilon$. It follows that

$$\begin{aligned}|(Tf)(x_0) &+ ((I - T)g)(x_0)| \\ &\leqslant |\lambda f(x_0) + (1 - \lambda)g(x_0)| + 2\varepsilon|f(x_0)| + 2\varepsilon|g(x_0)| + 64\varepsilon \\ &\leqslant 1 + 68\varepsilon.\end{aligned}$$

The point $x_0 \in \Omega$ was arbitrary, and so $\|Tf + (I - T)g\| \leqslant 1 + 68\varepsilon$. □

This set of lemmas makes possible the following characterization of M-summands in $B(C_0(\Omega))^{**}$.

THEOREM 2.5. *A closed subspace J of $B(C_0(\Omega))^{**}$ is an M-summand if and only if it is equal to $zB(C_0(\Omega))^{**}$ for some Hermitian projection $z \in B(C_0(\Omega))^{**}$.*

PROOF. Let z be an Hermitian projection in $B(C_0(\Omega))^{**}$. If it can be established that $\|U\| = \max\{\|zU\|, \|(I - z)U\|\}$ for all $U \in B(C_0(\Omega))^{**}$, then $zB(C_0(\Omega))^{**}$ and $(I - z)B(C_0(\Omega))^{**}$ are complementary M-summands. Inequality in one direction is straightforward. Since z is Hermitian, its norm and spectral radius are equal [**17**], and so $\|z\| = 1$. Then $\|zU\| \leqslant \|U\|$ and a similar argument works with z replaced by $I - z$. It suffices, then, to prove that $\|U\| \leqslant \max\{\|zU\|, \|(I - z)U\|\}$.

First consider operators X, $Y \in B(C_0(\Omega))$ and set $m = \max\{\|X\|, \|Y\|\}$. By [**21**] there exists a net $\{\varepsilon_\alpha\}$ of positive numbers and a net $\{T_\alpha\}$ of operators from $B(C_0(\Omega))$ such that $\lim_\alpha \varepsilon_\alpha = 0$, w^* - $\lim_\alpha T_\alpha = z$ and $W(T_\alpha) \subseteq R_{\varepsilon_\alpha}$. If $f \in C_0(\Omega)$, then, by Lemma 2.4,

$$\|T_\alpha(Xf) + (I - T_\alpha)(Yf)\| \leqslant m\|f\|(1 + 68\varepsilon_\alpha)$$

from which it follows that $\|T_\alpha X + (I - T_\alpha)Y\| \leqslant m(1 + 68\varepsilon_\alpha)$.

Now consider elements K, $L \in B(C_0(\Omega))^{**}$ with $\max\{\|K\|, \|L\|\} = m$. Choose nets $\{K_\beta\}$ and $\{L_\beta\}$ from $B(C_0(\Omega))$ with w^*-limits, respectively K and L, satisfying $\max\{\|K_\beta\|, \|L_\beta\|\} = m$. The previous estimates apply to K_β and L_β, and so

$$\|T_\alpha K_\beta + (I - T_\alpha)L_\beta\| \leqslant m(1 + 68\varepsilon_\alpha)$$

for all α and β. Since T_α and $I - T_\alpha$ are elements of $B(C_0(\Omega))$ we may apply (2) to obtain $\|T_\alpha K + (I - T_\alpha)L\| \leqslant m(1 + 68\varepsilon_\alpha)$ for all α, and may then apply (3) to conclude that $\|zK + (I - z)L\| \leqslant m$. This can be restated as

$$\|zK + (I - z)L\| \leqslant \max\{\|K\|, \|L\|\}$$

for all $K, L \in B(C_0(\Omega))^{**}$.

For an element $U \in B(C_0(\Omega))^{**}$ define K and L to be, respectively, zU and $(I - z)U$. Then

$$\|U\| = \|zK + (I - z)L\| \leqslant \max\{\|K\|, \|L\|\} = \max\{\|zU\|, \|(I - z)U\|\}.$$

This proves that $zB(C_0(\Omega))^{**}$ and $(I - z)B(C_0(\Omega))^{**}$ are complementary M-summands, and it only remains to show that every M-summand arises in this way.

Given an M-summand J in $B(C_0(\Omega))^{**}$, let z be the associated Hermitian projection [**20**, Proposition 3.1]. This projection is also associated with $zB(C_0(\Omega))^{**}$, from which it follows that J and $zB(C_0(\Omega))^{**}$ are equal. □

3. M-ideals in $B(C_0(\Omega))$. In $B(C_0(\Omega))^{**}$ the M-summands have been shown to be in one-to-one correspondence with the Hermitian projections. While it is true that every double annihilator $J^{\perp\perp}$ of an M-ideal J in $B(C_0(\Omega))$ is an M-summand in $B(C_0(\Omega))^{**}$, not every M-summand need arise in this way. In a general Banach algebra A with identity I the Hermitian projections in A^{**} corresponding to M-ideals in A are all lower semicontinuous in the sense discussed in §1 [**21**]. It is interesting to note that the converse is also true.

PROPOSITION 3.1. *An M-summand in A^{**} is the double annihilator of an M-ideal in A if and only if its associated Hermitian projection is lower semicontinuous.*

PROOF. Only the sufficiency of the condition need be established [**21**]. let X be an M-summand in A^{**}. Since X is w^*-closed the preannihilator $X_\perp$ of X is an L-summand in A^* (see the opening remarks of §2). Let P: $A^* \to A^*$ be the associated L-projection. Our aim is to show that $X_\perp$ is w^*-closed so that $X_{\perp\perp}$ is an M-ideal in A. This will follow from the assumption that the Hermitian projection $z = (I - P^*)(e)$ associated with X is lower semicontinuous.

The L-projection P induces a pair of complementary split faces $F_1 = S \cap PA^*$ and $F_2 = S \cap (I - P)A^*$ of the state space S (i.e. $S = F_1 \oplus_{\text{conv}} F_2$). $\ker z$ is w^*-closed since θz is lower semicontinuous on $K = \text{co}(S \cup -iS)$ and $z(S) \subseteq [0, 1]$. An easy computation shows that $F_1 = \ker z \cap S$ so that F_1 is w^*-closed.

Let ϕ be an arbitrary element of PA^*. By [**16**] there exists an absolute constant $c > 0$ so that ϕ may be expressed as $\phi = \sum_{i=1}^4 \lambda_i \phi_i$ with $\phi_i \in S$ and $|\lambda_i| \leqslant c\|\phi\|$. Apply P to this decomposition to obtain

$$\phi = P\phi = \sum_{i=1}^{4} \lambda_i P\phi_i = \sum_{i=1}^{4} \lambda_i \|P\phi_i\| \left(P\phi_i / \|P\phi_i\|\right)$$

where $P\phi_i/\|P\phi_i\| \in F_1$ [**20**]. Thus, if $\psi \in PA^*$ there exist four states $\psi_i \in F_1$ such that $\psi = \sum_{i=1}^4 \mu_i \psi_i$ with $|\mu_i| \leqslant c\|\psi\|$. Now suppose that $\{\psi_\alpha\}$ is a net from PA^* with w^*-limit $\psi \in A^*$. The net is bounded in norm by some constant m and so, for each α,

there exist states $\psi_{i,\alpha} \in F_1$ with

$$\psi_\alpha = \sum_{i=1}^{4} \mu_{i,\alpha}\psi_{i,\alpha}$$

and $|\mu_{i,\alpha}| \leqslant cm$. Passing to convergent subnets if necessary, there exist states $\psi_i \in F_1$ and constants μ_i which are, respectively, the limits of $\{\psi_{i,\alpha}\}$ and $\{\mu_{i,\alpha}\}$. The linear functional ψ is thus in the span of F_1 and lies in $PA^* = X$. This L-summand is w^*-closed and the proof is complete. □

Consider the special case in which Ω is a compact Hausdorff space. Recall from §1 that π: $C_{\mathbf{R}}(\Omega) \to B(C(\Omega))$ given by $\pi f = M_f$ is an isometry of $C_{\mathbf{R}}(\Omega)$ onto $\mathcal{H}(B(C(\Omega)))$. The second adjoint π^{**}: $C(\Omega)^{**} \to B(C(\Omega))^*$ is then an isometric embedding. Notice that $C(\Omega)^{**}$ contains the lower semicontinuous functions on Ω.

PROPOSITION 3.2. *Let $z \in B(C(\Omega))^{**}$ be a lower semicontinuous Hermitian projection. Then there exists a lower semicontinuous projection $f \in C(\Omega)^{**}$ for which $\pi^{**}f = z$.*

PROOF. Since $\theta(z)$ is lower semicontinuous on $K = \mathrm{co}\{S \cup -iS\}$ and $\theta(z)|_{-iS} = 0$ there exists by [**1**, Corollary I.1.4] nets $\{T_\alpha\} \subseteq B(C(\Omega))$ and $\{\varepsilon_\alpha\}$ of positive numbers with the following properties.

(i) $\lim_\alpha \varepsilon_\alpha = 0$.

(ii) $W(T_\alpha) \subseteq R_{\varepsilon_\alpha}$ where R_{ε_α} is the rectangle with vertices $-\varepsilon_\alpha \pm \varepsilon_\alpha i$, $1 + \varepsilon_\alpha \pm \varepsilon_\alpha i$.

(iii) $\lim_\alpha T_\alpha = z$ in the w^*-topology.

(iv) For any state ψ on $B(C(\Omega))$ the net $\{\mathrm{Re}\,\psi(T_\alpha)\}$ is increasing with limit $z(\psi)$.

Let 1 denote the unit in $C(\Omega)$, write $g_\alpha = T_\alpha 1$ and $f_\alpha = \mathrm{Re}\, g_\alpha$. If $\omega \in \Omega$ is an arbitrary point, define a state ϕ_ω on $B(C(\Omega))$ by $\phi_\omega T = (T1)(\omega)$ for $T \in B(C(\Omega))$. Then $f_\alpha(\omega) = \mathrm{Re}(T_\alpha 1)(\omega) = \mathrm{Re}\,\phi_\omega(T_\alpha)$ and so $\{f_\alpha(\omega)\}$ is an increasing net by (iv) and, hence, the pointwise limit f is lower semicontinuous on Ω.

For any fixed $\omega_0 \in \Omega$ and $h \in C(\Omega)$ the function $h - h(\omega_0)1$ vanishes at ω_0 and has norm at most $2\|h\|$. By Lemma 2.3

$$|T_\alpha h(\omega_0) - h(\omega_0)(T_\alpha 1)(\omega_0)| \leqslant 32\|h\|\varepsilon_\alpha.$$

It follows that $\|T_\alpha - \pi(g_\alpha)\| \leqslant 32\varepsilon_\alpha$. In addition, $\sup_{\omega\in\Omega} \mathrm{Im}\,|g_\alpha(\omega)| \leqslant \varepsilon_\alpha$ by Lemma 2.1 so that $\|\pi(f_\alpha) - \pi(g_\alpha)\| \leqslant \varepsilon_\alpha$ and $\|T_\alpha - \pi(f_\alpha)\| \leqslant 33\varepsilon_\alpha$. Since z is the w^*-limit of $\{T_\alpha\}$ it is also the w^*-limit of $\{\pi(f_\alpha)\}$. The w^*-continuity of π^{**} yields $\pi^{**}(f) = z$ which completes the proof. □

THEOREM 3.3. *Let Ω be a compact Hausdorff space. There is a one-to-one correspondence between each pair of the following sets*:

(i) *M-ideals in $B(C(\Omega))$*,

(ii) *lower semicontinuous Hermitian projections in $B(C(\Omega))^{**}$*,

(iii) *lower semicontinuous selfadjoint projections in $C(\Omega)^{**}$*,

(iv) *open subsets of Ω*,

(v) *closed ideals in $C(\Omega)$*.

PROOF. The equivalence of (i) and (ii) is an immediate consequence of Proposition 3.1. A lower semicontinuous projection in $C(\Omega)^{**}$ is simply the characteristic

function of an open set, and it is well known that all closed ideals in $C(\Omega)$ are of the form $\{f \in C(\Omega)\colon f|_E = 0\}$ where $E \subseteq \Omega$ is closed. Thus (iii), (iv) and (v) are equivalent. Finally, the map $\pi^{**}\colon C(\Omega)^{**} \to B(C(\Omega))^{**}$ clearly takes lower semicontinuous and selfadjoint projections to lower semicontinuous Hermitian projections and Proposition 3.2 shows that all such Hermitian projections arise in this way. This proves that (ii) and (iii) are equivalent. □

REMARK 3.4. This theorem states that, given an M-ideal J, there exists an open subset U of Ω with $J = \pi^{**}(\chi_U)B(C(\Omega))^{**} \cap B(C(\Omega))$. One might think that the correspondence between M-ideals in $B(C(\Omega))$ and closed ideals in $C(\Omega)$ should be

$$N \to \{T \in B(C(\Omega)) \mid T(C(\Omega)) \subseteq N\}$$

where N is a closed ideal in $C(\Omega)$. In general, this is incorrect. Let Ω be the unit interval and consider the ideal N of functions vanishing at 0. Let $\{f_n\}_{n=1}^{\infty}$ be an increasing approximate identity for N with pointwise limit f, the characteristic function of $(0, 1]$. Define $T\colon C(\Omega) \to N$ by $Tk = k - k(0)1$ for all $k \in C(\Omega)$. If T is in the M-ideal corresponding to N then

$$T = \pi^{**}(f)T = w^*\text{-}\lim_n \pi(f_n)T.$$

Thus T is the weak limit of the sequence $\{\pi(f_n)T\}_{n=1}^{\infty}$ and so in the norm closure of its convex hull. There must exist functions g_n from the unit ball of N for which $\|T - \pi(g_n)T\| \leqslant 1/n$. For each $h \in N$, $Th = h$, and so $\|h - g_n h\| \leqslant \|h\|/n$. This may be rewritten as $\|(f - g_n)h\| \leqslant \|h\|/n$ for each integer n and for all $h \in N$, from which it follows that f is the uniform limit of continuous functions. This is clearly a contradiction, and so the M-ideal corresponding to N is not $\{T \in B(C(\Omega)) \mid T(C(\Omega)) \subseteq N\}$.

The situation for a locally compact Hausdorff space Ω is more complicated. The algebra $C^b(\Omega)$ of bounded continuous functions on Ω embeds naturally into $B(C_0(\Omega))$ as an algebra of multipliers, and so the set of Hermitian projections in $B(C_0(\Omega))^{**}$ contains at least the projections of $C^b(\Omega)^{**}$. It is natural to conjecture that these constitute the entire set of such projections, but the authors have only succeeded in establishing this under an additional topological hypothesis on Ω.

Recall that a topological space is said to be countably paracompact if each countable open cover has a locally finite refinement. Metrizable spaces, pseudometrizable spaces and σ-compact spaces are all countably paracompact [**13**]. Dowker [**9**], has established the following equivalences.

THEOREM 3.5. *The following are equivalent.*

(i) *Ω is countably paracompact and normal.*

(ii) *The product of Ω with the closed unit interval is normal.*

(iii) *If u and l are, respectively, upper and lower semicontinuous functions satisfying $u(\omega) < l(\omega)$ for all $\omega \in \Omega$, then there exists a continuous function f on Ω with $u < f < l$.*

The relevance of this result to the problem at hand will become apparent in the proof of the following proposition. Recall that the distance $d(x, Y)$ of an element x to a subspace Y of a Banach space X is defined by $d(x, Y) = \inf_{y \in Y}\|x - y\|$.

PROPOSITION 3.6. *Let Ω be a countably paracompact normal space and let f be a bounded real valued function on Ω with the property that $d(fh, C_0(\Omega)) \leqslant \varepsilon\|h\|$ for some $\varepsilon > 0$ and for all $h \in C_0(\Omega)$. Then there exists a real valued function $g \in C^b(\Omega)$ satisfying $\|f - g\| \leqslant 5\varepsilon$.*

PROOF. First suppose that $f \geqslant 0$. Choose an increasing approximate identity $\{e_\alpha\}$ for $C_0(\Omega)$ so that $0 \leqslant e_\alpha \leqslant 1_\Omega$ and, for each $x \in \Omega$, there exists α with $e_\alpha(x) = 1$.

From the hypothesis there exist functions $h_\alpha \in C_0(\Omega)$ with the property that $\|h_\alpha - fe_\alpha\| \leqslant 2\varepsilon$. It may be assumed that each h_α is nonnegative since otherwise, $h_\alpha \vee 0$ may be used as a replacement. Write $l_\alpha = h_\alpha - 2\varepsilon 1_\Omega \in C^b(\Omega)$ and observe that $l_\alpha \leqslant fe_\alpha \leqslant f$.

For a fixed but arbitrary $x \in \Omega$, choose e_α so that $e_\alpha(x) = 1$. Then

$$h_\alpha(x) - f(x) = h_\alpha(x) - e_\alpha(x)f(x) \leqslant 2\varepsilon \quad \text{and} \quad l_\alpha(x) - f(x) \leqslant 4\varepsilon.$$

Consequently, if the lower semicontinuous function l is defined to be $\sup_\alpha l_\alpha$, then $l \leqslant f \leqslant l + 4\varepsilon 1_\Omega$.

The function $\|f\|1_\Omega - f$ is nonnegative and satisfies the hypothesis of the proposition. From above there exists a lower semicontinuous function l_1 with

$$l_1 \leqslant \|f\|1_\Omega - f \leqslant l_1 + 4\varepsilon 1_\Omega.$$

If an upper semicontinuous function is defined by $u = \|f\|1_\Omega - l_1$, then $u - 4\varepsilon 1_\Omega \leqslant f \leqslant u$, and so $u - 5\varepsilon 1_\Omega < f \leqslant l + 4\varepsilon 1_\Omega$. By Theorem 3.5 there exists a function $g \in C^b(\Omega)$ satisfying $u - 5\varepsilon 1_\Omega < g < l + 4\varepsilon 1_\Omega$ from which it follows that $f - 5\varepsilon 1_\Omega < g < f + 4\varepsilon 1_\Omega$. The norm inequality $\|f - g\| \leqslant 5\varepsilon$ is immediate.

To obtain the result for general functions, apply the preceding work to $\|f\|1_\Omega - f$, which is nonnegative. □

THEOREM 3.7. *Let Ω be countably paracompact and normal. There is a one-to-one correspondence between any pair of the following sets*:

(i) *M-ideals in $B(C_0(\Omega))$*,

(ii) *lower semicontinuous Hermitian projections in $B(C_0(\Omega))^{**}$*

(iii) *lower semicontinuous projections in $C^b(\Omega)^{**}$*,

(iv) *closed ideals in $C^b(\Omega)$*,

(v) *open subsets of the Stone-Čech compactification $\beta\Omega$ of Ω.*

PROOF. By definition, $\beta\Omega$ is the maximal ideal space of $C^b(\Omega)$. The equivalence of (iii), (iv) and (v) then follows from Theorem 3.3. Proposition 3.1 establishes the correspondence between (i) and (ii), and it only remains to demonstrate the connection between (ii) and (iii).

Denote by π: $C^b(\Omega) \to B(C_0(\Omega))$ the isomorphism which represents a bounded continuous function as a multiplication operator on $C_0(\Omega)$. The image under π^{**} of the set of lower semicontinuous projections in $C^b(\Omega)^{**}$ is contained in the set of lower semicontinuous Hermitian projections in $B(C_0(\Omega))^{**}$. It will now be shown that every such projection arises in this way.

Let z be a lower semicontinuous Hermitian projection in $B(C_0(\Omega))^{**}$. From Proposition 3.2 there exists a net $\{\varepsilon_\alpha\}$ of positive numbers and a net $\{T_\alpha\}$ from

$B(C_0(\Omega))$ with the properties:

(i) $\lim_\alpha \varepsilon_\alpha = 0$,

(ii) $W(T_\alpha)$ is contained in R_{ε_α},

(iii) $\lim_\alpha T_\alpha = z$ in the w^*-topology,

(iv) for any state ψ on $B(C_0(\Omega))$ the net $\{\operatorname{Re}\psi(T_\alpha)\}$ is increasing with limit $z(\psi)$.

The operators T_α and T_α^{**} have the same numerical range and $C_0(\Omega)^{**}$ may be regarded as an algebra of continuous functions on a compact space. By Proposition 3.2 there exist selfadjoint elements $h_\alpha \in C_0(\Omega)^{**}$ so that

$$\lim_\alpha \|T_\alpha^{**} - \pi^{**}(h_\alpha)\| = 0.$$

Each h_α, when restricted to the point measures in $C_0(\Omega)^*$, can be regarded as a function on Ω. By Proposition 3.6 there exists a net $\{f_\alpha\}$ from the selfadjoint part of $C^b(\Omega)$ satisfying $\lim_\alpha \|T_\alpha - \pi(f_\alpha)\| = 0$.

Write $k_\alpha = f_\alpha - \|T_\alpha - \pi(f_\alpha)\| 1_\Omega \in C^b(\Omega)$ and define a new net $\{g_\beta\}$ indexed by finite subsets $\beta = (\alpha_1, \ldots, \alpha_n)$ of the old net by $g_\beta = \sup_{\alpha \in \beta} k_\alpha$. The net $\{g_\beta\}$ is then increasing with a lower semicontinuous limit $g \in C^b(\Omega)^{**}$. Clearly $\pi^{**}(g) = z$, from which it follows that g is a projection. The proof is complete. □

REMARK 3.8. Without any extra topological assumptions on Ω it is easy to see that the M-summands of $B(C_0(\Omega))$ are of the form $\{T \in B(C_0(\Omega)) \colon \chi_U T = T\}$ where U is both open and closed in $\beta\Omega$. For if M is an M summand of $B(C_0(\Omega))$ and P is the associated M-projection, then $Pe = z \in \mathcal{K}(B(C_0(\Omega)))$. Hence $z = M_h$ for some $h \in C^b_{\mathbf{R}}(\Omega) = C(\beta\Omega)$. Clearly $M = \{T \in B(C_0(\Omega)) \colon zT = T\} = zB(C_0(\Omega))$ and $h = \chi_U$ since z is a projection.

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DEPARTMENT OF MATHEMATICS, TEXAS A & M UNIVERSITY; COLLEGE STATION, TEXAS 77843